Some nonlinear dispersive waves arising in compressible hyperelastic plates

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Abstract

In this paper we study finite deformations in a pre-stressed, hyperelastic compressible plate. For small-amplitude nonlinear waves, we obtain equations that involve an amplitude parameter ε. Using an asymptotic perturbation technique, we derive a new family of two-dimensional nonlinear dispersive equations. This family includes the KdV, Kadomtsev-Petviashvili and Camassa-Holm equations.

1 Introduction

A fundamental problem in nonlinear elasticity is to understand the relationship between the three-dimensional theory and the theories for lower-dimensional objects (plates, shells, rods, ...). There are many such theories (see e.g. [2], [8]). In this paper we consider a thin compressible hyperelastic plate and derive the two-dimensional nonlinear dispersive wave equation

$$\{u_t - u_{xxt} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx})\}_x - \alpha u_{yy} + \beta u_{xxyy} = 0, \tag{1.1}$$

where u represents the displacement in the $Z-$direction relative to a pre-stressed state. There are four material parameters (denoted by γ, α, β) in the new equation which play an important role. Equation (1.1) generalizes several well-known equations:

(1) When $\gamma = \beta = 0$, (1.1) becomes the regularized long-wave Kadomtsev-Petviashvili (RLW-KP) equation (see [5]).

(2) When $\alpha = \beta = 0$ and $\gamma = 1$, there is no $y-$dependence in the equation, and it becomes the Camassa-Holm equation ([7]). So we can think of (1.1) as the Camassa-Holm equation in two dimensions.

(3) When $\alpha = \beta = \gamma = 0$, (1.1) simply reduces to the regularized long-wave Korteweg-de Vries (RLW-KdV) equation or BBM equation ([4]).

Furthermore, if we suppose that the plate is sufficiently stiff so that it resists bending, then we obtain the more general equation

$$\{u_t - u_{xxt} + \delta u_{xxxxt} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx})\}_x - \alpha u_{yy} + \beta u_{xxyy} = 0, \tag{1.2}$$

where the parameter δ is related to the stiffness.

There are many one-dimensional models of nonlinear axial-radial deformation waves in hyperelastic rods. To overcome the difficulty of nonlinearity, the assumption that the wave amplitude is small but finite (which means small amplitude nonlinear waves are being considered) is adopted in order to derive simpler model equations for both incompressible and compressible materials (e.g. [19], [20], [23], [24], [9], [10], [11]). In particular, in [11], Dai considered thin cylindrical rods and derived the KdV equation

$$u_t + \delta_1 u u_x + \delta_2 u_{xxx} = 0 \tag{1.3}$$

as a model equation for waves of large wavelength, and

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + u u_{xxx}) \tag{1.4}$$

as a model when the wavelength is not too large. In the two last equations, $u(x,t)$ represents the radial stretch relative to a pre-stressed state.

In this paper we use a similar technique as in the study of rods to study nonlinear small but finite-amplitude traveling waves in a general compressible hyperelastic plate. To reduce the three-dimensional field equations to an approximate two-dimensional plate equation, we assume that the thickness of the plate is small in comparison to the other dimensions. Other kinematic assumptions will be presented in Section 2.

The deformations we consider in the first part are small perturbations superimposed on a pre-stressed state as follows:

$$\begin{cases} \tilde{x} = CX + W(X,Y,T), \\ \tilde{y} = BY, \\ \tilde{z} = [A + U(X,Y,T)]Z, \end{cases} \tag{1.5}$$

where A, B, C are constants which characterize the unperturbed pre-stressed state, and W, U are small perturbations. Using Hamilton's principle, we get two very complicated fully nonlinear governing equations. Scaling the variables in a certain way and using a perturbation technique, we derive less complicated nonlinear dispersive equations. The coefficients of these equations depend on the thickness of the plate in an undisturbed configuration, on the prestress and on material properties. Here are the main results:

(1) If we assume that the wavelength in the $X-$direction is large and the variation in the $Y-$direction is gradual (this is called the unidimensional wave), we obtain the governing equation

$$(u_\tau + \kappa_1 u u_\xi + \kappa_2 u_{\xi\xi\xi})_\xi + \kappa_3 u_{\eta\eta} = 0, \tag{1.6}$$

which is the well-known Kadomtsev-Petviashvili (KP) equation that has been developed in the contexts of shallow-water theory and plasmas ([1], [12], [14], [15], [16]). In (1.6), u represents the displacement in the $Z-$direction relative to a pre-stressed state. Furthermore, $\tau = \varepsilon_1 t'$, ($\varepsilon_1 \ll 1$ is a small positive parameter) where t' is the dimensionless time, $\xi = x' - t'$ and $\eta = \sqrt{\varepsilon_1} y'$ (where x' and y' are the dimensionless space variables), and the κ_i's are constants determined by the pre-stress and material parameters.

(2) If we consider the case when the dispersive term in (1.6) vanishes and use a different scaling, we obtain

$$\{u_\tau + \sigma_1 u_{\xi\xi\tau} + \sigma_3 u u_\xi + \sigma_4(2u_\xi u_{\xi\xi} + u u_{\xi\xi\xi})\}_\xi + \sigma_5 u_{\eta\eta} + \sigma_6 u_{\xi\xi\eta\eta} = 0. \tag{1.7}$$

Using the conditions $\sigma_3 \neq 0$ and $\sigma_1 < 0$, we can of course transform (1.7) into (1.1). In (1.1), the parameters α, β and γ are determined by the pre-stress and material parameters. Some of the solutions of (1.1) are shock waves with infinitely steep profiles. For instance , it is well-known in the Camassa-Holm case that there are shock wave solutions ([11]).

We point out that in the completely different context of shallow water theory, Johnson [13] derived another version of two-dimensional Camassa-Holm equation:

$$(u_\tau + \kappa u_x + 3\varepsilon u u_x - \varepsilon\delta^2 u_{xx\tau})_x + \varepsilon^2 u_{yy} = \varepsilon^2\delta^2(2u_x u_{xx} + u u_{xxx})_x \tag{1.8}$$

where u is the fluid velocity along the $x-$direction, $\kappa = \frac{2}{5}\sqrt{\frac{3}{5}}$, ε is the amplitude parameter, δ is the shallowness parameter and τ is the scaled time. (1.8) appears to be a special case of (1.1). However, unlike the derivation of (1.8) which makes use of the Hamiltonian, we derive (1.1) by a consistent perturbation procedure. In (1.8), the right-hand side is of higher order (since both ε and δ are small.), while in (1.1) all terms are of the same order. In (1.1) we have the βu_{xxyy} term which is a higher derivative in x and y while in (1.8) there is only one term involving the $y-$derivative. (1.8) was derived in the context of shallow-water waves by assuming that both u and u_x tend to zero sufficiently fast as $|x|$ tends to infinity. Such a limitation excludes the possibility of periodic traveling waves. By contrast, for (1.1), which is derived in the context of nonlinear dispersive waves in hyperelastic plates, the only requirement is that u and u_x are bounded; hence periodic solutions may arise.

Later in the paper we make a modification of the deformation in the $Z-$direction in order to include a stiffness effect. Motivated in part by the common use of second-gradient models to compensate for the loss of dependence on the Z variable and to overcome the lack of regularity in nonlinear elasticity (see e.g. [17], [3], [21], [22]), we describe the deformations as

$$\begin{cases} \tilde{x} = CX + W(X, Y, T), \\ \tilde{y} = BY, \\ \tilde{z} = [A + U(X, Y, T) + \lambda U_X(X, Y, T)]Z, \end{cases} \tag{1.9}$$

where $\lambda \geq 0$ is the stiffness parameter. More detailed explanations will be presented in Section 5. This model will lead to equation (1.2). The advantage of including stiffness is that the solutions will be more regular. In fact, we expect that there are no shocks if $\lambda > 0$. An analysis of the solutions of (1.2) will appear in a forthcoming paper.

The outline of this paper is as follows: In Section 2 we start with (1.5) and use the variational principle to derive the general governing equations for small but finite-amplitude waves. The obtained equations are fully nonlinear and contain many terms. In Section 3 we restrict our attention to long waves and obtain the model equation (1.6) which is the Kadomtsev-Petviashvili equation, and which reduces to the Korteweg-de Vries (KdV) equation when restricted to one-dimension. Then we turn our attention to the finite-wavelength and small-amplitude waves in Section 4. We obtain an equation which involves an amplitude parameter ε. Using an asymptotic perturbation in ε, we derive the two-dimensional nonlinear dispersive equation (1.1). In the last section, we assume the more general deformation (1.9) and repeat what we did in Section 2, 3 and 4 to get the more regular equation (1.2).

