CENTER MANIFOLDS WITHOUT A PHASE SPACE FOR QUASILINEAR PROBLEMS IN ELASTICITY, BIOLOGY, AND HYDRODYNAMICS

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Abstract. In this paper, we present a center manifold reduction theorem for quasilinear elliptic equations posed on infinite cylinders that is done without a phase space in the sense that we avoid explicitly reformulating the PDE as an evolution problem. Under suitable hypotheses, the resulting center manifold is finite dimensional and captures all sufficiently small bounded solutions. Compared with classical methods, the reduced ODE on the manifold is more directly related to the original physical problem and also easier to compute. The analysis is conducted directly in Hölder spaces, which is often desirable for elliptic equations.

We then use this machinery to construct small bounded solutions to a variety of systems. These include heteroclinic and homoclinic solutions of the anti-plane shear problem from nonlinear elasticity; exact slow moving invasion fronts in a two-dimensional Fisher–KPP equation; and hydrodynamic bores with vorticity in a channel. The last example is particularly interesting in that we find solutions with critical layers and distinctive "half cat's eye" streamline patterns.

Contents

1. Introduction

Our basic objective in this paper relates to a classical problem: characterizing small bounded solutions of a quasilinear elliptic PDE posed on an unbounded cylinder $\Omega = \mathbb{R} \times \Omega'$. The base of the cylinder $\Omega' \subset \mathbb{R}^{n-1}$ is a bounded and connected $C^{2+\alpha}$ domain for some $\alpha \in (0,1)$, and the dimension $n \geq 2$. For simplicity, say that $0 \in \overline{\Omega'}$.

As a fairly representative example, we initially focus on the following quasilinear PDE:

$$\begin{cases} \nabla \cdot \mathcal{A}(y,u,\nabla u,\lambda) + \mathcal{B}(y,u,\nabla u,\lambda) = 0 & \text{in } \Omega \\ \mathcal{G}(y,u,\nabla u,\lambda) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where spatial coordinates in Ω are written (x,y) for $x \in \mathbb{R}$ and $y \in \Omega'$. Here, $\lambda \in \mathbb{R}$ is a parameter, while $u = u(x,y) \in C^{2+\alpha}(\overline{\Omega})$ is the unknown. We ask that the functions $\mathcal{A} = \mathcal{A}(y,z,p,\lambda)$,

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$\mathcal{B} = \mathcal{B}(y, z, p, \lambda)$, and $\mathcal{G} = \mathcal{G}(y, z, p, \lambda)$ are uniformly C^{M+4} in their arguments[1] for a fixed integer $M \geq 2$. Moreover, we assume that the interior equation is uniformly elliptic in the sense that there exists $\theta > 0$ such that

$$\sum_{i,j} \partial_{p_j} \mathcal{A}_i(y, z, p, \lambda) q_i q_j \geq \theta |q|^2 \qquad \text{for all } y \in \Omega',\ p, q \in \mathbb{R}^n,\ z, \lambda \in \mathbb{R}. \tag{1.2}$$

The boundary condition is taken to be uniformly oblique in that there exists $\chi > 0$ such that

$$-N(y) \cdot \nabla_p \mathcal{G}(y, z, p, \lambda) \geq \chi \qquad \text{for all } y \in \Omega',\ p \in \mathbb{R}^n,\ z, \lambda \in \mathbb{R}, \tag{1.3}$$

where $N = (0, N') \in \mathbb{R}^n$ denotes the outward unit normal to Ω on $\partial\Omega = \mathbb{R} \times \partial\Omega'$. Note that since the coefficients in (1.1) are independent of x, the full nonlinear problem is invariant under axial translation. We also extend this to include nonlinear transmission problems, Dirichlet conditions, and diagonal elliptic systems in Section 3.

Borrowing terminology from dynamical systems, we say a solution (u, λ) of (1.1) is *homoclinic* if u limits to a fixed function as $|x| \to \infty$, and we call it *heteroclinic* provided u has distinct limits as $x \to \pm\infty$. Beyond their intrinsic mathematical importance, equations of the form (1.1) arise in a surprisingly diverse array of physical settings. Of particular interest to us is their connection to traveling waves in nonlinear elasticity, mathematical biology, and especially hydrodynamics. In those contexts, homoclinic solutions are referred to variously as *pulses*, *solitons*, or *solitary waves*, and heteroclinics correspond to *fronts* or *bores*. Although the techniques we develop are equally well-suited to both these types of solutions, our emphasis will be on fronts because they are more difficult to construct. An ulterior motive for this choice is that, in a companion paper [8, 9], we present a global bifurcation theory for heteroclinics.

The unboundedness of Ω seriously complicates the task of finding these solutions. For example, it is well-known that the relevant linearized operators fail to be Fredholm in unweighted Hölder spaces, which precludes the direct application of bifurcation theoretic techniques. For semilinear problems, monotonicity methods have proven to be effective; see, for example, Berestycki and Nirenberg [6], A. Volpert, V. Volpert, and V. Volpert [51], and the references therein. By contrast, in the quasilinear setting, the predominant approach is to reformulate (1.1) as a *spatial* dynamical system (that is, treating x as an evolution variable), and use infinite-dimensional invariant manifold theory. Seeking small bounded solutions, we might hope to construct a finite-dimensional center manifold and study the bounded orbits of a reduced equation there. Beginning with the pioneering work of Kirchgässner [28] and Mielke [39, 40] in the 1980s, this basic strategy has been built upon and applied to great effect by many authors; see, for example, the book of Haragus and Iooss [20] for historical overview or [12] for applications to water waves.

While the Mielke–Kirchgässner approach is quite general and very powerful, the way it is traditionally phrased is not perfectly suited to all applications. For many systems, such as (1.1), the process of recasting it as an evolution equation contorts the PDE in an inconvenient way. In particular, accommodating nonlinear boundary conditions requires one or more implicit changes of dependent variables. This is certainly possible to do, but it adds an additional layer of complexity to the already involved process of computing the reduced ODE. More importantly, it obscures the relationship between the equation on the center manifold and the physical problem, complicating for instance the task of establishing properties such as symmetry and monotonicity. When using the center manifold reduction as a preparatory step towards an existence theory for large solutions, being able to efficiently deduce this type of qualitative information is extremely desirable. Lastly, the Mielke–Kirchgässner machinery is formulated in relatively weak Sobolev spaces in the transversal variable y due to its reliance on so-called maximal regularity estimates. When studying

[1]Throughout the paper, we use the convention that ∇ denotes the full gradient with respect to (x, y) unless indicated otherwise via a subscript. In particular, $(y, z, p, \lambda) \in \Omega' \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$. We reserve D for Fréchet derivatives or total derivatives. For a function $v = v(x)$, we will often write v' as a shorthand for $\partial_x v$.

elliptic PDEs, it is sometimes preferable to work directly in spaces of Hölder continuous functions. An extension of Mielke–Kirchgässner to this setting was given by Kirrmann [29] in his unpublished PhD thesis.

Recently, Faye and Scheel [15] introduced an alternative technique that ameliorates some of the issues with the spatial dynamics approach. Rather than reformulate the problem as an evolution equation, they instead perform a delicate fixed point argument in exponentially weighted Sobolev spaces. This furnishes what they call a center manifold "without a phase space." Indeed, the manifold is parameterized by the components of the solution in the kernel of the linearized operator rather than initial data. This permits them to treat certain non-local problems — which was their original intent — and also greatly simplifies the arduous task of computing the reduced equation. Unfortunately, the Faye–Scheel method is fundamentally restricted to semilinear problems, and it appears to be ill-adapted to Hölder spaces.

As one contributions of this paper, we present a new variant of the center manifold reduction theorem that is specialized to treat quasilinear elliptic problems of the form (1.1), though it can be extended to more general ones. The analysis is conducted entirely in Hölder spaces and, like Faye–Scheel, the reduced equation can be computed with comparatively elementary methods. For heteroclinic solutions, one must expand the reduction function to cubic order, and so these differences in complexity are especially salient. We deliberately choose the projection involved in the definition of the center manifold so that one obtains an ODE for the restriction $v(x) = u(x, 0)$. For instance, when we study surface water waves, we arrange for the reduced equation to directly govern the free boundary. In this way, the physical context remains in view even as we restrict to the center manifold. One substantial advantage of this choice is that we are able to prove monotonicity properties of the solutions relatively easily, laying the groundwork for subsequent global bifurcation theoretic analysis. Indeed, the authors [8, 10] and Hogancamp [24] use exactly this strategy to construct large solutions for two of the three applications considered in this paper; these works all rely in an essential way on the qualitative information garnered from the local theory.

The most technically challenging part of constructing a center manifold invariably involves solving a fixed point problem in weighted spaces and then verifying that the solution depends smoothly on the parameters. For this, we rely on the work of Amick and Turner [3], where bounds and Fréchet differentiability of superposition operators in exponentially weighted Hölder spaces is painstakingly worked out. In fact, these authors developed their own center manifold reduction based on the above estimates and a point-wise in x spectral splitting approach. We use their ideas to construct a preliminary center manifold, and then reconfigure it in the style of Faye and Scheel, thereby obtaining the simplified expansion procedure and freedom of projection choice.

The second part of the present paper consists of three nontrivial applications of our center manifold reduction theorem. These problems were selected both for their physical relevance and to illustrate different aspects of the methodology. First, we prove the existence of homoclinic and heteroclinic solutions to the anti-plane shear equations from nonlinear elasticity. While this model has been studied extensively, the class of front-type equilibria that we exhibit appear to be completely new. Second, we verify the existence of slow moving fronts in a two-dimensional Fisher–KPP system with absorbing boundary conditions. The one-dimensional Fisher–KPP system is classical and thoroughly studied; the model we consider was recently formulated in [42] as an explanation for experimental data suggesting that the presence of obstacles may reduce the speed of invasion fronts in certain biological systems. We give the first rigorous existence theory for traveling wave solutions to this equation.

Finally, and most significantly, we construct small rotational bores in a channel. These are heteroclinic solutions of the full two-dimensional incompressible Euler equations, with two immiscible layers of constant density fluid separated by a free boundary. A major novelty is that we allow for constant vorticity as well as critical layers.

Notation. Here we record some notational conventions followed throughout the rest of the paper. Let $\Sigma \subset \mathbb{R}^n$ be a cylinder in dimension $n \geq 2$, $k \in \mathbb{N}$, and $\alpha \in (0,1)$. We define the usual Hölder norm

$$\|f\|_{C^{k+\alpha}(\Sigma)} := \sum_{|\beta| \leq k} \|\partial^\beta f\|_{C^0(\Sigma)} + \sup_{\substack{(x_1,y_1),(x_2,y_2) \in \Sigma \\ (x_1,y_1) \neq (x_2,y_2)}} \frac{|f(x_1,y_1) - f(x_2,y_2)|}{|(x_1-x_2, y_1-y_2)|^\alpha}$$

and denote by $C_{\mathrm{b}}^{k+\alpha}(\overline{\Sigma})$ the Banach space of $f \in C^k(\overline{\Sigma})$ for which $\|f\|_{C^{k+\alpha}(\Sigma)} < \infty$. On the other hand we say that $f \in C^{k+\alpha}(\overline{\Sigma})$ if $\varphi f \in C_{\mathrm{b}}^{k+\alpha}(\overline{\Sigma})$ for any smooth function φ with support compactly contained in $\overline{\Sigma}$. We call functions in $C_{\mathrm{b}}^{k+\alpha}(\overline{\Sigma})$ uniformly Hölder continuous, and functions in $C^{k+\alpha}(\overline{\Sigma})$ locally Hölder continuous up to the boundary.