2 Governing equations

The object we study is a wide but thin plate composed of a compressible hyperelastic material, which has thickness $2a$ in its undistorted state. Material points in the solid are identified by their position vectors $\mathbf{X} = (X, Y, Z)$ in the reference configuration (see Fig.2.1), while the corresponding current position vector is denoted by $\mathbf{x}(\mathbf{X}) = (\tilde{x}, \tilde{y}, \tilde{z})$. The second rank tensor $F = \nabla \mathbf{x}$ is called the deformation gradient and satisfies $\det F > 0$ in the solid. The product $G = F^T F$ is called the right Cauchy-Green tensor.

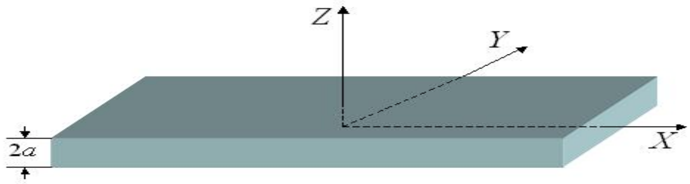

Figure 2.1: Reference configuration

In this paper we shall consider wave motions superimposed on a pre-stressed state. For simplicity, we begin with a uniformly pre-stressed state described by

$$\tilde{x} = CX, \tilde{y} = BY, \tilde{z} = AZ, \tag{2.1}$$

where A, B, C are constants. Then we superimpose small disturbances $\tilde{W}(X, Y, Z, T)$, $\tilde{U}(X, Y, Z, T)$ along the $X, Z-$axes on the stretched state (2.1) (see Fig.2.2), where T is the time variable.

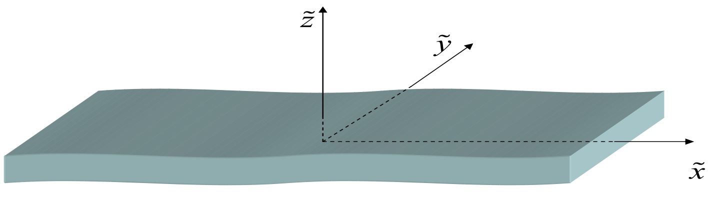

Figure 2.2: Deformed configuration

To obtain two-dimensional plate equations, we make similar kinematic assumptions as in the literature of classical theory of elasticity ([23], [24], [9], [11], [18]):

(a) If we expand the incremental X-directional displacement $\tilde{W}(X,Y,Z,T)$ with respect to Z (assuming that the plate is thin), we get

$$\tilde{W}(X,Y,Z,T) = W(X,Y,T) + W_1(X,Y,T)Z + \cdots \tag{2.2}$$

Here we only consider small disturbances so that as a first approximation we can retain the first term in the above expansion. That is,

$$\tilde{W} = W(X,Y,T) \tag{2.3}$$

and hence

$$\tilde{x} = CX + W(X,Y,T). \tag{2.4}$$

This implies the kinematic assumption that each cross-sectional material plane normal to the middle surface remains planar and normal to the $X-$axis. It will be a good approximation if, along the $X-$axis, the bending effects in the $Z-$direction are negligible.

(b) Taylor expanding $\tilde{U}(X,Y,Z,T)$ in Z at $Z=0$, we get

$$\tilde{U}(X,Y,Z,T) = \tilde{U}(X,Y,0,T) + \tilde{U}_Z(X,Y,0,T)Z + \cdots \tag{2.5}$$

If we assume the middle surface remains unstrained, the midplane $\{Z=0\}$ of the plate remains fixed and the first term in (2.5) vanishes. Also assuming that the thickness of the plate is small, we can drop the higher powers of Z so that

$$\tilde{U} = U(X,Y,T)Z. \tag{2.6}$$

Thus such a motion is described by the equations

$$\left\{\begin{array}{l} \tilde{x} = CX + W(X,Y,T), \\ \tilde{y} = BY, \\ \tilde{z} = [A + U(X,Y,T)]Z. \end{array}\right. \tag{2.7}$$

Given such a deformation, we can compute the deformation gradient F and the right Cauchy-Green strain tensor $G = F^T F$. The three principal invariants of G are defined as $I_1 = \mathrm{tr}G$, $I_2 = \mathrm{tr}(\mathrm{cof}(G))$ and $I_3 = \det G$ and are given explicitly by

$$\begin{array}{rl} I_1 = & (A+U)^2 + B^2 + Z^2U_Y^2 + Z^2U_X^2 + (C+W_X)^2 + W_Y^2, \\ I_2 = & (A+U)^2[B^2 + W_Y^2 + (C+W_X)^2] + B^2(C+W_X)^2 \\ & +Z^2\{B^2U_X^2 + [W_YU_X - (C+W_X)U_Y]^2\}, \\ I_3 = & (A+U)^2B^2(C+W_X)^2. \end{array}$$

Now we suppose that the plate is composed of an isotropic compressible hyperelastic material. For such a material, the strain energy Φ is a function of the three invariants I_1, I_2, I_3. From the above three expressions, it can be seen that Φ can also be regarded as a function of 6 variables, namely

$$\Phi = \phi(q_1, q_2, q_3, q_4, q_5, q_6), \tag{2.8}$$

where

$$q_1 = U, q_2 = W_Y^2, q_3 = W_X, q_4 = Z^2U_Y^2, q_5 = Z^2U_X^2, q_6 = Z^2U_XU_YW_Y.$$

In this paper, we consider small but finite-amplitude waves. For small q_i's, we Taylor expand (2.8) to get the complicated expression:

$$\Phi = (\mathbf{A}) + Z^2 \cdot (\mathbf{B}) + \cdots , \tag{2.9}$$

where

$$\begin{aligned}
(\mathbf{A}) &= \phi_1 U + \phi_2 W_Y^2 + \phi_3 W_X + \frac{1}{2}\phi_{11} U^2 + \phi_{12} U W_Y^2 + \phi_{13} U W_X + \frac{1}{2}\phi_{22} W_Y^4 \\
&\quad + \phi_{23} W_Y^2 W_X + \frac{1}{2}\phi_{33} W_X^2 + \frac{1}{6}\phi_{111} U^3 + \frac{1}{2}\phi_{112} U^2 W_Y^2 + \frac{1}{2}\phi_{113} U^2 W_X \\
&\quad + \frac{1}{2}\phi_{122} U W_Y^4 + \phi_{123} U W_Y^2 W_X + \frac{1}{2}\phi_{133} U W_X^2 + \frac{1}{6}\phi_{222} W_Y^6 \\
&\quad + \frac{1}{2}\phi_{223} W_Y^4 W_X + \frac{1}{2}\phi_{233} W_Y^2 W_X^2 + \phi_{237} W_Y^2 W_X U_X + \frac{1}{6}\phi_{333} W_X^3, \\
(\mathbf{B}) &= \phi_4 U_Y^2 + \phi_5 U_X^2 + \phi_6 U_X U_Y W_Y \\
&\quad + \phi_{14} U U_Y^2 + \phi_{15} U U_X^2 + \phi_{16} U U_X U_Y W_Y \\
&\quad + \phi_{24} W_Y^2 U_Y^2 + \phi_{25} W_Y^2 U_X^2 + \phi_{26} W_Y^3 U_X U_Y \\
&\quad + \phi_{34} W_X U_Y^2 + \phi_{35} W_X U_X^2 + \phi_{36} W_X U_X U_Y W_Y .
\end{aligned} \tag{2.10}$$

Here

$$\phi_i = \frac{\partial \phi}{\partial q_i}\Big|_0, \phi_{ij} = \frac{\partial^2 \phi}{\partial q_i \partial q_j}\Big|_0, \phi_{ijk} = \frac{\partial^3 \phi}{\partial q_i \partial q_j \partial q_k}\Big|_0$$

are constant coefficients related to A, B and C.

We shall use Hamilton's principle to derive the governing equations for the two unknowns W, U. The potential energy per unit area of the plate is

$$\Psi = \int_{-a}^{a} \Phi dZ = 2\Big(a \cdot (\mathbf{A}) + \frac{a^3}{3} \cdot (\mathbf{B})\Big) + \cdots . \tag{2.11}$$

The kinetic energy per unit area is

$$J = \int_{-a}^{a} \frac{1}{2}\rho (W_T^2 + Z^2 U_T^2) dZ = \rho a (W_T^2 + \frac{a^2}{3} U_T^2), \tag{2.12}$$

where ρ is the material density in the reference configuration.