We will also have occasion to work with exponentially weighted Hölder spaces. For $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$, and a function $f \in C^k(\overline{\Sigma})$, we define the exponentially weighted Hölder norm

$$\|f\|_{C_\mu^{k+\alpha}(\Sigma)} := \sum_{|\beta| \leq k} \|w_\mu \partial^\beta f\|_{C^0(\Sigma)} + \sum_{|\beta|=k} \|w_\mu |\partial^\beta f|_\alpha\|_{C^0(\Sigma)},$$

where $w_\mu(x) := \operatorname{sech}(\mu x)$ is an exponential weight function and $|f|_\alpha$ is the local Hölder seminorm

$$|f|_\alpha(x,y) := \sup_{\substack{|(r,s)|<1 \\ (x+r,y+s) \in \Sigma}} \frac{|f(x+r,y+s) - f(x,y)|}{|(r,s)|^\alpha}.$$

We denote by

$$C_\mu^{k+\alpha}(\overline{\Sigma}) := \left\{ f \in C^{k+\alpha}(\overline{\Sigma}) : \|f\|_{C_\mu^{k+\alpha}(\Sigma)} < \infty \right\}.$$

Note that $C_{\mathrm{b}}^{k+\alpha}(\overline{\Sigma}) = C_{\mu=0}^{k+\alpha}(\overline{\Sigma})$. For $k \geq 1$, and μ and α as above, we also define the seminorm

$$|f|_{\mathring{C}_\mu^{k+\alpha}(\Sigma)} := \sum_{1 \leq |\beta| \leq k} \|w_\mu \partial^\beta f\|_{C^0(\Sigma)} + \sum_{|\beta|=k} \|w_\mu |\partial^\beta f|_\alpha\|_{C^0(\Sigma)},$$

and say $f \in \mathring{C}_\mu^{k+\alpha}(\overline{\Sigma})$ provided that $|f|_{\mathring{C}_\mu^{k+\alpha}(\Sigma)} < \infty$. Finally, we may append a subscript of "e" to any of these spaces to indicate that we are restricting to the subspace of functions that are even in the x-variable, and a subscript "c" when the functions have support that is a compact subset of the stated domain.

1.1. **Statement of results.**

Written as an abstract operator equation, the elliptic problem (1.1) takes the form

$$\mathscr{F}(u,\lambda) = 0, \tag{1.4}$$

where

$$\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \colon C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega}) \times \mathbb{R} \longrightarrow C_{\mathrm{b}}^{0+\alpha}(\overline{\Omega}) \times C_{\mathrm{b}}^{1+\alpha}(\partial\Omega).$$

By this convention, $\mathscr{F}_1$ represents the equation in the interior, whereas $\mathscr{F}_2$ corresponds to the boundary condition.

It is well-known that families of "long waves" can be found bifurcating from "trivial" x-independent solutions at certain critical parameter values (often connected to so-called dispersion relations). This intuition motivates the following structural assumptions on $\mathscr{F}$. First, suppose that there exists a family of trivial solutions parameterized by λ; for simplicity, this can be stated as

$$\mathscr{F}(0,\lambda) = 0 \qquad \text{for all } \lambda \in \mathbb{R}. \tag{1.5}$$

We will study solutions near $(u,\lambda) = (0,0)$, which leads us to consider the linearized operator $L := D_u \mathscr{F}(0,0)$. We make two hypotheses on L. First,

$$L \text{ is formally self-adjoint and commutes with reflections in } x. \tag{1.6}$$

Explicitly, this means that L takes the form (2.1). Second, we make a spectral assumption on the transversal linearized operator

$$L' := L|_{C^{2+\alpha}(\overline{\Omega'})} : C^{2+\alpha}(\overline{\Omega'}) \to C^{0+\alpha}(\overline{\Omega'}) \times C^{1+\alpha}(\partial\Omega'),$$

which results from restricting L to acting on x-independent functions. As is typical for PDE operators, we say that ν is an eigenvalue of L' provided there exists a nontrivial solution $\varphi \in C^{2+\alpha}(\overline{\Omega'})$ to the spectral problem $L'\varphi = (\nu\varphi, 0)$. Because Ω' is a bounded and smooth domain in $\mathbb{R}^{n-1}$, standard elliptic theory ensures that L' has a (unique) principal eigenvalue $\nu_0 \in \mathbb{R}$ for which the corresponding eigenfunction φ_0 is strictly positive on $\overline{\Omega'}$. In fact, ν_0 must be simple and it lies strictly to the right of all other eigenvalues of L'; see, for example, [1, Theorem 12.1] or [36]. Our final assumption is that

$$\nu_0 = 0 \text{ is the principal eigenvalue of } L'. \tag{1.7}$$

This is the sense in which the parameter value $\lambda = 0$ is critical. As we will see in Lemma 2.1, hypothesis (1.7) is equivalent to asking that $\ker L$ is spanned by φ_0.

Theorem 1.1 (Center manifold reduction)**.** *Consider the quasilinear elliptic PDE* (1.1) *posed on the infinite cylinder* Ω. *Assume that it has a family of trivial solutions* (1.5)*, its linearization satisfies* (1.6)*, and that* $\lambda = 0$ *is a critical parameter value in that the corresponding transversal linearized problem has the spectral behavior* (1.7)*. There exist* $\overline{\mu} > 0$ *such that for any fixed* $\mu \in (0, \overline{\mu})$ *and integer* $M \geq 2$*, there exist neighborhoods* $U \subset C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R}$ *and* $V \subset \mathbb{R}^3$ *of the origin and a coordinate map* $\Psi = \Psi(A, B, \lambda)$ *satisfying*

$$\Psi \in C^{M+1}(\mathbb{R}^3, C^{2+\alpha}_{\mu}(\overline{\Omega})), \quad \Psi(0,0,\lambda) = 0 \text{ for all } \lambda, \quad \nabla_{(A,B)}\Psi(0,0,0) = (0,0), \tag{1.8}$$

and such that the following hold.

(a) *Suppose that* $(u, \lambda) \in U$ *solves* (1.1)*. Then* $v(x) := u(x, 0)$ *solves the second-order ODE*

$$v'' = f(v, v', \lambda) \tag{1.9}$$

where $f\colon \mathbb{R}^3 \to \mathbb{R}$ *is the* C^{M+1} *mapping*

$$f(A, B, \lambda) := \frac{d^2}{dx^2}\bigg|_{x=0} \Psi(A, B, \lambda)(x, 0). \tag{1.10}$$

(b) *Conversely, if* $v\colon \mathbb{R} \to \mathbb{R}$ *satisfies the ODE* (1.9) *and* $(v(x), v'(x), \lambda) \in V$ *for all* x*, then* $v = u(\,\cdot\,, 0)$ *for a solution* $(u, \lambda) \in U$ *of the PDE* (1.1)*. Moreover,*

$$u(x+\tau, y) = \frac{v(x)}{\varphi_0(0)}\varphi_0(y) + \frac{v'(x)}{\varphi_0(0)}\tau\varphi_0(y) + \Psi(v(x), v'(x), \lambda)(\tau, y), \tag{1.11}$$

for all $\tau \in \mathbb{R}$*. Here, recall that* φ_0 *generates the kernel of* L'*.*

Remark 1.2. Setting $\tau = 0$ in (1.11) and normalizing φ_0 so that $\varphi_0(0) = 1$, we obtain

$$u(x, y) = v(x)\varphi_0(y) + \Phi(v(x), v'(x), \lambda, y),$$

where $\Phi \in C^{2+\alpha}(\mathbb{R}^4, \mathbb{R})$ is given by

$$\Phi(A, B, \lambda, y) := \Psi(A, B, \lambda)(0, y) = O\Big((|A| + |B|)(|A| + |B| + |\lambda|)\Big). \tag{1.12}$$

Remark 1.3. It is useful to note that the reduction function also satisfies

$$\Psi(A, B, \lambda)(0, 0) = \partial_x\Psi(A, B, \lambda)(0, 0) = 0.$$

Remark 1.4. If the PDE (1.1) exhibits symmetries, then one expects that they will be inherited in some form by the coordinate map Ψ and hence the ODE (1.9). In Section 3.1, we prove that this does indeed hold for a class of discrete symmetries that includes the important case of reflections in x, sometimes called "reversibility."

Remark 1.5. Through a straightforward modification of the proof, versions of Theorem 1.1 can be obtained that treat more general spectral behavior than (1.7). For instance, it is natural to consider scenarios where L' has finitely many positive eigenvalues or 0 is an eigenvalue of higher multiplicity. In the interest of keeping the presentation concise, we will focus here on the simpler situation in (1.7).

Let us draw attention again to the fact that the ODE (1.9) relates in a transparent way to the original PDE (1.1). For example, when studying free boundary problems, we may pick coordinates on Ω' so that the graph of v parametrizes the interface.

Another advantage of our approach — which it inherits from Faye–Scheel — is the comparative simplicity of deriving the reduced equation. This can be seen in the next result, which says essentially that Ψ in (1.8) and f in (1.9) can be determined through a naïve formal asymptotic expansion.

Theorem 1.6 (Reduced equation). *In the setting of Theorem 1.1, the coordinate map Ψ admits the Taylor expansion*

$$\Psi(A,B,\lambda) = \sum_{\substack{2\le i+j+k\le M\\ i+j\ge 1}} \Psi_{ijk} A^i B^j \lambda^k + O\big((|A|+|B|)(|A|+|B|+|\lambda|)^M\big) \qquad \text{in } C^{2+\alpha}_\mu(\overline{\Omega}), \tag{1.13}$$

where the coefficients Ψ_{ijk} are the unique functions in $C^{2+\alpha}_\mu(\overline{\Omega})$ that satisfy

(i) $\Psi_{ijk}(0,0) = \partial_x \Psi_{ijk}(0,0) = 0$.
(ii) *For all $i+j+k \le M$, the formal Gâteaux derivative*

$$\partial_A^i \partial_B^j \partial_\lambda^k \Big|_{(A,B,\lambda)=0} \mathscr{F}\left(\frac{A}{\varphi_0(0)}\varphi_0 + \frac{B}{\varphi_0(0)} x\varphi_0 + \Psi(A,B,\lambda)\right) = 0. \tag{1.14}$$

Remark 1.7. By introducing an appropriate cut-off function, we may consider the Gâteaux derivative of $\mathscr{F}$ in (1.14) as the Fréchet derivative of a modified $\mathscr{F}$. In practice, however, this distinction is unimportant when using (1.14) to calculate the Ψ_{ijk}. Further details can be found in Lemma 2.4 and Section 4.

Remark 1.8. As mentioned in the introduction, we actually have considerable freedom in choosing the linear relationship $v = \mathcal{V}u$ between the original unknown u and the quantity v governed by the reduced ODE (1.9) in Theorem 1.1. Like Faye and Scheel [15], we have found pointwise evaluation $\mathcal{V}u(x) := u(x,0)$ to be the most convenient for calculations, but our proofs also apply to, for instance,

$$\mathcal{V}u(x) := \int_{\Omega'} u(x,y)\,dy \quad \text{or} \quad \mathcal{V}u(x) := \frac{1}{2}\int_{x-1}^{x+1}\int_{\Omega'} u(s,y)\,dy\,ds. \tag{1.15}$$

Besides slightly alterning the very final step in the proof of Theorem 1.6 in Section 2.5, the only other modification is that Theorem 1.6 (i) becomes $\mathcal{Q}\Psi_{ijk} = 0$, where the operator

$$\mathcal{Q}w(x,y) := \frac{\mathcal{V}u(0)}{\int_{\Omega'} \varphi_0(s)\,ds}\varphi_0(y) + \frac{\mathcal{V}\partial_x u(0)}{\int_{\Omega'} \varphi_0(s)\,ds} x\varphi_0(y)$$

is a bounded projection from $C^{2+\alpha}_\mu(\overline{\Omega})$ onto the kernel of L, here thought of as a mapping between weighted Hölder spaces.

We also obtain the following theorem relating the linearized problem at any small non-trivial solution of the PDE (1.1) to the linearization of the reduced ODE (1.9).

Theorem 1.9 (Linearization and reduction)**.** *In the setting of Theorem 1.1* (b)*, if* $\dot{u} \in C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega})$ *is a solution to the linearized PDE*[2]

$$D_u \mathscr{F}(u, \lambda)\dot{u} = 0,$$

then $\dot{v} := \dot{u}(\,\cdot\,, 0)$ *satisfies the linearized reduced ODE*

$$\dot{v}'' = \nabla_{(A,B)} f(v, v', \lambda) \cdot \left(\dot{v}, \dot{v}'\right). \tag{1.16}$$

The above theorem allows us to, among other things, calculate the dimension of the kernel of $D_u\mathscr{F}(u, \lambda)$ using only information about the planar system (1.16). Indeed, Theorem 1.9 tells us that the linearizations of the PDE and reduced ODE are compatible in that uniqueness of bounded solutions to the latter implies invertibility properties for the former.

Analogous results to Theorem 1.9 can be found in [54, Theorem 4.1(ii)] and [7, Theorem 5.1(ii)], for example. There the authors must carefully linearize each step in the center manifold construction. By contrast, our proof of Theorem 1.9 relies on a soft analysis argument that avoids this rather tedious process through an extension of Theorem 1.1 to diagonal elliptic systems.

While the Amick–Turner theory [3] seems particularly well-suited for our purposes, we note that one should in principle be able to prove versions of Theorems 1.1, 1.6, and 1.9 through the classical Mielke–Kirchgässner theory [28, 39, 40] or its variant due to Kirrmann [29]. This would involve the implicit change of dependent variable mentioned above in order to absorb the nonlinear boundary conditions, a further change of coordinates to achieve the desired projection, the reinterpretation of the center manifold in the spirit of Faye and Scheel, and finally the application of embedding theorems to obtain a result in Hölder spaces. To our knowledge, no results of this type appear in the literature. Another intention of the present paper is to rekindle interest in the Amick–Turner theory, as it has unfortunately received little attention in recent years. On the other hand, this choice does force us to make the additional stipulation in (1.6) that the linearized operator commutes with reflections. A spatial dynamics approach would not necessarily require this.

In the remainder of the paper, we use Theorem 1.1 and Theorem 1.6 to construct homoclinic and heteroclinic solutions to three quasilinear elliptic problems arising in quite different physical settings. This includes anti-plane shear equilibria for a nonlinear elastic model with live body forces, and slow-moving invasion fronts for a two-dimensional Fisher–KPP equation with reactive boundary conditions. To keep the presentation here compact, we defer stating these results and discussing the relevant history until later.

Our last application is to water waves. Specifically, we study a system consisting of two incompressible fluids at constant density governed by the Euler equations. They are separated by a free boundary and confined to a infinitely long horizontal channel. Steady traveling solutions to this problem are often referred to as *internal waves*, and they are observed frequently in coastal flows [44]. We prove the existence of several families of front-type internal waves, which in hydrodynamics are known as (smooth) bores. From a physical standpoint, bores are interesting because they are a genuinely stratified phenomenon: one can show that no bores exist in constant density fluids [54]. As heteroclinic connections, they also require considerably more finesse to construct.

Numerical studies of bores have been carried out by a number of authors [50, 35, 19], but very few rigorous results are currently available. The earliest work is due to Amick and Turner [2], who used a precursor to the center manifold reduction in [3] to characterize all small bounded solutions to the system assuming the flow in each layer is irrotational. Later, Mielke [41] obtained an analogous result by applying traditional spatial dynamics techniques. Using direct fixed point arguments, Makarenko [37] gave an alternative construction for small-amplitude bores in the same setting, and later studied the continuously stratified case [38].

We not only prove the existence of irrotational bores, but in addition allow constant vorticity in the upper layer. In the latter case, many of these waves will have *critical layers* — curves in

[2] Here and elsewhere in the paper, we use a dot to denote a variation. This should not be mistaken for a time derivative.

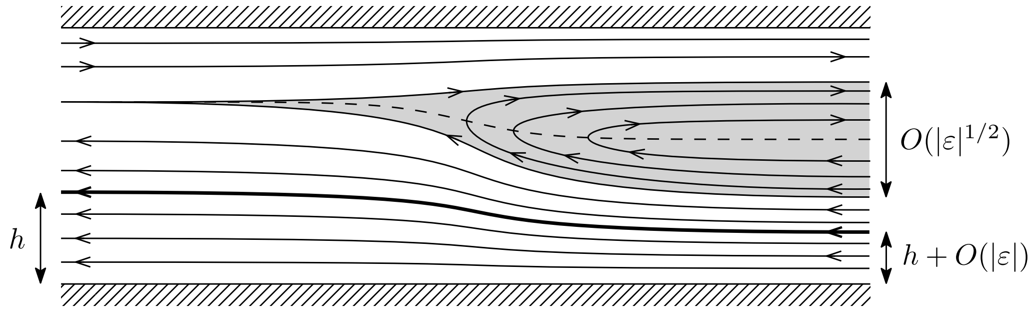

FIGURE 1. A smooth bore with a "half cat's eye" streamline pattern. The two fluid regions are bounded above and below by rigid walls and separated by a sharp interface (shown in bold). The dashed curve is the critical layer, above which particles move to the right and below which they move to the left (in the moving frame). Inside the shaded region (the "eye"), the streamlines are bounded from the left and unbounded to the right, whereas outside they are unbounded in both directions.

the fluid where particles move with the same horizontal velocity as the wave itself. It is well-known that this can create interesting streamline patterns, such as the famous cat's eyes in the periodic setting [53, 14, 11, 46]. We find many families of waves feature a striking "half cat's eye"; see Figure 1 and Theorem 7.8. To the best of our knowledge, this configuration has never been observed before. Indeed, it is commonly thought that surface solitary waves in constant density water can *never* have unbounded critical layers (but see [32]). Our results show that this heuristic does not extend to internal fronts.

The analysis required for the water wave problem is several orders of magnitude more involved than the previous two examples. It is here that the elegance of the expansion in Theorem 1.6 and the choice of projection in the definition of the manifold are exploited most fully. For instance, we are able to give a very simple proof that the free surface is monotonically decreasing and the streamlines have the expected pattern.

An essential part of each of the above problems is identifying a parameter regime that admits front-type solutions. For elasticity, we are able to exploit symmetry properties of the equation, whereas for the Fisher–KPP we take advantage of the robustness of the well-studied one-dimensional model. Neither of these simplifications are available for water waves. Instead, we make strong use of the theory of *conjugate flows*; see Section 7.3.

1.2. **Plan of the article.** The proofs of Theorem 1.1 and Theorem 1.6 are carried out in Section 2. First, in Section 2.1, we establish some basic facts regarding the linear elliptic operator $L = D_u\mathscr{F}(0,0)$. Then, in Section 2.2, the PDE is rewritten as a fixed point problem in the style of Amick–Turner. Over the course of Section 2.3 and Section 2.4, we verify the hypotheses of that general theory, which yields a center manifold, but does not directly furnish the reduced equation for v in (1.9). In Section 2.5, we complete the proof using a near-identity change of variables to convert locally to the Faye–Scheel formulation, which gives us the liberty to choose the projection in the definition of the manifold, and also leads to the Taylor expansion (1.13).

In Section 3, we consider a number of extensions of Theorem 1.1 and Theorem 1.6 to other types of elliptic equations. We also provide the proof of Theorem 1.9.

For the benefit of the reader, Section 4 contains a gentle explanation of the general strategy for actually computing the reduced equation and finding heteroclinic or homoclinic solutions. While this is in principle deducible from (1.9) and (1.13), there are certain choices that are not immediately obvious but greatly simplify the process. Readers who are more interested in applying the theory than the specifics of its proof are encouraged to read this section first.

The application to nonlinear elasticity can be found in Section 5, while Section 6 contains our results on invasion fronts in two-dimensional Fisher–KPP. We devote Section 7 to proving the existence of internal bores with vorticity.

Finally, two appendices are included. Appendix A provides a brief statement of Amick and Turner's fixed point theory that is sufficient for proving Theorem 1.1. In Appendix B, we collect some details regarding the calculation of the reduced equations for the elasticity problem.

2. Center manifolds for quasilinear elliptic PDE on a cylinder

In this section we prove the general results Theorem 1.1 and Theorem 1.6. The main tool is the fixed-point theory of Amick–Turner [3], which is recalled in Appendix A for the reader's convenience.

Let us begin by fixing some notation. Recalling that $\alpha \in (0,1)$ is the Hölder exponent introduced earlier, we set

$$\mathscr{X} := C^{2+\alpha}(\overline{\Omega}), \qquad \mathscr{Y} = \mathscr{Y}_1 \times \mathscr{Y}_2 := C^{0+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\partial\Omega).$$

Recall that these are spaces of functions that are only *locally* Hölder up to the boundary; the corresponding spaces of uniformly bounded functions are designated with a subscript b. Likewise, for $\mu \geq 0$, we write $\mathscr{X}_\mu$ and $\mathscr{Y}_\mu$ to indicate the associated exponentially weighted Hölder spaces.

2.1. **Linear theory.** Recall that

$$\mathscr{F}_1(u,\lambda) := \nabla \cdot \mathcal{A}(y,u,\nabla u,\lambda) + \mathcal{B}(y,u,\nabla u,\lambda),$$
$$\mathscr{F}_2(u,\lambda) := \big(\mathcal{G}(y,u,\nabla u,\lambda)\big)\big|_{\partial\Omega}.$$

Direct computation yields that the linearized operator of $\mathscr{F}_1$ at (u,λ) is given by

$$D_u\mathscr{F}_1(u,\lambda)v = \nabla \cdot [\nabla_p \mathcal{A} \nabla v + (\partial_z \mathcal{A} + \nabla_p \mathcal{B})\, v] + [\partial_z \mathcal{B} - \nabla \cdot \nabla_p \mathcal{B}]\, v,$$

where the coefficients are all of class $C^{M+3}_{\mathrm{b}}(\overline{\Omega'})$ and their arguments are being suppressed for readability. Invoking assumption (1.6), the self-adjointness of L forces

$$\partial_z \mathcal{A} + \nabla_p \mathcal{B} = 0 \quad \text{at} \quad (y,0,0,0).$$

Writing $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}')$ with $\mathcal{A}_1$ taking values in $\mathbb{R}$ and $\mathcal{A}'$ taking values in $\mathbb{R}^{n-1}$, and likewise denoting $p = (p_1, p') \in \mathbb{R} \times \mathbb{R}^{n-1}$, we then have

$$\nabla \cdot (\nabla_p \mathcal{A} \nabla v) = \partial_{p_1}\mathcal{A}_1 \partial_x^2 v + \nabla_{p'}\mathcal{A}_1 \cdot \nabla' \partial_x v + \nabla' \cdot \big(\partial_{p_1}\mathcal{A}' \partial_x v\big) + \nabla' \cdot \big(\nabla' \mathcal{A}' \nabla' v\big)$$

where ∇' and $\nabla'\cdot$ indicate the gradient and divergence in y, respectively. The reflection symmetry in (1.6) implies that

$$\nabla_{p'}\mathcal{A}_1 = \partial_{p_1}\mathcal{A}' = 0 \quad \text{at} \quad (y,0,0,0).$$