Then the Lagrangian is given by

$$L = J - \Psi = L(U, U_T, U_X, U_Y, W_T, W_X, W_Y). \tag{2.13}$$

The Euler-Lagrange equations become

$$\frac{\partial L}{\partial U} - \frac{\partial}{\partial T}\frac{\partial L}{\partial U_T} - \frac{\partial}{\partial X}\frac{\partial L}{\partial U_X} - \frac{\partial}{\partial Y}\frac{\partial L}{\partial U_Y} = 0, \tag{2.14}$$

$$\frac{\partial}{\partial T}\frac{\partial L}{\partial W_T} + \frac{\partial}{\partial X}\frac{\partial L}{\partial W_X} + \frac{\partial}{\partial Y}\frac{\partial L}{\partial W_Y} = 0. \tag{2.15}$$

Hence we obtain

$$
\begin{aligned}
&\Big(\phi_1 + \phi_{11}U + \phi_{13}W_X\Big) + \Big(\phi_{12}W_Y^2 + \frac{1}{2}\phi_{111}U^2 + \phi_{113}UW_X + \frac{1}{2}\phi_{133}W_X^2\Big) \\
&+\frac{a^2}{3}\Big\{\rho U_{TT} - 2(\phi_4 U_{YY} + \phi_5 U_{XX}) - [\phi_{14}U_Y^2 + \phi_{15}U_X^2 + 2\phi_{14}(UU_Y)_Y + 2\phi_{34}(U_Y W_X)_Y \\
&\qquad + 2\phi_{15}UU_{XX} + 2\phi_{35}(U_X W_X)_X + \phi_6(U_X W_Y)_Y + \phi_6(W_Y U_Y)_X]\Big\} \\
&+\cdots\cdots = 0
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
&-\rho W_{TT} + \Big(\phi_{13}U_X + \phi_{33}W_{XX} + 2\phi_2 W_{YY}\Big) + \Big\{\phi_{113}UU_X + \phi_{133}(UW_X)_X + \phi_{333}W_X W_{XX}\Big\} \\
&+ \Big[2\phi_{12}(UW_Y)_Y + 2\phi_{23}(W_X W_Y)_Y + 2\phi_{23}W_Y W_{XY}\Big] \\
&+\frac{a^2}{3}\Big\{2\phi_{35}U_X U_{XX} + 2\phi_{34}U_Y U_{XY} + \phi_6(U_X U_Y)_Y\Big\} + \cdots\cdots = 0,
\end{aligned}
\tag{2.17}
$$

which are the governing equations for small but finite-amplitude waves. Later on we shall derive much simplified forms of these equations.

When $W = U = 0$, we see from (2.16) that $\phi_1 = 0$. This is a relation that must be satisfied by the constants A, B, C. Throughout this paper, we shall consider waves propagating principally along one direction (the $X-$axis) with weak dispersive transverse effects, often called essentially unidimensional waves.

We shall always use the following type of scaling to analyze the governing equations

$$
W = hw,\ U = \frac{h}{l}u,\ X = lx',\ Y = ly',\ T = \frac{l}{c}t', \tag{2.18}
$$

where h is a characteristic displacement in the $X-$direction and l is the characteristic wave-length. We shall always assume that

$$
h \ll l. \tag{2.19}
$$

This means that in the $Z-$direction the wave amplitude is small. The value of the characteristic velocity c is motivated as follows. As a first crude approximation, we consider the linearized equations of (2.16) and (2.17) and neglect the $O(a^2)$ terms. We thus obtain

$$
\begin{cases}
\phi_{11}U + \phi_{13}W_X = 0 \\
-\rho W_{TT} + \phi_{13}U_X + \phi_{33}W_{XX} + 2\phi_2 W_{YY} = 0.
\end{cases}
\tag{2.20}
$$

Since we consider only essentially unidimensional waves, we drop the W_{YY} term. Then the principal propagation speed c of small disturbances for (2.20) is

$$
c^2 = \frac{1}{\rho}\frac{\phi_{11}\phi_{33} - \phi_{13}^2}{\phi_{11}}. \tag{2.21}
$$

This is our choice of c.

The motivation for the scaling (2.18) is based on the assumption that

$$
U = O(W_X), \tag{2.22}
$$

as in [11]. To explain this assumption, we first consider an incompressible material, for which we know that $I_3 = 1$, i.e.

$$(A+U)^2 B^2 (C+W_X)^2 = 1. \tag{2.23}$$

We should have $A^2B^2C^2 = 1$ (for $U = W = 0$). Hence for small disturbances, we obtain for an incompressible material that

$$U \approx -\frac{A}{C} W_X. \tag{2.24}$$

In this paper we are considering a compressible material, and relation (2.24) does not hold. However, it is reasonable to expect that the magnitude of the disturbances will not change drastically when compressibility comes into play. Therefore, for the present problem we assume that (2.22) holds.

3 Long waves

In this section, we further assume that

(a) The wavelength l is large compared to the thickness $2a$.

(b) hl and a^2 have the same order of magnitude.

Substituting (2.18) into (2.16) and (2.17), we get

$$\begin{aligned}
&\Big(\phi_{11}u + \phi_{13}w_{x'}\Big) + \varepsilon_1\Big(\phi_{12}w_{y'}^2 + \frac{1}{2}\phi_{111}u^2 + \phi_{113}uw_{x'} + \frac{1}{2}\phi_{133}w_{x'}^2\Big)\\
&+\frac{\theta}{3}\Big\{\rho c^2 u_{t't'} - 2(\phi_4 u_{y'y'} + \phi_5 u_{x'x'}) - \varepsilon_1[\phi_{14}u_{y'}^2 + \phi_{15}u_{x'}^2 + 2\phi_{14}(uu_{y'})_{y'}\\
&\qquad +2\phi_{34}(u_{y'}w_{x'})_{y'} + 2\phi_{15}uu_{x'x'} + 2\phi_{35}(u_{x'}w_{x'})_{x'} + \phi_6(u_{x'}w_{y'})_{y'} + \phi_6(w_{y'}u_{y'})_{x'}]\Big\}\\
&+\cdots\cdots = 0,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&-\rho c^2 w_{t't'} + \Big(\phi_{13}u_{x'} + \phi_{33}w_{x'x'} + 2\phi_2 w_{y'y'}\Big) + \varepsilon_1\Big[\phi_{113}uu_{x'} + \phi_{133}(uw_{x'})_{x'} + \phi_{333}w_{x'}w_{x'x'}\Big]\\
&+\varepsilon_1\Big[2\phi_{12}(uw_{y'})_{y'} + 2\phi_{23}(w_{x'}w_{y'})_{y'} + 2\phi_{23}w_{y'}w_{x'y'}\Big] + \frac{\varepsilon_1\theta}{3}\Big(2\phi_{35}u_{x'}u_{x'x'}\Big)\\
&+\cdots\cdots = 0,
\end{aligned} \tag{3.2}$$

where $\varepsilon_1 = h/l$, $\theta = a^2/l^2$. By assumption (b), the parameters ε_1 and θ have the same order of magnitude.

In order to derive an essentially unidimensional model equation which is asymptotically valid in the far-field, we shall scale x' and y' differently. Following the approach usually adopted for the Kadomtsev-Petviashvili equation ([12], [13]), we introduce the transformations

$$\xi = x' - t',\ \eta = \sqrt{\varepsilon_1}y',\ \tau = \varepsilon_1 t'. \tag{3.3}$$

Plugging (3.3) into (3.1), (3.2), we obtain

$$\phi_{11}u + \phi_{13}w_\xi + \frac{\theta}{3}(\rho c^2 - 2\phi_5)u_{\xi\xi} + \varepsilon_1\Big(\frac{1}{2}\phi_{111}u^2 + \phi_{113}uw_\xi + \frac{1}{2}\phi_{133}w_\xi^2\Big) + \cdots\cdots = 0, \quad (3.4)$$

and

$$-\rho c^2 w_{\xi\xi} + \Big(\phi_{13}u_\xi + \phi_{33}w_{\xi\xi}\Big) + \varepsilon_1\Big\{2\rho c^2 w_{\xi\tau} + 2\phi_2 w_{\eta\eta} + \phi_{113}uu_\xi + \phi_{133}(uw_\xi)_\xi + \phi_{333}w_\xi w_{\xi\xi}\Big\}$$
$$+\cdots\cdots = 0, \quad (3.5)$$

where we have dropped the quadratic terms in (ε_1, θ).

Furthermore, we assume that u and w have the following asymptotic expansions in ε_1:

$$u = u_0 + \varepsilon_1 u_1 + \varepsilon_1^2 u_2 + \cdots, \quad (3.6)$$
$$w = w_0 + \varepsilon_1 w_1 + \varepsilon_1^2 w_2 + \cdots \quad (3.7)$$

and plug them into the above two equations. At the lowest order $O(1)$, we have

$$\mathbf{M_0 S_0} = \mathbf{0}, \quad (3.8)$$

where

$$\mathbf{M_0} = \begin{bmatrix} \phi_{11} & \phi_{13} \\ \phi_{13}\partial_\xi & (\phi_{33} - \rho c^2)\partial_\xi \end{bmatrix}, \quad \mathbf{S_0} = \begin{bmatrix} u_0 \\ w_{0\xi} \end{bmatrix}. \quad (3.9)$$

Hence we obtain the relation

$$w_{0\xi} = s u_0, \quad \text{where} \quad s = -\frac{\phi_{11}}{\phi_{13}}. \quad (3.10)$$

By (2.21) the left nullvector of $\mathbf{M_0}$ is

$$\mathbf{L_e} = \begin{bmatrix} \phi_{13}\partial_\xi & -\phi_{11} \end{bmatrix}. \quad (3.11)$$

At the next order $O(\varepsilon_1)$, we have

$$\mathbf{M_0 S_1} + \mathbf{T_1} u_{0\tau} + \mathbf{T_2} w_{0\eta\eta} + \mathbf{R_1} u_{0\xi\xi} + \mathbf{Q_1} u_0^2 = 0, \quad (3.12)$$

where

$$\mathbf{S_1} = \begin{bmatrix} u_1 \\ w_{1\xi} \end{bmatrix}, \quad \mathbf{T_1} = \begin{bmatrix} 0 \\ 2\rho c^2 s \end{bmatrix}, \quad \mathbf{T_2} = \begin{bmatrix} 0 \\ 2\phi_2 \end{bmatrix},$$

$$\mathbf{R_1} = \begin{bmatrix} (\theta/\varepsilon_1)\frac{1}{3}(\rho c^2 - 2\phi_5) \\ 0 \end{bmatrix}, \quad \mathbf{Q_1} = \begin{bmatrix} \frac{1}{2}\phi_{111} + \phi_{113}s + \frac{1}{2}\phi_{133}s^2 \\ (\frac{1}{2}\phi_{113} + \phi_{133}s + \frac{1}{2}\phi_{333}s^2)\partial_\xi \end{bmatrix}.$$

We then multiply the left hand side of (3.12) by the left nullvector $\mathbf{L_e}$, use (3.10) and take an additional derivative in ξ to obtain

$$\left(u_{0\tau} + \kappa_1 u_0 u_{0\xi} + \kappa_2 u_{0\xi\xi\xi}\right)_{\xi} + \kappa_3 u_{0\eta\eta} = 0, \tag{3.13}$$

where

$$\kappa_1 = \frac{\phi_{111} + 3\phi_{113}s + 3\phi_{133}s^2 + \phi_{333}s^3}{2\rho c^2 s^2}, \quad \kappa_2 = \frac{(\theta/3\varepsilon_1)(\rho c^2 - 2\phi_5)}{2\rho c^2 s^2}, \quad \kappa_3 = \frac{\phi_2}{\rho c^2}.$$

Equation (3.13) the well-known Kadomtsev-Petviashvili (KP) equation, which is the asymptotically valid far-field equation which models the essentially unidimensional, long, small but finite-amplitude waves in a pre-stressed stiff plate composed of a compressible hyperelastic material.

Furthermore, if we restrict our considerations to the unidirectional waves, the $\eta-$dependence in (3.13) disappears, and we obtain the one-dimensional KdV equation

$$u_{0\tau} + \kappa_1 u_0 u_{0\xi} + \kappa_2 u_{0\xi\xi\xi} = 0. \tag{3.14}$$

4 Short waves

In this section, we assume that

(c) The wavelength l is of the same order of magnitude as the thickness of the plate $2a$.

(d) The material constants satisfy:

$$\rho c^2 = 2\phi_5. \tag{4.1}$$

We first explain assumption (d). In the previous section we derived the KP equation as a model equation. It has been proved that if the KP equation admits solitary-wave solutions, then the solutions must be smooth (see [6]). Hence the KP model equation does not solve the question of whether non-smooth solitary wave solutions can arise. It is known that the nonlinearity could be the source of a discontinuity, and the dispersive terms could help to smooth out the solution. This suggests consideration of the case that the dispersive term vanish. For example, our assumption (4.1) is consistent with the vanishing of κ_2 in the long-wave case (3.13).

Effectively, by (c) we can take l to be a in (2.18) and hence the scaling becomes

$$W = hw,\ U = \frac{h}{a}u,\ X = ax',\ Y = ay',\ T = \frac{a}{c}t'. \tag{4.2}$$

Let $\varepsilon = h/a$. The governing equations (2.16) and (2.17) now become

$$\begin{aligned}
&\left(\phi_{11}u + \phi_{13}w_{x'}\right) + \varepsilon\left(\phi_{12}w_{y'}^2 + \frac{1}{2}\phi_{111}u^2 + \phi_{113}uw_{x'} + \frac{1}{2}\phi_{133}w_{x'}^2\right) \\
&+\frac{1}{3}\Big\{\rho c^2 u_{t't'} - 2(\phi_4 u_{y'y'} + \phi_5 u_{x'x'}) - \varepsilon[\phi_{14}u_{y'}^2 + \phi_{15}u_{x'}^2 + 2\phi_{14}(uu_{y'})_{y'} \\
&\qquad +2\phi_{34}(u_{y'}w_{x'})_{y'} + 2\phi_{15}uu_{x'x'} + 2\phi_{35}(u_{x'}w_{x'})_{x'} + \phi_6(u_{x'}w_{y'})_{y'} + \phi_6(w_{y'}u_{y'})_{x'}]\Big\} \\
&+\cdots\cdots = 0
\end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
&-\rho c^2 w_{t't'} + \Big(\phi_{13}u_{x'} + \phi_{33}w_{x'x'} + 2\phi_2 w_{y'y'}\Big) + \varepsilon\Big[\phi_{113}uu_{x'} + \phi_{133}(uw_{x'})_{x'} + \phi_{333}w_{x'}w_{x'x'}\Big] \\
&+ \varepsilon\Big[2\phi_{12}(uw_{y'})_{y'} + 2\phi_{23}(w_{x'}w_{y'})_{y'} + 2\phi_{23}w_{y'}w_{x'y'}\Big] + \frac{\varepsilon}{3}\Big(2\phi_{35}u_{x'}u_{x'x'}\Big) \\
&+ \cdots\cdots = 0. \qquad (4.4)
\end{aligned}
$$

As before, we shall regard ε as a small parameter but retain the nonlinear terms. Hence we are considering small but finite-amplitude waves. Because the characteristic scale is a, the waves are of finite length.

To obtain the far-field model equation for an essentially unidimensional wave equation, we substitute (3.3) (with ε_1 replaced by ε) into (4.3) and (4.4) we obtain

$$
\begin{aligned}
&\phi_{11}u + \phi_{13}w_\xi + \frac{1}{3}(\rho c^2 - 2\phi_5)u_{\xi\xi} + \varepsilon\Big\{(\frac{1}{2}\phi_{111}u^2 + \phi_{113}uw_\xi + \frac{1}{2}\phi_{133}w_\xi^2) - \frac{2}{3}\rho c^2 u_{\xi\tau} \\
&- \frac{2}{3}\phi_4 u_{\eta\eta} - \frac{1}{3}[\phi_{15}u_\xi^2 + 2\phi_{15}uu_{\xi\xi} + 2\phi_{35}(u_\xi w_\xi)_\xi]\Big\} + \cdots\cdots = 0 \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\rho c^2 w_{\xi\xi} + \Big(\phi_{13}u_\xi + \phi_{33}w_{\xi\xi}\Big) \\
&+\varepsilon\Big\{2\rho c^2 w_{\xi\tau} + 2\phi_2 w_{\eta\eta} + \phi_{113}uu_\xi + \phi_{133}(uw_\xi)_\xi + \phi_{333}w_\xi w_{\xi\xi} + \frac{2}{3}\phi_{35}u_\xi u_{\xi\xi}\Big\} \\
&+ \cdots\cdots = 0, \qquad (4.6)
\end{aligned}
$$

where we have dropped the quadratic terms in ε.