From the above we conclude that the linearized operator at $(u,\lambda) = (0,0)$ takes the form

$$\begin{aligned} D_u\mathscr{F}_1(0,0)v &= a_{11}(y)\partial_x^2 v + \nabla' \cdot \big(a'(y)\nabla' v\big) + c(y)v, \\ D_u\mathscr{F}_2(0,0)v &= \Big(-N'(y) \cdot \big(a'(y)\nabla' v\big) + g(y)v\Big)\Big|_{\partial\Omega'}. \end{aligned} \tag{2.1}$$

As required by assumption (1.6), the linearized boundary condition is co-normal. The coefficients are all of class $C^{M+3}_{\mathrm{b}}(\overline{\Omega'})$, and related to the nonlinear problem via

$$a_{11}(y) := \nabla_{p_1}\mathcal{A}_1(y,0,0,0), \qquad a'(y) := \nabla_{p'}\mathcal{A}'(y,0,0,0),$$
$$c(y) := (\partial_z \mathcal{B} - \nabla \cdot \nabla_p \mathcal{B})\,(y,0,0,0), \quad g(y) := \partial_z \mathcal{G}(y,0,0,0).$$

From (2.1), we also see that the transversal linear operator $L' = (L_1', L_2')$ has the explicit form

$$\begin{aligned} L_1' w &:= \nabla' \cdot \big(a'(y)\nabla' w\big) + c(y)w, \\ L_2' w &:= \big(-N'(y) \cdot \big(a'(y)\nabla' w\big) + g(y)w\big)\,\Big|_{\partial\Omega'}, \end{aligned} \tag{2.2}$$

for all $w = w(y) \in C^{2+\alpha}(\overline{\Omega'})$.

Projections. The uniform ellipticity assumption implies that $a_{11}, 1/a_{11} \in C^{M+3}(\overline{\Omega'})$. For integers $k \geq 0$, let $\underline{H}^k(\Omega')$ denote the weighted L^2-based Sobolev space corresponding to the inner product

$$(u, v)_{\underline{H}^k(\Omega')} := \sum_{|\beta| \leq k} \int_{\Omega'} (\partial_y^\beta u)(\partial_y^\beta v)\, a_{11} dy.$$

As usual, we denote $\underline{L}^2(\Omega') := \underline{H}^0(\Omega')$ and $(\,\cdot\,,\,\cdot\,)_{\underline{L}^2(\Omega')} := (\,\cdot\,,\,\cdot\,)_{\underline{H}^0(\Omega')}$. Observe that

$$\underline{L}' := \frac{1}{a_{11}} L'$$

is an elliptic PDE operator that is symmetric with respect to the $\underline{L}^2(\Omega')$ inner product. Moreover, by the spectral assumption 1.7, its kernel is spanned by φ_0, which recall is strictly positive on $\overline{\Omega'}$. It follows that 0 is also the principal eigenvalue of $\underline{L}'$, and hence the spectrum of $\underline{L}'$ consists of a sequence of finite-multiplicity eigenvalues $\{\nu_k\}_{k=0}^\infty$ such that $\nu_0 = 0 > \nu_1 \geq \nu_2 \geq \ldots$, and $\nu_k \searrow -\infty$.

Let the corresponding eigenfunctions functions be $\{\varphi_k\}$, so that

$$\underline{L}'\varphi_k = (\nu_k \varphi_k, 0), \qquad \|\varphi_k\|_{\underline{L}^2(\Omega')} = 1, \qquad \text{and} \qquad (\varphi_j, \varphi_k)_{\underline{L}^2(\Omega')} = 0 \quad \text{for } j \neq k.$$

In particular, we assume here and in the rest of this section that $\|\varphi_0\|_{\underline{L}^2(\Omega')} = 1$. As it is not φ_0 itself but the ratio $\varphi_0/\varphi_0(0)$ which appears in Theorems 1.1 and 1.6, this assumption can be made without loss of generality. Let P_0' denote the continuous orthogonal projection onto the eigenspace corresponding to φ_0. It is also important to mention that there is a variational characterization of our spectral assumption (1.7) in terms of the Rayleigh quotient defined by

$$\mathcal{R}(w) := \frac{\int_{\Omega'} \left(-a'(y)\nabla' w \cdot \nabla' w + c(y)w^2\right)\, dy + \int_{\partial\Omega'} g(y) w^2\, dS(y)}{\int_{\Omega'} w^2\, a_{11}(y) dy},$$

for all $w \in \underline{H}^1(\Omega') \setminus \{0\}$ when $n \geq 3$. For the case $n = 2$, we have $\partial\Omega' = \{y_{\min}, y_{\max}\}$ for some points $y_{\min} < y_{\max}$, and so the boundary integral above is replaced by $g(y_{\max})w^2(y_{\max}) + g(y_{\min})w^2(y_{\min})$. It is easy to verify that any critical point of $\mathcal{R}$ is in the kernel of $\underline{L}'$ (which is also the kernel of L'). By classical elliptic theory, we have further that

$$\max_{\substack{w \in \underline{H}^1(\Omega') \\ w \not\equiv 0}} \mathcal{R}(w) = \nu_0 = 0, \qquad \max_{\substack{w \in P_{\geq 1}' \underline{H}^1(\Omega') \\ w \not\equiv 0}} \mathcal{R}(w) = \nu_1 < 0, \tag{2.3}$$

where $P_{\geq 1}' := 1 - P_0'$. Likewise, a standard argument shows that (2.3) implies there exists $\kappa > 0$ such that

$$\int_{\Omega'} \left(a'(y)\nabla' w \cdot \nabla' w - c(y)w^2\right)\, dy - \int_{\partial\Omega'} g(y) w^2\, dS(y) \geq \kappa \|w\|_{\underline{H}^1(\Omega')}^2, \tag{2.4}$$

for all $w \in P_{\geq 1}' \underline{H}^1(\Omega')$.

We can therefore introduce the projection P_0 point-wise in x onto the 0 eigenvalue for the transverse problem. For functions $u = u(x, y)$, it is defined by

$$(P_0 u)(x, y) := P_0' u(x, \cdot).$$

One can confirm that P_0 is a bounded projection on $\mathscr{X}_\mu$ for any $\mu \geq 0$. Continuing the above convention, set

$$P_{\geq 1} := 1 - P_0.$$

Boundedness of the partial Green's function. In this section, we seek to understand the solvability properties of the linearized problem $Lu = f$ for f lying in an exponentially weighted Hölder space. We begin by characterizing the kernel of $L|_{\mathscr{X}_\mu}$ when $\mu \in (0, \sqrt{|\nu_1|})$.

Lemma 2.1 (Kernel)**.** *For all* $\mu \in (0, \sqrt{|\nu_1|})$,

$$\ker L|_{\mathscr{X}_\mu} = \{(x,y) \mapsto A\varphi_0(y) + Bx\varphi_0(y) : A, B \in \mathbb{R}\}.$$

Proof. Suppose that $Lu = 0$ for some $u \in \mathscr{X}_\mu$. It follows that u can be represented via the eigenfunctions (for the homogeneous boundary condition) as

$$u(x,y) = \sum_{k=0}^{\infty} \hat{u}_k(x)\varphi_k(y)$$

where the series above converges in $C^2(\mathbb{R}; \underline{L}^2(\Omega'))$. Thus, $\hat{u}_k$ satisfies the ODE

$$\partial_x^2 \hat{u}_k = -\nu_k \hat{u}_k, \qquad \text{for all } k \geq 0. \tag{2.5}$$

Recalling that $\nu_k < 0$ for $k \geq 1$, this ensures that $\hat{u}_k$ grows exponentially as $x \to \infty$ with rate $\sqrt{|\nu_k|}$. Thus, when $\mu \in (0, \sqrt{|\nu_1|})$, it must be that $\hat{u}_k \equiv 0$ for $k \geq 1$, and hence the kernel of $L|_{\mathscr{X}_\mu}$ must lie in $P_0\mathscr{X}_\mu$. Setting $k = 0$ in (2.5) then gives

$$u(x,y) = A\varphi_0(y) + Bx\varphi_0(y),$$

for some A, $B \in \mathbb{R}$. □

One consequence of the above lemma is that composing with the projection $P_{\geq 1}$ eliminates the kernel of $L|_{\mathscr{X}_\mu}$. Before considering the inhomogeneous problem for L, it will therefore be useful to define a projection on $\mathscr{Y}$ (which is then inherited by $\mathscr{Y}_\mu$) that agrees in a natural way with P_0. More precisely, we will need to normalize again by dividing by the coefficient a_{11}. Let

$$\underline{L} := \frac{1}{a_{11}} L,$$

and for $v = (v_1, v_2) \in \mathscr{Y}$, define

$$(Q_0 v)(x,y) := \left((P_0 v_1)(x,y) + \varphi_0(y) \int_{\partial\Omega'} v_2(x,s)\varphi_0(s)\, a_{11}(s) dS(s),\, 0 \right).$$

Thus $Q_0\mathscr{Y} \subset (P_0\mathscr{Y}_1) \times \{0\} \subset \mathscr{Y}$, and, for any $u \in \mathscr{X}$,

$$Q_0\underline{L}u = \left(\varphi_0(y) \left(\int_{\Omega'} (L_1 u)(x,s)\varphi_0(s)\, ds + \int_{\partial\Omega'} (L_2 u)(x,s)\varphi_0(s)\, dS(s) \right), 0 \right).$$

But,

$$\begin{aligned}(L_1 u, \varphi_0)_{L^2(\Omega')} &= \left(\partial_x^2 u, \varphi_0\right)_{\underline{L}^2(\Omega')} + \left(L_1' u, \varphi_0\right)_{L^2(\Omega')} \\ &= \left(\partial_x^2 u, \varphi_0\right)_{\underline{L}^2(\Omega')} - (L_2 u, \varphi_0)_{L^2(\partial\Omega')},\end{aligned}$$

and so combining this with the line above yields

$$Q_0\underline{L}u = \left(\varphi_0(y) \int_{\Omega'} (\partial_x^2 u)(x,s)\varphi_0(s)\, a_{11}(s) ds,\, 0 \right) = \underline{L}P_0 u. \tag{2.6}$$

In keeping with the notation above, let $Q_{\geq 1} \colon \mathscr{Y} \to \mathscr{Y}$ be defined by

$$Q_{\geq 1} v := (1 - Q_0)v \qquad \text{for all } v \in \mathscr{Y}.$$

With Q_0 and $Q_{\geq 1}$ in hand, we now establish the following elementary fact about the solvability of $Lu = f$ when the data $f \in Q_{\geq 1}\mathscr{Y}_\mu$.

Proposition 2.2 (Partial Green's function)**.** *For any $\mu \in [0, \sqrt{|\nu_1|/2})$ and $f = (f_1, f_2) \in \mathscr{Y}_\mu$ such that $Q_0 f = 0$, there exists a unique $u \in P_{\geq 1}\mathscr{X}_\mu$ such that $\underline{L}u = f$. Moreover,*

$$\|u\|_{\mathscr{X}_\mu} \lesssim \|f\|_{\mathscr{Y}_\mu},$$

with the implied constant above uniform in μ as $\mu \to 0$. Equivalently, for all $\mu \in [0, \sqrt{|\nu_1|/2})$,

$$\underline{L}|_{P_{\geq 1}\mathscr{X}_\mu} \colon P_{\geq 1}\mathscr{X}_\mu \to Q_{\geq 1}\mathscr{Y}_\mu$$

is invertible with bounded inverse $G \colon Q_{\geq 1}\mathscr{Y}_\mu \to P_{\geq 1}\mathscr{X}_\mu$ that we call the partial Green's function*.*

Proof. Fix μ as above and let $f \in \mathscr{Y}_\mu$ be given with $Q_0 f = 0$. Following the general strategy of [2, Theorem 3.1], we introduce a smooth partition of unity $\{\zeta^{(m)}\}_{m\in\mathbb{Z}}$ on $\mathbb{R}$ such that

$$\zeta \in C^\infty(\mathbb{R}), \quad \operatorname{supp}\zeta \subset [-2,2], \quad \zeta = 1 \text{ on } [-1,1], \quad \zeta \text{ even}, \quad \zeta - \tfrac{1}{2} \text{ odd about } x = \tfrac{3}{2} \text{ on } [1,2],$$

and taking $\zeta^{(m)} := \zeta(\,\cdot - 3m)$.

For each $m \in \mathbb{Z}$, consider the cut-off problem

$$\underline{L}u^{(m)} = f^{(m)} := \zeta^{(m)} f. \tag{2.7}$$

Observe that, because the projectors are pointwise in x,

$$Q_0 f^{(m)} = \zeta^{(m)} Q_0 f = 0.$$

Thus, $f^{(m)} \in Q_{\geq 1}\mathscr{Y}_\mu$, and the commutation identity (2.6) implies that any solution $u^{(m)}$ of (2.7) necessarily lies in $P_{\geq 1}\mathscr{X}_\mu \oplus \ker L|_{P_0\mathscr{X}_\mu}$.

As a starting point, we show that there exists weak solutions to (2.7) by introducing the Hilbert space $\mathscr{H} := P_{\geq 1}\underline{H}^1(\Omega)$. Let $\mathscr{B}$ be the bilinear form associated to (2.7),

$$\mathscr{B}[u,v] := \int_\Omega (-a\nabla u \cdot \nabla v + cuv)\, dx\, dy + \int_{\partial\Omega} guv\, dS,$$

for all $u, v \in \mathscr{H}$, and where $a = a(y) := \nabla_p \mathcal{A}(y,0,0,0)$. The weak formulation of (2.7) is

$$\mathscr{B}[u^{(m)}, \psi] = (f_1^{(m)}, \psi)_{\underline{L}^2(\Omega)} + (f_2^{(m)}, \psi)_{\underline{L}^2(\partial\Omega)} \qquad \text{for all } \psi \in \mathscr{H}.$$

Notice that, because $f^{(m)}$ is compactly supported, the right-hand side above does indeed represent an element of $\mathscr{H}^*$ acting on ψ. As the coefficients are C_{b}^{M+3}, $\mathscr{B}$ is bounded. On the other hand,

$$\begin{aligned} -\mathscr{B}[u,u] &= \int_\Omega a_{11}(\partial_x u)^2\, dx\, dy + \int_\Omega \left(a'\nabla' u \cdot \nabla' u - cu^2\right)\, dx\, dy - \int_{\partial\Omega} gu^2\, dS \\ &\geq \|\partial_x u\|^2_{\underline{L}^2(\Omega)} + \kappa \int_{\mathbb{R}} \|u(x,\cdot)\|^2_{\underline{H}^1(\Omega')}\, dx \gtrsim \|u\|^2_{\underline{H}^1(\Omega)}, \end{aligned}$$

where we have used (2.4) to derive the inequality on the second line. It follows that $-\mathscr{B}$ is coercive on $\mathscr{H}$, and thus Lax–Milgram implies that there exists a weak solution $u^{(m)} \in \mathscr{H}$ to the cut-off problem (2.7) for each $m \in \mathbb{Z}$.

We must now improve this to classical solutions and estimate their norm in $\mathscr{X}_\mu$. Let an integer $\ell \in \mathbb{Z}$ be given and put

$$\mathscr{X}_\mu^{(\ell)} := C_\mu^{2+\alpha}([\ell,\ell+1] \times \overline{\Omega'}), \qquad \mathscr{Y}_\mu^{(\ell)} := C_\mu^{0+\alpha}([\ell,\ell+1] \times \overline{\Omega'}) \times C_\mu^{1+\alpha}([\ell,\ell+1] \times \partial\overline{\Omega'}).$$

The next stage of the argument involves deriving a priori estimates for $u^{(m)}$ in $\mathscr{X}_\mu^{(\ell)}$. This will follow by elliptic regularity theory, but first we must expand the class of admissible test functions to all of $\underline{H}^1(\Omega)$. In particular, observe that if $\psi \in C_{\mathrm{c}}^1(\overline{\Omega})$, we may use the splitting above to write

$$\psi = \hat{\psi}_0(x)\varphi_0(y) + \psi_{\geq 1} \in P_0 C_{\mathrm{c}}^1(\overline{\Omega}) \oplus P_{\geq 1} C_{\mathrm{c}}^1(\overline{\Omega}).$$

It is easy to verify that $\mathscr{B}[u^{(m)}, \cdot\,]$ extends to a bounded linear functional on $C^1_{\mathrm{c}}(\overline{\Omega})$, and indeed

$$\begin{aligned}\mathscr{B}[u^{(m)}, \psi] &= \mathscr{B}[u^{(m)}, \psi_{\geq 1}] = (f_1^{(m)}, \psi_{\geq 1})_{\underline{L}^2(\Omega)} + (f_2^{(m)}, \psi_{\geq 1})_{\underline{L}^2(\partial\Omega)} \\ &= (f_1^{(m)}, \psi)_{\underline{L}^2(\Omega)} + (f_2^{(m)}, \psi)_{\underline{L}^2(\partial\Omega)} - (f_1^{(m)}, \hat{\psi}_0\varphi_0)_{\underline{L}^2(\Omega)} - (f_2^{(m)}, \hat{\psi}_0\varphi_0)_{\underline{L}^2(\partial\Omega)} \\ &= (f_1^{(m)}, \psi)_{\underline{L}^2(\Omega)} + (f_2^{(m)}, \psi)_{\underline{L}^2(\partial\Omega)},\end{aligned}$$

since $Q_0 f^{(m)} = 0$, by hypothesis. Thus $u^{(m)}$ is a weak solution of (2.7) in the $\underline{H}^1(\Omega)$ sense.

Let $\Omega^{(\ell)} := [\ell - 1/4, \ell + 5/4] \times \Omega'$, which is a slight enlargement of the domain associated to $\mathscr{X}_\mu^{(\ell)}$. There are precisely two integers m for which $\operatorname{supp}\zeta^{(m)}$ and $\Omega^{(\ell)}$ have non-empty intersection; as they are consecutive, let us call them $\tilde{m}$ and $\tilde{m}+1$. Conjugating (2.7) with the exponential weight $\operatorname{sech}(\mu x)$, and applying standard elliptic regularity theory on bounded domains (see, for example, [18, Chapter 8]), we infer that $u^{(m)} \in \mathscr{X}_\mu^{(\ell)}$. Moreover, it obeys the bound

$$\|u^{(m)}\|_{\mathscr{X}_\mu^{(\ell)}} \lesssim \begin{cases} e^{-\mu|\ell|}\|u^{(m)}\|_{\underline{L}^2(\Omega^{(\ell)})} & \text{for } m \in \mathbb{Z} \setminus \{\tilde{m}, \tilde{m}+1\} \\ e^{-\mu|\ell|}\|u^{(m)}\|_{\underline{L}^2(\Omega^{(\ell)})} + \|f\|_{\mathscr{Y}_\mu} & \text{for } m = \tilde{m},\ \tilde{m}+1. \end{cases} \tag{2.8}$$

Here, the constants are uniform in μ, ℓ, and m. In order to complete the argument we must justify the convergence of the series $\sum_m u^{(m)}$ in $\mathscr{X}_\mu$. Looking at (2.8), it is apparent that this hinges on having sufficiently refined bounds on the $L^2(\Omega^{(\ell)})$ norm of $u^{(m)}$.

First, suppose that $m < \tilde{m}$, so that $\underline{L}u^{(m)} = 0$ on $\Omega^{(\ell)}$. In fact, this holds on the semi-infinite strip $(3m+2, \infty) \times \Omega'$, and so we may apply elliptic regularity again to conclude that $u^{(m)} \in C^{2+\alpha}_{\mathrm{b}}$ on this set. By construction, $u^{(m)}$ is also in $\underline{H}^1(\Omega)$, so in particular this also ensures that $u^{(m)}$ and $\nabla u^{(m)}$ decay to 0 as $x \to +\infty$.

We are therefore justified in taking the equation $\underline{L}u^{(m)} = 0$, multiplying by $u^{(m)}$, and then integrating over the strip $(x, \infty) \times \Omega'$. This procedure yields the identity

$$\begin{aligned}\frac{1}{2}\partial_x \int_{\Omega'} |u^{(m)}(x,y)|^2\, a_{11}(y)dy = &\int_x^\infty\!\!\int_{\Omega'} a'(y)\nabla' u^{(m)}(s,y) \cdot \nabla' u^{(m)}(s,y)\, dy\, ds \\ &- \int_x^\infty\!\!\int_{\Omega'} c(y)|u^{(m)}(s,y)|^2\, dy\, ds \\ &- \int_x^\infty\!\!\int_{\partial\Omega'} g(y)|u^{(m)}(s,y)|^2\, dS(y)\, ds,\end{aligned}$$

which holds for all $x > 3m+2$. Using the Rayleigh–Ritz characterization of ν_1 in (2.3), this furnishes the integro-differential inequality

$$\frac{1}{2}\partial_x \|u^{(m)}(x,\cdot\,)\|^2_{\underline{L}^2(\Omega')} \leq \nu_1 \int_x^\infty \|u^{(m)}(s,\cdot\,)\|^2_{\underline{L}^2(\Omega')}\, ds \qquad \text{for all } x > 3m+2.$$

From this, we may further estimate that

$$-\int_x^\infty \|u^{(m)}(s,\cdot\,)\|^2_{\underline{L}^2(\Omega')}\, ds \leq e^{\sqrt{2|\nu_1|}(3m+2-x)} \int_{3m+2}^\infty \|u^{(m)}(s,\cdot\,)\|^2_{\underline{L}^2(\Omega')}\, ds,$$

for all $x > 3m+2$. Thus,

$$\|u^{(m)}\|_{\underline{L}^2(\Omega^{(\ell)})} = \left(\int_{\ell-\frac{1}{4}}^{\ell+\frac{5}{4}} \|u^{(m)}(s,\cdot\,)\|^2_{\underline{L}^2(\Omega')}\, ds \right)^{1/2} \lesssim e^{\sqrt{|\nu_1|/2}(3m-\ell)} \|u^{(m)}\|_{\underline{L}^2(\Omega)}, \tag{2.9}$$

where recall we have assumed $m < \tilde{m}$. Now, $u^{(m)}$ is bounded in $\underline{L}^2(\Omega)$ in terms of the data $f^{(m)}$ via Lax–Milgram. Relating this back to f, we find that

$$\|u^{(m)}\|_{\underline{L}^2(\Omega)} \lesssim e^{3\mu|m|}\|f\|_{\mathscr{Y}_\mu} \qquad \text{for all } m \in \mathbb{Z}.$$