At the lowest order $O(1)$, the equations are almost identical to those in Section 3 and we have

$$\mathbf{M_0 S_0} = \mathbf{0}, \qquad (4.7)$$

where $\mathbf{M_0}, \mathbf{S_0}$ are the same as in (3.9). Hence we obtain the relation

$$w_{0\xi} = su_0, \quad \text{where} \quad s = -\frac{\phi_{11}}{\phi_{13}}. \qquad (4.8)$$

The left eigenvector of $\mathbf{M_0}$ is the same as in (3.11). At the next order $O(\varepsilon)$, we find

$$\mathbf{M_0 S_1} + \mathbf{T_1}u_{0\tau} + \mathbf{T_2}u_{0\xi\tau} + \mathbf{T_3}w_{0\eta\eta} + \mathbf{T_4}u_{0\eta\eta} + \mathbf{Q_1}u_0^2 + \mathbf{Q_2}u_{0\xi}^2 + \mathbf{Q_3}u_0u_{0\xi\xi} = 0, \qquad (4.9)$$

where

$$\mathbf{S_1} = \begin{bmatrix} u_1 \\ w_{1\xi} \end{bmatrix}, \mathbf{T_1} = \begin{bmatrix} 0 \\ 2\rho c^2 s \end{bmatrix}, \mathbf{T_2} = \begin{bmatrix} -\frac{2}{3}\rho c^2 \\ 0 \end{bmatrix}, \mathbf{T_3} = \begin{bmatrix} 0 \\ 2\phi_2 \end{bmatrix}, \mathbf{T_4} = \begin{bmatrix} -\frac{2}{3}\phi_4 \\ 0 \end{bmatrix},$$

$$\mathbf{Q_1} = \begin{bmatrix} \frac{1}{2}\phi_{111} + \phi_{113}s + \frac{1}{2}\phi_{133}s^2 \\ (\frac{1}{2}\phi_{113} + \phi_{133}s + \frac{1}{2}\phi_{333}s^2)\partial_\xi \end{bmatrix},$$

$$\mathbf{Q_2} = \begin{bmatrix} -\frac{1}{3}(\phi_{15} + 2\phi_{35}s) \\ \frac{1}{3}\phi_{35}\partial_\xi \end{bmatrix}, \quad \mathbf{Q_3} = \begin{bmatrix} -\frac{2}{3}(\phi_{15} + \phi_{35}s) \\ 0 \end{bmatrix}.$$

Multiplying the left-hand side of (4.9) by $\mathbf{L_e}$, we obtain

$$u_{0\tau} + \sigma_1' u_{0\xi\xi\tau} + \sigma_2' u_0 u_{0\xi} + \sigma_3'(2u_{0\xi}u_{0\xi\xi} + u_0 u_{0\xi\xi\xi}) + \sigma_4' w_{0\eta\eta} + \sigma_5' u_{0\xi\eta\eta} = 0, \tag{4.10}$$

where

$$\begin{aligned}
\sigma_1' &= -\frac{1}{3s^2}, \\
\sigma_2' &= \frac{\phi_{111}s^{-1} + 3\phi_{113} + 3\phi_{133}s + \phi_{333}s^2}{2\rho c^2 s}, \\
\sigma_3' &= -\frac{\phi_{15} + \phi_{35}s}{3\rho c^2 s}, \\
\sigma_4' &= \frac{\phi_2}{\rho c^2 s}, \\
\sigma_5' &= -\frac{\phi_4}{3\rho c^2 s^2}.
\end{aligned}$$

Taking ∂_ξ of (4.10), we have

$$\{u_\tau + \sigma_1' u_{\xi\xi\tau} + \sigma_2' u u_\xi + \sigma_3'(2u_\xi u_{\xi\xi} + u u_{\xi\xi\xi})\}_\xi + \sigma_4'' u_{\eta\eta} + \sigma_5' u_{\xi\xi\eta\eta} = 0, \tag{4.11}$$

where

$$\sigma_4'' = \frac{\phi_2}{\rho c^2}.$$

Using the assumption that $\sigma_2' \neq 0$ and the fact that $\sigma_1' < 0$, we can introduce the rescaling

$$\tau = \frac{3\sqrt{-\sigma_1'}}{\sigma_2'}t, \qquad \xi = \sqrt{-\sigma_1'}x, \qquad y = \eta.$$

Then (4.11) turns into

$$\{u_t - u_{xxt} + 3uu_x - \gamma(2u_x u_{xx} + u u_{xxx})\}_x - \alpha u_{yy} + \beta u_{xxyy} = 0, \tag{4.12}$$

which is (1.1), where

$$\gamma = \frac{3\sigma_3'}{\sigma_1'\sigma_2'}, \qquad \alpha = \frac{3\sigma_1'\sigma_4'}{\sigma_2'}, \qquad \beta = \frac{3\sigma_5'}{\sigma_2'}.$$

Equation (4.12) is the far-field model equation for unidimensional finite-length small but finite-amplitude waves in a pre-stressed plate composed of a compressible hyperelastic material when the material constants and prestress satisfy (4.1).

5 Elastic material with internal variables

With the same setup as in Section 2, we study wave motions superimposed on a uniformly pre-stressed state which is described by

$$\begin{cases} \tilde{x} = CX + \tilde{W}(X,Y,Z,T), \\ \tilde{y} = BY, \\ \tilde{z} = AZ + \tilde{U}(X,Y,Z,T). \end{cases} \tag{5.1}$$

Our aim is to study the vertical disturbance $\tilde{U}(X,Y,Z,T)$. In Section 2 we used (2.6) as an approximation so that the problem becomes two-dimensional. However, such a drastic simplification may be too crude as it may leave out some physical effects induced by $Z-$dependence. In order to compensate for that, we introduce an internal variable $P(X,Y,T)$ in the deformation. Such a motion is described by the equations

$$\begin{cases} \tilde{x} = CX + W(X,Y,T), \\ \tilde{y} = BY, \\ \tilde{z} = [A + U(X,Y,T) + P(X,Y,T)]Z. \end{cases} \tag{5.2}$$

The next step is to determine the relation between the internal variable P and the function U. Motivated by the theories of higher order gradient in elasticity (see, e.g., [3], [21], [22]), we may consider that P is related to derivatives of U. Let us assume that the effects in the $Y-$direction are weak. Therefore the simple model that we choose is

$$P(X,Y,T) = \lambda U_X(X,Y,T), \tag{5.3}$$

where $\lambda \geq 0$ is a parameter. Physically, relation (5.3) states that the stress along the vertical direction also depends on the geometry of the plate, say, the curvature. In other words, the plate exhibits notable stiffness along the $Z-$axis, resisting bending. Thus the motion is now described by

$$\begin{cases} \tilde{x} = CX + W(X,Y,T), \\ \tilde{y} = BY, \\ \tilde{z} = [A + U(X,Y,T) + \lambda U_X(X,Y,T)]Z, \end{cases} \tag{5.4}$$

where λ is the stiffness parameter (possibly zero).

5.1 Governing equations

We shall proceed exactly the same way as in the previous three sections. We compute the three principal invariants of the new right Cauchy-Green strain tensor as

$$\begin{aligned} I_1 = \;& (A+U+\lambda U_X)^2 + B^2 + Z^2(U_Y+\lambda U_{XY})^2 + Z^2(U_X+\lambda U_{XX})^2 + (C+W_X)^2 \\ & + W_Y^2, \\ I_2 = \;& (A+U+\lambda U_X)^2[B^2 + W_Y^2 + (C+W_X)^2] + B^2(C+W_X)^2 \\ & + Z^2\{B^2(U_X+\lambda U_{XX})^2 + [W_Y(U_X+\lambda U_{XX}) - (C+W_X)(U_Y+\lambda U_{XY})]^2\}, \\ I_3 = \;& (A+U+\lambda U_X)^2 B^2 (C+W_X)^2. \end{aligned}$$

Hence for an isotropic compressible hyperelastic material, we can write its strain energy function as

$$\Phi = \phi(q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}), \tag{5.5}$$

where

$$q_1 = U, q_2 = W_Y^2, q_3 = W_X, q_4 = Z^2U_Y^2, q_5 = Z^2U_X^2,$$

$$q_6 = Z^2(U_X+\lambda U_{XX})(U_Y+\lambda U_{XY})W_Y, q_7 = \lambda U_X, q_8 = Z^2\lambda U_Y U_{XY},$$

$$q_9 = Z^2\lambda U_X U_{XX}, q_{10} = \lambda^2 Z^2 U_{XY}^2, q_{11} = \lambda^2 Z^2 U_{XX}^2.$$