Combining this with (2.9) yields

$$\|u^{(m)}\|_{\underline{L}^2(\Omega^{(\ell)})} \lesssim e^{\sqrt{|\nu_1|/2}(3m-\ell)+3\mu|m|}\|f\|_{\mathscr{Y}_\mu} \qquad \text{for all } m < \tilde{m}.$$

The same type of reasoning applied to the case $m > \tilde{m}+1$ gives a similar bound. For the exceptional values $m = \tilde{m}, \tilde{m}+1$, we may use (2.8), so that in total

$$\|u^{(m)}\|_{\underline{L}^2(\Omega^{(\ell)})} \lesssim e^{-\sqrt{|\nu_1|/2}|3m-\ell|+3\mu|m|}\|f\|_{\mathscr{Y}_\mu} \qquad \text{for all } m \in \mathbb{Z}.$$

Returning to the preliminary a priori estimate (2.8), we can now conclude that

$$\sum_{m=-M}^{M} u^{(m)} \longrightarrow u \text{ in } \mathscr{X}_\mu^{(\ell)} \text{ as } M \to \infty \qquad \text{for all } \ell \in \mathbb{Z},$$

and

$$\|u\|_{\mathscr{X}_\mu^{(\ell)}} \le C\|f\|_{\mathscr{Y}_\mu},$$

with a constant C that depends only on $\sqrt{|\nu_1|/2}-\mu$. As ℓ on the left-hand side above is arbitrary, this gives the desired bounds on G. ☐

In applications, it will often be convenient to use alternative projections onto the kernel of L. For instance, looking at the statement of Theorem 1.1, we see that the coefficients A and B are found by evaluating u and $\partial_x u$ at $(0,0)$. With that in mind, suppose that $\mathcal{Q}$ is a given bounded projection from $\mathscr{X}_\mu$ to $\ker L$ which is independent of μ. As in the partial Green's function analysis, we expect that L is invertible on the kernel of $\mathcal{Q}$. To make this precise, we adopt the approach of Faye–Scheel [15] and consider a so-called "bordered" operator where one appends $\mathcal{Q}$ to $\underline{L}$. The result is the following.

Lemma 2.3 (Bordered operator)**.** *The bordered operator*

$$(\underline{L}, \mathcal{Q}) : \ \mathscr{X}_\mu \longrightarrow \mathscr{Y}_\mu \times \ker L, \qquad u \longmapsto (\underline{L}u,\ \mathcal{Q}u)$$

is invertible with a bounded inverse.

Proof. From Lemma 2.1 and Proposition 2.2, we see that $\underline{L} : \mathscr{X}_\mu \to \mathscr{Y}_\mu$ has a two-dimensional kernel and its range includes $Q_{\ge 1}\mathscr{Y}_\mu$. If $f \in Q_0\mathscr{Y}_\mu$, then it must take the form $f = (v\varphi_0, 0)$ for some $v = v(x) \in C_\mu^{0+\alpha}(\mathbb{R})$. Since $w \mapsto \int_0^{(\,\cdot\,)} w(s)\,ds$ is a bounded linear mapping $C_\mu^{k+\alpha}(\mathbb{R}) \to C_\mu^{k+1+\alpha}(\mathbb{R})$ for any $k \ge 0$, we may let $V \in C_\mu^{2+\alpha}(\mathbb{R})$ be given so that $V'' = v$. It follows that $u(x,y) := V(x)\varphi_0(y)$ satisfies $\underline{L}u = f$. Thus $\underline{L}$ is surjective, and so it must be Fredholm index 2.

A standard dimension counting argument shows that the bordered operator has Fredholm index 0; see, for example, [48, Lemma 4.4]. Now, if $u \in \mathscr{X}_\mu$ satisfies $(\underline{L}u, \mathcal{Q}u) = 0$, then in particular $u \in \ker L$. On the other hand, $\mathcal{Q}u = 0$, and so it must be that $u = 0$. Thus, the bordered operator is injective and Fredholm index 0. It follows that it is invertible, and the boundedness of its inverse is a consequence of Proposition 2.2. ☐

2.2. **Reformulation as a fixed point.** Now, let us return to the full nonlinear problem. The abstract operator equation (1.4) can be rewritten as

$$\underline{L}w = \mathcal{N}(w,\lambda), \tag{2.10}$$

where $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$ is defined by

$$\mathcal{N}(w,\lambda) := \underline{L}w - \frac{1}{a_{11}}\mathscr{F}(w,\lambda).$$

Thus, $\mathcal{N}$ is "flat" in the sense that $\mathcal{N}(0,0)=0$, $D_u\mathcal{N}(0,0)=0$, and $\partial_\lambda\mathcal{N}(0,0)=0$. The commutation identity (2.6) permits us to perform a spectral decomposition in both the domain and codomain to rewrite (2.10) as the system

$$\begin{cases} \underline{L}(P_0u) = Q_0\mathcal{N}(P_0u + P_{\geq 1}u, \lambda) \\ \underline{L}(P_{\geq 1}u) = Q_{\geq 1}\mathcal{N}(P_0u + P_{\geq 1}u, \lambda). \end{cases}$$

Applying the partial Green's function G of Proposition 2.2 to the second equation then gives

$$P_{\geq 1}u = GQ_{\geq 1}\mathcal{N}(P_0u + P_{\geq 1}u, \lambda),$$

while, recalling the explicit form of P_0 and $\underline{L}$, we see that

$$\partial_x^2 P_0 u = Q_0\mathcal{N}(P_0u + P_{\geq 1}u, \lambda). \tag{2.11}$$

Integrating (2.11) twice we get the full system

$$\begin{cases} P_0u = \xi_1\varphi_0 + \displaystyle\int_0^x (\partial_x P_0 u)(s,y)\,ds \\ P_0\partial_x u = \xi_2\varphi_0 + \displaystyle\int_0^x Q_0\mathcal{N}(P_0u + P_{\geq 1}u, \lambda)(s,y)\,ds \\ P_{\geq 1}u = GQ_{\geq 1}\mathcal{N}(P_0u + P_{\geq 1}u, \lambda). \end{cases}$$

for some constants $\xi_1, \xi_2 \in \mathbb{R}$ (the "initial data"). Introducing a parameter β (representing a rescaling of the axial variable), defining

$$(P_0u)(x,y) =: U_1(x)\varphi_0(y), \qquad (P_0\partial_x u)(x,y) =: \beta U_2(x)\varphi_0(y), \qquad R := P_{\geq 1}u, \tag{2.12}$$

and scaling ξ_2, we finally obtain the following integro-differential fixed-point equation in the spirit of Amick and Turner [3]:

$$\begin{cases} U_1(x) = \xi_1 + \beta\displaystyle\int_0^x U_2(s)\,ds \\ U_2(x) = \xi_2 + \dfrac{1}{\beta}\displaystyle\int_0^x\int_{\Omega'} \varphi_0(y)\mathcal{N}_1(U_1\varphi_0 + R, \lambda)(s,y)\,a_{11}(y)dy\,ds \\ \qquad\qquad + \dfrac{1}{\beta}\displaystyle\int_0^x\int_{\partial\Omega'} \varphi_0(y)\mathcal{N}_2(U_1\varphi_0 + R, \lambda)(s,y)\,a_{11}(y)dS(y)\,ds \\ \qquad R = GQ_{\geq 1}\mathcal{N}(U_1\varphi_0 + R, \lambda). \end{cases} \tag{2.13}$$

In terms of regularity, we ultimately seek solutions of (2.13) with

$$(U_1, U_2, R) \in C_{\mathrm{b}}^{2+\alpha}(\mathbb{R}) \times C_{\mathrm{b}}^{1+\alpha}(\mathbb{R}) \times C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega}) =: \mathbb{X}_{\mathrm{b}}.$$

Unraveling definitions, this will imply that $u \in \mathscr{X}_{\mathrm{b}}$. In view of Lemma 2.2, define $\overline{\mu} := \sqrt{|\nu_1|/2}$. In order to obtain a fixed point, we cannot work directly in $\mathbb{X}_{\mathrm{b}}$, but must instead consider the problem posed in the corresponding exponentially weighted space $\mathbb{X}_\mu$ for $\mu \in (0, \overline{\mu})$.

2.3. **Analysis of the nonlinear term.** We wish to eventually apply the fixed-point theorem for systems of the type (2.13) given by Amick and Turner, which is recalled in Appendix A. Towards that end, it is necessary to look more closely at the form of the nonlinear terms $\mathcal{N}$.

Splitting into bulk and boundary operators as usual, we have $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2)$ where

$$\begin{aligned} \mathcal{N}_1(u,\lambda) &= \tfrac{1}{a_{11}(y)}\nabla\cdot\big(\mathcal{A}(y,u,\nabla u,\lambda) - \nabla_p\mathcal{A}(y,0,0,0)\nabla u\big) \\ &\qquad + \tfrac{1}{a_{11}(y)}\left(\mathcal{B}(y,u,\nabla u,\lambda) - \partial_z\mathcal{B}(y,0,0,0)u - \nabla_p\mathcal{B}(y,0,0,0)\nabla u\right), \\ \mathcal{N}_2(u,\lambda) &= \tfrac{1}{a_{11}(y)}\big(\mathcal{G}(y,u,\nabla u,\lambda) - \partial_z\mathcal{G}(y,0,0,0)u - \nabla_p\mathcal{G}(y,0,0,0)\nabla u\big). \end{aligned}$$

Substituting

$$u = U_1\varphi_0 + R, \qquad \partial_x u = \beta U_2 \varphi_0 + R_x, \qquad \nabla' u = U_1 \nabla'\varphi_0 + \nabla' R,$$

we can rewrite this as

$$\begin{aligned} a_{11}\mathcal{N}_1 &= \nabla \cdot \tilde{\mathcal{A}}\big(y, U, \nabla R, \lambda, \beta\big) + \tilde{\mathcal{B}}\big(y, U, R, \nabla R, \lambda, \beta\big), \\ a_{11}\mathcal{N}_2 &= \tilde{\mathcal{G}}\big(y, U, R, \nabla R, \lambda, \beta\big), \end{aligned}$$

for some functions $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{G}}$ that are C^{M+4} in all of their arguments. Moreover, they are flat with respect to $(U, R, \nabla R, \lambda)$ in that their Taylor expansions in these variables at the origin contain only quadratic and higher-order terms. Note that here, and in the sequel, we write $U := (U_1, U_2)$ to shorten the equations.

With this in mind, let's analyze the terms in the fixed-point equation (2.13) and ensure they take the required form (A.2) for Theorem A.1.

Equation for U_1. Looking at (2.13), we see the right-hand side of the equation for U_1 is $\beta\mathfrak{L}_1(U,R)$ where $\mathfrak{L}_1(U,R) = \int_0^x U_2\,dx$. Since the operator $f \mapsto \int_0^x f\,ds$ is bounded and linear $C^{1+\alpha}_\mu \to C^{2+\alpha}_\mu$ and $C^{1+\alpha}_{\mathrm{b}} \to \mathring{C}^{2+\alpha}_{\mathrm{b}}$, $\mathfrak{L}_1$ satisfies the first component of (A.3). In particular, F_1 in (A.1) has the form (A.2) with $\mathcal{H}_1 = 0$.

Equation for U_2. In the equation for $U_2 \in C^{1+\alpha}$ in (2.13), consider the term

$$\frac{1}{\beta}\int_0^x\!\!\int_{\Omega'} \varphi_0 \left(\nabla \cdot \tilde{\mathcal{A}}\big(y, U, \nabla R, \lambda, \beta\big) + \tilde{\mathcal{B}}\big(y, U, R, \nabla R, \lambda, \beta\big)\right) dy\,ds.$$

The boundary integral can be handled in a similar fashion. Writing $\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}_1, \tilde{\mathcal{A}}')$, the contribution due to $\tilde{\mathcal{A}}_1$ can be rewritten as

$$\frac{1}{\beta}\int_0^x\!\!\int_{\Omega'} \varphi_0 \partial_x \tilde{\mathcal{A}}_1\big(y, U, \nabla R, \lambda, \beta\big)\,dy\,ds = \frac{1}{\beta}\int_{\Omega'} \varphi_0 \tilde{\mathcal{A}}_1\big(y, U, \nabla R, \lambda, \beta\big)\,dy\bigg|_0^x.$$

Stripping away the evaluations bars, we recognize this as having the form

$$\frac{1}{\beta}\mathcal{I}S_g(\mathcal{D}(U,R); \lambda, \beta),$$

where

$$\mathcal{D}(U,R) := (U, \nabla R), \qquad \mathcal{I}f := \int_{\Omega'} \varphi_0(y) f(\,\cdot\,, y)\,dy,$$

and S_g is the superposition operator defined by (A.6) for the function

$$g(x,y,u,r,\lambda,\beta) := \tilde{\mathcal{A}}_1(y,u,r,\lambda,\beta).$$

As $\mathcal{D}$ only evaluates derivatives in the R variables, it is easy to confirm that

$$\mathcal{D} \text{ bounded and linear } \mathbb{X}_\mu \longrightarrow C^{1+\alpha}_\mu(\mathbb{R};\mathbb{R}^2) \times C^{1+\alpha}_\mu(\overline{\Omega};\mathbb{R}^2) =: \mathbb{Y}_\mu,$$

for any $\mu \geq 0$, with bounds uniform in μ on compact subsets of $[0, \overline{\mu})$. Clearly, then, $\mathcal{D}$ satisfies (A.5). The function g is C^{M+3} and flat as required by (A.7) in light of the regularity assumed on the coefficients and the presence of the trivial solution family in (1.5). Finally, $\mathcal{I}$ simply integrates in the transversal direction, and hence

$$\mathcal{I} \text{ bounded and linear } \mathbb{Y}_\mu \to C^{1+\alpha}_\mu(\mathbb{R}) \text{ and } \mathbb{Y}_{\mathrm{b}} \to C^{1+\alpha}_{\mathrm{b}}(\mathbb{R}),$$

with bounds uniform in μ on compact subsets of $(0, \overline{\mu})$. In particular, the structural assumption (A.8) is satisfied.

Now let us move on to the contribution of $\tilde{\mathcal{A}}'$ to the U_2 equation:

$$\frac{1}{\beta}\int_0^x\int_{\Omega'}\varphi_0\nabla'\cdot\tilde{\mathcal{A}}'\big(y,U,\nabla R,\lambda,\beta\big)\,dy\,ds.$$

Stripping off the $1/\beta$ and integrating by parts in y, we get

$$-\int_0^x\int_{\Omega'}\nabla'\varphi_0\cdot\tilde{\mathcal{A}}'\big(y,U,\nabla R,\lambda,\beta\big)\,dy\,ds+\int_0^x\int_{\partial\Omega'}\varphi_0 N'\cdot\tilde{\mathcal{A}}'\big(y,U,\nabla R,\lambda,\beta\big)\,dS(y)\,ds.$$

These are analogous to the contribution of $\tilde{\mathcal{A}}_1$ considered above, except that the operator $\mathcal{I}$ is post-composed with $f\mapsto\int_0^x f\,ds$. While this is *not* a bounded map from $C^{1+\alpha}_{\mathrm{b}}(\mathbb{R})\to C^{1+\alpha}_{\mathrm{b}}(\mathbb{R})$, it *is* bounded $C^{1+\alpha}_{\mathrm{b}}(\mathbb{R})\to\mathring{C}^{1+\alpha}_{\mathrm{b}}(\mathbb{R})$, which is all that is required in (A.8). The contribution from $\tilde{\mathcal{B}}$ is treated in exactly the same manner; indeed, it is even simpler since no integration by parts is needed.

Equation for R. The work for the R equation has mostly been done through the study of the operator G. We know in particular that G, and hence the composition $\mathcal{I}:=GQ_{\geq 1}$, are bounded $\mathscr{Y}_\mu\to P_{\geq 1}\mathscr{X}_\mu$ for any $\mu\in[0,\overline{\mu})$ by Proposition 2.2. The argument of $GQ_{\geq 1}$ is the interior and boundary components of $\mathcal{N}(U_1\varphi_0+R,\lambda)$, each of which can be written as

$$S_g(U,\partial_x U_2,R,\nabla R,D^2R;\lambda,\beta)$$

for some suitably flat g that is independent of x. Thus for $\mathcal{D}$ we can take

$$\mathcal{D}(U,R):=(U,\partial_x U_2,R,\nabla R,D^2R),$$

which satisfies

$$\mathcal{D}\text{ is bounded and linear }\mathbb{X}_\mu\longrightarrow C^\alpha_\mu(\mathbb{R};\mathbb{R}^3)\times C^\alpha_\mu(\overline{\Omega};\mathbb{R}^{1+n+n^2})$$

for any $\mu\in[0,\overline{\mu})$ with bounds uniform on compact subsets of this interval. This will certainly satisfy (A.5), and we have

$$(U,R)\mapsto S_g(\mathcal{D}(U,R);\lambda,\beta)\text{ bounded }\mathbb{X}_\mu\to C^\alpha_\mu(\overline{\Omega}),$$

for all $\mu\in[0,\overline{\mu})$ and uniformly on compact subsets. Applying Proposition 2.2, we conclude that $\mathcal{I}$ will then satisfy (A.8):

$$\mathcal{I}\text{ is bounded and linear }C^\alpha_\mu(\overline{\Omega})\to C^{2+\alpha}_\mu(\overline{\Omega})\text{ and }C^\alpha_{\mathrm{b}}(\overline{\Omega})\to C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}),$$

for all $\mu\in(0,\overline{\mu})$.

2.4. **Truncation and fixed point mapping.** We have verified all of the hypotheses of Theorem A.1. As it stands, however, this only tells us about solutions to a truncated version of (2.13) where the nonlinear terms have been precomposed with cutoff functions. Undoing the various changes of variable, this leads us to a cut-off version $\mathscr{F}^r$ (in the sense of (A.9)) of the nonlinear elliptic operator $\mathscr{F}$, where $r>0$ measures the scale of the cut-off function. An advantage of $\mathscr{F}^r$ is that it is defined as a mapping $\mathscr{X}_\mu\times\mathbb{R}\to\mathscr{Y}_\mu$ between weighted spaces. If we increase the weight on the target space relative to the domain, then we can arrange for $\mathscr{F}^r$ to have any finite degree of smoothness: $\mathscr{F}^r\in C^{M+3}(\mathscr{X}_\mu\times\mathbb{R};\mathscr{Y}_{(M+6)\mu})$. This rather technical fact is a consequence of [3, Theorem 2.1]. A slight complication is that the operator $\mathscr{F}^r$ is no longer local, since it is defined with reference to the spectral splitting (2.12). But this splitting only occurs in the transverse variable y, and so $\mathscr{F}^r$ *is* local in x. Moreover, $\mathscr{F}$ and $\mathscr{F}^r$ agree in a sufficiently small ball in $\mathscr{X}_{\mathrm{b}}$.

Lemma 2.4. *In the setting of Theorem 1.1, suppose that $\|u\|_{\mathscr{X}}<r$. Then*

(i) $\mathscr{F}^r(u,\lambda)=\mathscr{F}(u,\lambda)$.
(ii) $D_u\mathscr{F}^r(u,\lambda)\big|_{\mathscr{X}_{\mathrm{b}}}=D_u\mathscr{F}(u,\lambda)$.
(iii) $D_u^\ell D_\lambda^k\mathscr{F}^r(u,\lambda)\big|_{\mathscr{X}_{\mathrm{b}}^\ell}=D_u^\ell D_\lambda^k\mathscr{F}(0,0)$.

Proof. Part (i) is obvious by the definition of $\mathscr{F}^r$ in the sense of (A.9). For (ii), we know that for any $v \in \mathscr{X}_{\mathrm{b}}$ and for $\|u\|_{\mathscr{X}} < r$ one can find ε sufficiently small so that $\|u+\varepsilon v\|_{\mathscr{X}} < r$. Thus

$$D_u\mathscr{F}^r(u,\lambda)v = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathscr{F}^r(u+\varepsilon v,\lambda) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathscr{F}(u+\varepsilon v,\lambda) = D_u\mathscr{F}(u,\lambda)v.$$

It is also easy to see that $D_\lambda^k\mathscr{F}^r(u,\lambda) = D_\lambda^k\mathscr{F}^r(u,\lambda)$ for $\|u\|_{\mathscr{X}} < r$ and $k \leq M$. Repeatedly differentiating with respect to u, (iii) then follows by induction on ℓ. □

In terms of $\mathscr{F}^r$, the result of applying Theorem A.1 is recorded in the following lemma.

Lemma 2.5 (Existence of a fixed point)**.** *For any integer $M \geq 2$, there exists $\mu \in (0,\overline{\mu})$, $r > 0$, $\beta \in (0,1]$, and a C^{M+1} mapping*

$$\mathscr{W} : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{X}_\mu \qquad (\xi_1,\xi_2,\lambda) \longmapsto (U_1,U_2,R) \tag{2.14}$$

so that, for all (ξ_1,ξ_2,λ), the function $u \in \mathscr{X}_\mu$ defined by

$$u(x,y) = U_1(x)\varphi_0(y) + R(x,y) \tag{2.15}$$

is the unique solution to the truncated problem $\mathscr{F}^r(u,\lambda) = 0$ that satisfies the initial conditions

$$\xi_1 = U_1(0) = \int_{\Omega'} u(0,y)\varphi_0(y)\,dy, \qquad \xi_2 = U_2(0) = \frac{1}{\beta}\int_{\Omega'} \partial_x u(0,y)\varphi_0(y)\,dy.$$

2.5. **Proof of main results.** We are now ready to prove the main results of this section.