Taylor expanding (5.5) and then using Hamilton's principle as we did in Section 2, we obtain the new, more complicated governing equations

$$
\begin{aligned}
&\Big(\phi_1+\phi_{11}U+\phi_{13}W_X-\lambda\phi_{37}W_{XX}-\lambda^2\phi_{77}U_{XX}\Big)+\Big(\phi_{12}W_Y^2+\frac{1}{2}\phi_{111}U^2+\phi_{113}UW_X+\frac{1}{2}\phi_{133}W_X^2\Big)\\
&-\lambda\Big(2\phi_{27}W_YW_{YZ}+\phi_{137}UW_{XX}+\phi_{337}W_XW_{XX}\Big)-\lambda^2\Big[\phi_{177}(\frac{1}{2}U_X^2+UU_{XX})+\phi_{377}(U_XW_X)_X\Big]\\
&+\frac{a^2}{3}\Big\{\rho(U_{TT}-\lambda^2U_{XXTT})-2(\phi_4U_{YY}+\phi_5U_{XX})-[\phi_{14}U_Y^2+\phi_{15}U_X^2+2\phi_{14}(UU_Y)_Y\\
&\qquad+2\phi_{34}(U_YW_X)_Y+2\phi_{15}UU_{XX}+2\phi_{35}(U_XW_X)_X+\phi_6(U_XW_Y)_Y+\phi_6(W_YU_Y)_X]\\
&\qquad-\lambda[\phi_8U_{XYY}+\phi_9U_{XXX}-(3\phi_{19}-6\phi_{75})U_XU_{XX}+3\phi_{39}(U_XW_{XX})_X]\\
&\qquad-\lambda^2[2(\phi_{10}U_{YY}+\phi_{11}U_{XX})_{XX}-\phi_{1,10}U_{XX}^2-2\phi_{1,11}(UU_{XX})_{XX}-2\phi_{3,11}(W_XU_{XX})_{XX}]\Big\}\\
&+\cdots\cdots=0
\end{aligned}
\tag{5.6}
$$

and

$$
\begin{aligned}
&-\rho W_{TT}+\Big(\phi_{13}U_X+\phi_{33}W_{XX}+2\phi_2W_{YY}+\lambda\phi_{37}U_{XX}\Big)\\
&+\Big\{\phi_{113}UU_X+\phi_{133}(UW_X)_X+\phi_{333}W_XW_{XX}+\lambda[\phi_{137}(UU_X)_X+\phi_{337}(U_XW_X)_X]+\lambda^2\phi_{337}U_XU_{XX}\Big\}\\
&+\Big[2\phi_{12}(UW_Y)_Y+2\phi_{23}(W_XW_Y)_Y+2\phi_{23}W_YW_{XY}+2\lambda\phi_{27}(U_XW_Y)_Y\Big]\\
&+\frac{a^2}{3}\Big\{2\phi_{35}U_XU_{XX}+\lambda\phi_{39}(U_XU_{XX})_X+2\lambda^2\phi_{3,11}U_{XX}U_{XXX}+2\phi_{34}U_YU_{XY}\\
&\qquad+\lambda\phi_{38}(U_YU_{XY})_X+\phi_6[(U_X+\lambda U_{XX})(U_Y+\lambda U_{XY})]_Y\Big\}+\cdots\cdots=0.
\end{aligned}
\tag{5.7}
$$

Plugging in $U=W=0$ we also get $\phi_1=0$.

We shall still use the same scaling (2.18) with the same c as expressed in (2.21). We make the same assumptions as before and further assume that

$$\text{(e)}\ \ \lambda\ll l. \tag{5.8}$$

5.2 Long waves

In the long-wave approach, we shall use assumptions (a) and (b) in Section 3 as well as (e). We substitute (2.18) into (5.6) and (5.7), use the transformation (3.3) and then plug in the asymptotic expansions (3.6), (3.7). Because we have introduced a new parameter λ, there is a new parameter $\mu=\lambda/l$ in the transformed equations. At the lowest order $O(1)$, we have

$$\mathbf{M_0'S_0}=\mathbf{0}, \tag{5.9}$$

where

$$\mathbf{M_0'}=\begin{bmatrix}\phi_{11}-\mu^2\phi_{77}\partial_\xi^2 & \phi_{13}-\mu\phi_{37}\partial_\xi\\ \phi_{13}\partial_\xi+\mu\phi_{37}\partial_\xi^2 & (\phi_{33}-\rho c^2)\partial_\xi\end{bmatrix},\quad \mathbf{S_0}=\begin{bmatrix}u_0\\ w_{0\xi}\end{bmatrix} \tag{5.10}$$

At this point, we further make the material assumption that at the equilibrium ($U = W = 0$) the energy function ϕ in (5.5) depends on q_1, q_7 through their sum $q_1 + q_7$. Consequently, at the equilibrium we have

$$\phi_{11} = \phi_{77} \quad \text{and} \quad \phi_{13} = \phi_{73}, \tag{5.11}$$

which implies that

$$\det \mathbf{M_0'} = 0. \tag{5.12}$$

The left nullvector of $\mathbf{M_0'}$ is

$$\mathbf{L_e'} = \left[\begin{array}{cc} \phi_{13}\partial_\xi & -\phi_{11} + \mu \frac{\phi_{11}\phi_{37}}{\phi_{13}}\partial_\xi \end{array} \right]. \tag{5.13}$$

and we obtain the relation

$$w_{0\xi} = s u_0 + \mu r u_{0\xi}, \tag{5.14}$$

where

$$s = -\frac{\phi_{11}}{\phi_{13}}, \qquad r = -\frac{\phi_{11}\phi_{37}}{\phi_{13}^2}. \tag{5.15}$$

At the next order $O(\varepsilon_1)$, we have

$$\begin{aligned} &\mathbf{M_0 S_1} + \mathbf{T_1} u_{0\tau} + \mathbf{T_2} w_{0\eta\eta} + \mu \mathbf{T_3} u_{0\xi\tau} + \mathbf{R_1} u_{0\xi\xi} + \mu \mathbf{R_2} u_{0\xi\xi\xi} + \mu^2 \mathbf{R_3} u_{0\xi\xi\xi\xi} \\ &+ \mathbf{Q_1} u_0^2 + \mu \mathbf{Q_2} u_0 u_{0\xi} + \mu^2 \mathbf{P_1} u_{0\xi}^2 + \mu^2 \mathbf{P_2} u_0 u_{0\xi\xi} = 0, \end{aligned} \tag{5.16}$$

where $\mathbf{S_1}, \mathbf{T_1}, \mathbf{T_2}, \mathbf{R_1}$ and $\mathbf{Q_1}$ are given as before in Section 3, and

$$\mathbf{T_3} = \left[\begin{array}{c} 0 \\ 2\rho c^2 r \end{array} \right], \quad \mathbf{R_2} = \left[\begin{array}{c} -(\theta/\varepsilon_1)\frac{1}{3}\phi_9 \\ 0 \end{array} \right], \quad \mathbf{R_3} = \left[\begin{array}{c} -(\theta/\varepsilon_1)\frac{1}{3}(\rho c^2 + 2\phi_{11}) \\ 0 \end{array} \right],$$

$$\mathbf{Q_2} = \left[\begin{array}{c} \phi_{113} r + \phi_{133} s r - (\phi_{137} s + \phi_{337} s^2) \\ (\phi_{133} r + \phi_{333} s r + \phi_{137} + \phi_{337} s)\partial_\xi \end{array} \right],$$

$$\mathbf{P_1} = \left[\begin{array}{c} (\frac{1}{2}\phi_{133} r^2 - \phi_{337} s r - \phi_{377} s - \frac{1}{2}\phi_{177}) - \mu(\frac{1}{2}\phi_{337} r^2 + \phi_{377} r)\partial_\xi \\ (\frac{1}{2}\phi_{377} + \phi_{337} r + \frac{1}{2}\phi_{333} r^2)\}\partial_\xi \end{array} \right],$$

$$\mathbf{P_2} = \left[\begin{array}{c} -(\phi_{137} r + \phi_{337} s r + \phi_{177} + \phi_{337} s) \\ 0 \end{array} \right].$$

Then we multiply the left hand side of (5.16) by the left eigenvector $\mathbf{L_e'}$ and take an additional derivative on ξ to obtain

$$\begin{aligned} &\Big\{ u_{0\tau} + \kappa_1 u_0 u_{0\xi} + \kappa_2 u_{0\xi\xi\xi} - \mu^2 \kappa_4 u_{0\xi\xi\tau} - \mu \kappa_5 u_{0\xi\xi\xi\xi} - \mu^2 \kappa_6 u_{0\xi\xi\xi\xi\xi} - \mu \kappa_7 (u_0^2)_{\xi\xi} \\ &\quad + \mu^2 [\kappa_8 u_{0\xi} u_{0\xi\xi} + \kappa_9 (u_0 u_{0\xi\xi})_\xi] + \mu^3 \kappa_{10} (u_{0\xi}^2)_{\xi\xi} \Big\}_\xi + \kappa_3 u_{0\eta\eta} - \mu^2 \kappa_{11} u_{0\xi\xi\eta\eta} = 0, \end{aligned} \tag{5.17}$$

where

$$\kappa_4 = \frac{r^2}{s^2}, \quad \kappa_5 = \frac{(\theta/3\varepsilon_1)\phi_9}{2\rho c^2 s^2}, \quad \kappa_6 = \frac{(\theta/3\varepsilon_1)(\rho c^2 + 2\phi_{11})}{2\rho c^2 s^2},$$
$$\kappa_7 = \frac{r(\phi_{113} + 2\phi_{133}s + \phi_{333}s^2)}{4\rho c^2 s^2}, \quad \kappa_8 = \frac{\phi_{333}sr^2 + \phi_{133}r^2 - \phi_{337}s - \phi_{177}}{2\rho c^2 s^2},$$
$$\kappa_9 = \frac{\phi_{177} + \phi_{337}s + \phi_{137}r + \phi_{337}sr}{2\rho c^2 s^2}, \quad \kappa_{10} = \frac{3\phi_{377}r + 3\phi_{337}r^2 + \phi_{333}r^3}{4\rho c^2 s^2},$$
$$\kappa_{11} = \frac{\phi_2 r^2}{\rho c^2 s^2}.$$

In (5.17), the parameter μ (related to the stiffness of the plate) is treated as an $O(1)$ quantity. Now we impose the smallness assumption on μ to simplify the equation. Let's assume that μ is $o(1)$ and $\mu^2 = O(\varepsilon_1^b)$, where $1 < b < 2$ such that the μ^2- and μ^3-terms can be omitted in (5.17). Thus we obtain the equation

$$\Big\{ u_{0\tau} + \kappa_1 u_0 u_{0\xi} + \kappa_2 u_{0\xi\xi\xi} - \mu\kappa_5 u_{0\xi\xi\xi\xi\xi} - \mu\kappa_7 (u_0^2)_{\xi\xi} \Big\}_\xi + \kappa_3 u_{0\eta\eta} = 0. \tag{5.18}$$

It is easy to see that when $\lambda = 0$, the long-wave equation (5.18) reduces to the KP equation (3.13).

5.3 Short waves

As we explained in Section 4, we now assume that (c) holds ($l \approx a$) and that the dispersive terms in (5.17) vanish ($\kappa_2 = \kappa_6 = 0$); that is,

$$\text{(d)} \quad \rho c^2 = 2\phi_5, \tag{5.19}$$
$$\text{(f)} \quad \lambda(\rho c^2 + 2\phi_{11}) = 0. \tag{5.20}$$

Hence similarly as in Section 4, we use the scaling (2.18) with l replaced by a.

Let $\varepsilon = h/a$ as before, $\nu = \lambda/a$. Plugging in the scaling and (3.6), (3.7), we obtain

$$\begin{aligned}
&\phi_{11}u + \phi_{13}w_\xi - \nu\phi_{37}w_{\xi\xi} - \nu^2\phi_{77}u_{\xi\xi} + \frac{1}{3}(\rho c^2 - 2\phi_5)u_{\xi\xi} - \frac{1}{3}\nu\phi_9 u_{\xi\xi\xi} - \frac{1}{3}\nu^2(\rho c^2 + 2\phi_{11})u_{\xi\xi\xi\xi} \\
&+\varepsilon\Big\{ (\frac{1}{2}\phi_{111}u^2 + \phi_{113}uw_\xi + \frac{1}{2}\phi_{133}w_\xi^2) - \frac{2}{3}\rho c^2(u_{\xi\tau} - \nu^2 u_{\xi\xi\xi\tau}) - \frac{2}{3}\phi_4 u_{\eta\eta} \\
&\qquad - \frac{1}{3}[\phi_{15}u_\xi^2 + 2\phi_{15}uu_{\xi\xi} + 2\phi_{35}(u_\xi w_\xi)_\xi] \\
&\qquad - \nu[\phi_{137}uw_{\xi\xi} + \phi_{337}w_\xi w_{\xi\xi} + \frac{1}{3}(\phi_8 u_{\xi\eta\eta} + 3\phi_{1,9}u_\xi uu_{\xi\xi} - 6\phi_{75}u_\xi uu_{\xi\xi} - 3\phi_{3,9}(u_\xi w_{\xi\xi})_\xi)] \\
&\qquad -\nu^2[\phi_{177}(\frac{1}{2}u_\xi^2 + uu_{\xi\xi}) + \phi_{377}(u_\xi w_\xi)_\xi \\
&\qquad\qquad - \frac{1}{3}(-2\phi_{10}u_{\xi\xi\eta\eta} + \phi_{1,10}u_{\xi\xi}^2 + 22\phi_{1,11}(uu_{\xi\xi})_{\xi\xi} + 2\phi_{3,11}(w_\xi u_{\xi\xi})_{\xi\xi})]\Big\} \\
&+ \cdots\cdots = 0
\end{aligned} \tag{5.21}$$

and

$$\Big(-\rho c^2 w_{\xi\xi}+\phi_{13}u_\xi+\phi_{33}w_{\xi\xi}+\nu\phi_{37}u_{\xi\xi}\Big)$$
$$+\varepsilon\Big\{2\rho c^2 w_{\xi\tau}+2\phi_2 w_{\eta\eta}+\phi_{113}uu_\xi+\phi_{133}(uw_\xi)_\xi+\phi_{333}w_\xi w_{\xi\xi}+\frac{2}{3}\phi_{35}u_\xi u_{\xi\xi}$$
$$+\nu[\phi_{137}(uu_\xi)_\xi+\phi_{337}(u_\xi w_\xi)_\xi+\frac{1}{3}\phi_{3,9}(u_\xi u_{\xi\xi})_\xi]+\nu^2[\phi_{377}u_\xi u_{\xi\xi}+\frac{2}{3}\phi_{3,11}u_{\xi\xi}u_{\xi\xi\xi}]\Big\}$$
$$+\cdots\cdots=0, \tag{5.22}$$

where we have dropped the quadratic terms in ε. We make the additional simplifying assumption that

$$\lambda\phi_9=0, \tag{5.23}$$

which is obviously satisfied in the non-stiff case ($\lambda=0$).

At the lowest order $O(1)$, we have

$$\mathbf{M_0''S_0}=\mathbf{0}, \tag{5.24}$$

where $\mathbf{M_0''},\mathbf{S_0}$ are similar as in (5.10) with μ replaced by ν. The left nullvector of $\mathbf{L_e''}$ of $\mathbf{M_0''}$ is similar to (5.13) with μ replaced by ν. Hence we obtain the relation

$$w_{0\xi}=su_0+\nu ru_{0\xi}, \tag{5.25}$$

where s and r are given in (5.15). At the next order $O(\varepsilon)$, we find

$$\mathbf{M_0''S_1}+\mathbf{T_1}u_{0\tau}+\mathbf{T_2'}u_{0\xi\tau}+\mathbf{T_3}w_{0\eta\eta}+\mathbf{T_4}u_{0\eta\eta}+\mathbf{T_5}u_{0\xi\xi\xi\tau}+\mathbf{T_6}u_{0\xi\xi\eta}+\mathbf{T_7}u_{0\xi\xi\eta\eta}$$
$$+\mathbf{Q_1}u_0^2+\mathbf{Q_2'}u_{0\xi}^2+\mathbf{Q_3'}u_0u_{0\xi\xi}+\nu\mathbf{Q_4}u_0u_{0\xi}+\mathbf{K}=0, \tag{5.26}$$

where $\mathbf{S_1},\mathbf{T_1},\mathbf{T_3},\mathbf{T_4}$ and $\mathbf{Q_1}$ are the same as given in Section 4, and

$$\mathbf{T_2'}=\begin{bmatrix}-\frac{2}{3}\rho c^2\\ 2\rho c^2\nu r\end{bmatrix},\mathbf{T_5}=\begin{bmatrix}\frac{2}{3}\rho c^2\nu^2\\ 0\end{bmatrix},\mathbf{T_6}=\begin{bmatrix}-\frac{1}{3}\nu\phi_8\\ 0\end{bmatrix},\mathbf{T_7}=\begin{bmatrix}-\frac{2}{3}\nu^2\phi_{10}\\ 0\end{bmatrix},$$

$$\mathbf{Q_2'}=\begin{bmatrix}-\frac{1}{3}(\phi_{15}+2\phi_{35}s)+\nu^2(\frac{\phi_{133}}{2}r^2-\phi_{337}sr-\phi_{377}s-\frac{\phi_{177}}{2})+\nu(\phi_{75}-\frac{\phi_{19}}{2}+\phi_{19}s-\frac{2}{3}\phi_{35}r)\partial_\xi\\ +\frac{1}{2}\nu^2\phi_{19}r\partial_{\xi\xi}-\nu^3(\frac{1}{2}\phi_{337}r^2+\phi_{377}r)\partial_\xi+\frac{1}{3}\nu^3\phi_{3,11}r\partial_{\xi\xi\xi}\\ \{\frac{1}{3}\phi_{35}+\nu^2(\frac{1}{2}\phi_{377}+\phi_{337}r+\frac{1}{2}\phi_{333}r^2)\}\partial_\xi\end{bmatrix},$$

$$\mathbf{Q_3'}=\begin{bmatrix}-\frac{2}{3}(\phi_{15}+\phi_{35}s)-\nu^2\{\phi_{137}r+\phi_{337}sr+\phi_{177}+\phi_{337}s+\frac{2}{3}(\phi_{1,11}+\phi_{3,11}s)\partial_{\xi\xi}\}\\ 0\end{bmatrix},$$

$$\mathbf{Q_4}=\begin{bmatrix}\phi_{113}r+\phi_{133}sr-(\phi_{137}s+\phi_{337}s^2)\\ (\phi_{133}r+\phi_{333}sr+\phi_{137}+\phi_{337}s)\partial_\xi\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}\frac{1}{3}\nu^2\phi_{1,10}u_{0\xi\xi}^2+\frac{1}{3}\nu^3\phi_{3,11}r(u_{0\xi}^2)_{\xi\xi\xi}\\ \frac{1}{6}\phi_{39}(u_{0\xi}^2)_{\xi\xi}+\frac{1}{3}\nu^2\phi_{3,11}(u_{0\xi\xi}^2)_\xi\end{bmatrix}.$$

Multiplying the left-hand side of (5.26) by $\mathbf{L}''_{\mathbf{e}}$, we obtain

$$
\begin{aligned}
&u_{0\tau} + \tilde{\sigma}'_1 u_{0\xi\xi\tau} + \sigma'_2 u_0 u_{0\xi} + \sigma'_3(2u_{0\xi}u_{0\xi\xi} + u_0 u_{0\xi\xi\xi}) + \sigma'_4 w_{0\eta\eta} + \sigma'_5 u_{0\xi\eta\eta} + \sigma'_6 u_{0\xi\xi\xi\xi\tau} + \sigma'_7 u_{0\eta\eta} \\
&-(\nu\frac{\phi_8}{6\rho c^2 s^2} + \nu^2\frac{\phi_2 r^2}{\rho c^2 s^2})u_{0\xi\xi\eta\eta} - \nu^2\frac{\phi_{10}}{3\rho c^2 s^2}u_{0\xi\xi\xi\eta\eta} + \nu\frac{\frac{1}{6}\phi_{39}s + \phi_{75} - \frac{1}{2}\phi_{19} + \phi_{19}s - \phi_{35}r}{2\rho c^2 s^2}(u_{0\xi}^2)_{\xi\xi} \\
&+\nu^2\Big[\frac{\phi_{1,11} + \phi_{3,11}s}{3\rho c^2 s^2}(u_0 u_{0\xi\xi})_{\xi\xi} - \frac{\phi_{137}r + \phi_{337}sr + \phi_{133}r^2 + \phi_{333}sr^2}{2\rho c^2 s^2}(u_0 u_{0\xi})_{\xi\xi} \\
&\quad - \frac{\phi_{177} + \phi_{377}s + \phi_{137}r + \phi_{337}sr}{2\rho c^2 s^2}(u_0 u_{0\xi\xi})_{\xi} + \frac{\phi_{1,10} + \phi_{3,11}}{6\rho c^2 s^2}(u_{0\xi\xi}^2)_{\xi} - \frac{\phi_{39}r}{12\rho c^2 s^2}(u_{0\xi}^2)_{\xi\xi\xi} \\
&\quad + \frac{\phi_{133}r^2 - \phi_{377}s - \phi_{177} + \phi_{333}sr}{4\rho c^2 s^2}(u_{0\xi}^2)_{\xi} - \frac{\phi_{19}r}{4\rho c^2 s^2}(u_{0\xi}^2)_{\xi\xi\xi}\Big] \\
&+\nu^3\Big\{\frac{\phi_{3,11}r}{6\rho c^2 s^2}[(2u_{0\xi}^2)_{\xi\xi\xi\xi} - (u_{0\xi\xi}^2)_{\xi\xi}] - \frac{3\phi_{377}r + \phi_{337}r^2 + \phi_{333}r^3}{4\rho c^2 s^2}(u_{0\xi}^2)_{\xi\xi}\Big\} = 0,
\end{aligned} \tag{5.27}
$$

where $\sigma'_2, \sigma'_3, \sigma'_4$ and σ'_5 are same as in Section 4, and

$$
\begin{aligned}
\tilde{\sigma}'_1 &= -(\frac{1}{3s^2} + \nu^2\frac{r^2}{s^2}), \\
\sigma'_6 &= \nu^2\frac{1}{3s^2}, \\
\sigma'_7 &= -\nu\frac{\phi_2 r}{\rho c^2 s}.
\end{aligned}
$$

In the non-stiff case, (5.27) becomes exactly (4.10). When $\lambda > 0$ hence $\nu > 0$, we take ∂_ξ of (5.27), use (5.25), and perform the transformation

$$
\xi' = \nu^{-\frac{1}{8}}\xi, \qquad \tau' = \nu^{-\frac{11}{8}}\tau, \qquad y = \eta.
$$

We thus get

$$
\begin{aligned}
\{u_{\tau'} + \sigma_1 u_{\xi'\xi'\tau'} + \sigma_6 u_{\xi'\xi'\xi'\xi'\tau'} + \sigma_2 u u_{\xi'} + \sigma_3(2u_{\xi'}u_{\xi'\xi'} + u u_{\xi'\xi'\xi'})\}_{\xi'} + \sigma_4 u_{yy} + \sigma_5 u_{\xi'\xi' yy} \\
+ O(\nu^{\frac{15}{8}}) = 0
\end{aligned} \tag{5.28}
$$

where

$$
\begin{aligned}
&\sigma_1 = -\frac{1}{3s^2}\nu^{-\frac{1}{4}}, \quad \sigma_6 = \frac{1}{3s^2}\nu^{\frac{3}{2}}, \quad \sigma_2 = \sigma'_2\nu^{\frac{5}{4}}, \\
&\sigma_3 = \sigma'_3\nu, \qquad\quad \sigma_4 = \sigma'_4\nu^{\frac{3}{2}}, \qquad \sigma_5 = \sigma'_5\nu^{\frac{5}{4}}.
\end{aligned}
$$

Using the assumption that $\sigma_2 \neq 0$ and the fact that $\sigma_1 < 0$, we introduce the scaling

$$
\tau' = \frac{3\sqrt{-\sigma_1}}{\sigma_2}t, \qquad \xi' = \sqrt{-\sigma_1}x.
$$

With the higher-order term omitted, (5.28) turns into

$$
\{u_t - u_{xxt} + \delta u_{xxxxt} + 3uu_x - \gamma(2u_x u_{xx} + u u_{xxx})\}_x - \alpha u_{yy} + \beta u_{xxyy} = 0 \tag{5.29}
$$

where

$$\delta = \frac{\sigma_6}{\sigma_1^2} = 3s^2\nu^2, \qquad \gamma = \frac{3\sigma_3}{\sigma_1\sigma_2}, \qquad \alpha = \frac{3\sigma_1\sigma_4}{\sigma_2}, \qquad \beta = \frac{3\sigma_5}{\sigma_2}.$$

One can easily see that the parameters γ, α, β in (5.29) are exactly the same as those in (4.12). Furthermore, δ is positive.

We want to remark that this approach by use of the internal variables results in a higher-order derivative term in the final equation, which makes the PDE more regular.This problem will be considerably easier to analyze.

Acknowledgements

The Author is very grateful to Professor W. Strauss for his encouragement and helpful comments. The author also wants to thank Professor C. Dafermos for his interest in this work and extremely valuable discussions at various stages of the modeling.

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