Proof of Theorem 1.1. Our first step is to change variables from the initial data $\xi = (\xi_1,\xi_2)$ in Lemma 2.5 to

$$a = (a_1,a_2) := \left(\frac{u(0,0)}{\varphi_0(0)}, \frac{\partial_x u(0,0)}{\beta\varphi_0(0)}\right).$$

Towards that end, fix $(\xi_1,\xi_2) \in \mathbb{R}$, and suppose that u is given by (2.15) and (2.14) in Lemma 2.5. Then we calculate

$$a_1 = \frac{u(0,0)}{\varphi_0(0)} = U_1(0) + \frac{1}{\varphi_0(0)}R(0,0) = \xi_1 + \frac{1}{\varphi_0(0)}\mathscr{W}_3(\xi_1,\xi_2,\lambda)(0,0),$$

and, similarly,

$$a_2 = \frac{\partial_x u(0,0)}{\beta\varphi_0(0)} = \frac{1}{\beta}\partial_x U_1(0) + \frac{1}{\beta\varphi_0(0)}\partial_x R(0,0) = \xi_2 + \frac{1}{\beta\varphi_0(0)}\partial_x\mathscr{W}_3(\xi_1,\xi_2,\lambda)(0,0).$$

Thanks to the estimates in Lemma A.2, the mapping $\xi \mapsto a$ is a C^{M+1} near-identity change of variables. In particular, it has a C^{M+1} inverse $\xi = a + g(a,\lambda)$ for some function $g \in C^{M+1}$ which is flat in that $g(0,\lambda) = 0$ for all λ and $\partial_a g(0,0) = 0$. Introducing the scaled variables $A := \varphi_0(0)a_1$ and $B := \beta\varphi_0(0)a_2$, we further rewrite this as

$$\xi_1 = \frac{A}{\varphi_0(0)} + G_1(A,B,\lambda), \qquad \xi_2 = \frac{B}{\beta\varphi_0(0)} + G_2(A,B,\lambda), \tag{2.16}$$

for some G_1, G_2 that are flat with respect to (A,B) in that $G_1(0,0,\lambda) = G_2(0,0,\lambda) = 0$ for all λ, while $\nabla_{(A,B)}G_1$ and $\nabla_{(A,B)}G_2$ vanish at $(0,0,0)$.

The Faye–Scheel reduction function Ψ can now be explicitly defined by

$$\Psi(A,B,\lambda)(x,y) = \left(\mathscr{W}_1(\xi_1,\xi_2,\lambda)(x) - \frac{A}{\varphi_0(0)} - \frac{B}{\varphi_0(0)}x\right)\varphi_0(y) + \mathscr{W}_3(\xi_1,\xi_2,\lambda)(x,y),$$

with (ξ_1,ξ_2) the functions of (A,B,λ) given by (2.16). All of its properties are straightforward to check, and we obtain the formula (1.11) with $x \neq 0$ by appealing to the translation invariance of the problem.

It remains to derive the ODE (1.9) for $v := u(\,\cdot\,,0)$. Differentiating (1.11) twice with respect to x we obtain

$$\partial_x^2 u(x+\tau,y) = \partial_\tau^2 \Psi(v(x),v'(x),\lambda)(\tau,y).$$

Setting $y=0$ and $\tau=0$ this becomes

$$v''(x) = \partial_x^2 u(x,0) = \frac{d^2}{d\tau^2}\bigg|_{\tau=0} \Psi(v(x),v'(x),\lambda)(\tau,0) =: f(v(x),v'(x),\lambda),$$

as desired. □

Proof of Theorem 1.6. Our proof of Theorem 1.1 ensures the existence of the Faye–Scheel reduction function Ψ which is uniquely determined by $\mathcal{Q}\Psi = 0$ and

$$\mathscr{F}^r\left(\frac{A}{\varphi_0(0)}\varphi_0 + \frac{B}{\varphi_0(0)}x\varphi_0 + \Psi(A,B,\lambda),\lambda\right) = 0. \tag{2.17}$$

Recall that here we are using

$$(\mathcal{Q}u)(x,y) := \frac{u(0,0)}{\varphi_0(0)}\varphi_0(y) + \frac{\partial_x u(0,0)}{\varphi_0(0)}x\varphi_0(y),$$

but this argument can be repeated for any choice of projection onto $\ker L|_{\mathscr{X}_\mu}$. From the regularity and flatness properties (1.8) of Ψ, we know that Ψ admits an expansion of the form (1.13), and (i) follows from applying $\mathcal{Q}$ to it. Next we differentiate (2.17) to obtain

$$\partial_A^i \partial_B^j \partial_\lambda^k\Big|_{(A,B,\lambda)=0} \mathscr{F}^r\left(\frac{A}{\varphi_0(0)}\varphi_0 + \frac{B}{\varphi_0(0)}x\varphi_0 + \Psi(A,B,\lambda),\lambda\right) = 0 \tag{2.18}$$

for $i+j+k \le M$. Since the implied partials of $\mathscr{F}^r$ are all being evaluated at $(u,\lambda) = (0,0)$, we can then use Lemma 2.4 to replace them with the desired partials of $\mathscr{F}$, proving (ii).

It remains to show that the Ψ_{ijk} are uniquely determined by these properties. Plugging (1.13) into (1.14) and recalling that $\underline{L} = \frac{1}{a_{11}} D_u\mathscr{F}(0,0)$, we find that

$$\underline{L}\Psi_{ijk} + \mathscr{R}_{ijk} = 0, \tag{2.19}$$

where $\mathscr{R}_{ijk}$ depends on $\Psi_{i'j'k'}$ for $i' \le i,\ j' \le j,\ k' \le k$ and $i'+j'+k' \le i+j+k-1$. Lemma 2.3 then allows one to solve Ψ_{ijk} uniquely from $\{\Psi_{i'j'k'}\}$. □

3. Extensions to other types of elliptic problems

3.1. **Symmetries.** Unsurprisingly, symmetries of the nonlinear operator $\mathscr{F}$ are reflected in the reduction function Ψ in Theorem 1.1, and hence in the ODE (1.9). For example, the elasticity application studied in Section 5 leads us to consider PDEs of the general form

$$\begin{cases} \Delta u + \nabla\cdot\left(|\nabla u|^2 \nabla u\right) + u - \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega := \mathbb{R}\times(-\pi/2,\pi/2)$ and a number of constants have been set to 1 for expositional purposes. Observe that this problem is invariant with respect to the reversal transformation $u(x,y) \mapsto u(-x,y)$, which can be stated abstractly as

$$\mathscr{F}(u\circ T,\lambda) = \mathscr{F}(u,\lambda)\circ T \qquad \text{for } T := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Intuitively, this invariance should mean that many terms in the expansion of Ψ must vanish. Knowing this in advance greatly simplifies the task of computing the reduced equation.

We will consider only a slightly larger class of symmetries than the above example, as this will permit us to give a fairly simple argument. Let $T \in \mathbb{R}^{n\times n}$ be a diagonal matrix with $T^2 = \mathrm{id}$, and let $\epsilon_0, \epsilon_1, \epsilon_2 \in \{+1, -1\}$. Suppose that $T\Omega = \Omega$, and that

$$\mathscr{F}_i(\epsilon_0 u \circ T, \lambda) = \epsilon_i \mathscr{F}_i(u, \lambda) \circ T \tag{3.1}$$

for all $u \in \mathscr{X}$ and $\lambda \in \mathbb{R}$. One can then check that the cutoff operator $\mathscr{F}^r$ introduced in Section 2.4 satisfies the same identity, where here the key facts are that the splitting of u into U_1, U_2, R respects these linear reflections, and that the even cutoff function η_r is ultimately applied component-wise to U_1, U_2, R and their partials.

Fix $A, B, \lambda \in \mathbb{R}$ and let $u = A\varphi_0 + Bx\varphi_0 + \Psi(A, B, \lambda)$. Then by the construction of Ψ we have $\mathscr{F}^r(u, \lambda) = 0$, and hence also $\mathscr{F}^r(\epsilon_0 u \circ T, \lambda) = 0$ by the above argument. By the simplicity of the eigenvalue 0 of L', $\varphi_0 \circ T = \varphi_0$, and so we calculate

$$\epsilon_0(u \circ T)(x, y) = \epsilon_0 A\varphi_0(y) + \epsilon_0 T_{00} Bx\varphi_0(y) + \epsilon_0(\Psi(A, B, \lambda) \circ T)(x, y).$$

Applying $\mathcal{Q}$ and appealing to the uniqueness of Ψ, we deduce that

$$\Psi(\epsilon_0 A, \epsilon_0 T_{00} B, \lambda) = \epsilon_0 \Psi(A, B, \lambda) \circ T, \tag{3.2}$$

and hence that

$$f(\epsilon_0 A, \epsilon_0 T_{00} B, \lambda) = \epsilon_0 f(A, B, \lambda). \tag{3.3}$$

3.2. **Other boundary conditions.** In formulating Theorems 1.1 and 1.6, we chose to focus on problems that linearize to co-normal boundary conditions. This is not essential: looking at the proof, it is clear that one can just as easily impose nonlinear Dirichlet conditions of the form

$$\mathcal{G}(y, u, \lambda) = 0 \qquad \text{on } \partial\Omega, \tag{3.4}$$

for $\mathcal{G}$ that is C^{M+4} in its arguments. Naturally, for this case the codomain of $\mathscr{F}$ should be redefined to be

$$\mathscr{Y} := C^{0+\alpha}(\overline{\Omega}) \times C^{2+\alpha}(\partial\Omega),$$

and likewise for $\mathscr{Y}_\mu$ and $\mathscr{Y}_{\mathrm{b}}$. In place of the obliqueness assumption (1.3), we now require that

$$\partial_z \mathcal{G}(y, 0, 0) \neq 0 \qquad \text{for all } y \in \overline{\Omega'}.$$

By assumption (1.5), we must have $\mathcal{G}(y, 0, \lambda) = 0$ for all $y \in \partial\Omega'$ and $\lambda \in \mathbb{R}$. The above hypothesis on $\mathcal{G}$ justifies using the implicit function theorem to recast (3.4) as $u = 0$ on $\partial\Omega$. Then the proof of Proposition 2.2 proceeds as before, only using a priori estimates for linear elliptic PDE with homogeneous Dirichlet conditions. The fixed point argument is essentially unchanged. One small modification is that eigenfunction φ_0 can now vanish on $\partial\Omega'$, and hence we must assume that $0 \in \Omega'$ or else choose a different projection $\mathcal{Q}$.

Likewise, if $\partial\Omega$ has two or more connected components, one can freely impose either Dirichlet or co-normal conditions on each, adjusting the regularity of $\mathscr{Y}$ accordingly.

3.3. **Internal interfaces and free boundaries.** We can also expand the scope of the reduction theorem to treat nonlinear transmission problems. Suppose that $n = 2$ and the base Ω' is the union of two open intervals:

$$\Omega' = \Omega'_1 \cup \Omega'_2, \qquad \Omega'_1 := (-1, 0),\ \Omega'_2 := (0, 1).$$

Let $\Omega := \mathbb{R} \times \Omega'$ be the (slitted) cylinder, and say $\Omega_i = \mathbb{R} \times \Omega'_i$.

Physically, one can for instance imagine this as representing two immiscible fluids confined to a channel with rigid walls; the interface between them is the line $\Gamma := \mathbb{R} \times \{0\}$. Of course this interface is only flat once we have performed a change of variables, and this may introduce terms in the interior equation relating to traces (or derivatives of traces) of quantities on the boundary. With that in mind, we consider the following quite general quasilinear elliptic problem:

$$\begin{cases} \nabla\cdot\mathcal{A}(y,u,\nabla u,u|_\Gamma,u_x|_\Gamma,\lambda)=0 & \text{in } \Omega \\ \mathcal{G}(u_1,u_2,\nabla u_1,\nabla u_2,\lambda)=0 & \text{on } \Gamma \\ \mathcal{K}(u_1,u_2,\lambda)=0 & \text{on } \Gamma \\ u=0 & \text{on } \{y=\pm 1\}, \end{cases} \tag{3.5}$$

where $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{G}$, $\mathcal{K}$ are C^{M+4} in their arguments. Here, we are breaking convention slightly by writing $u_i := u|_{\Omega_i}$ and likewise for $\mathcal{A}$. As before, assume that $\mathcal{A}_i$ is uniformly elliptic (1.2). In place of obliqueness (1.3), we now ask that

$$N(y)\cdot(\nabla_{p_1}\mathcal{G}(z_1,z_2,p_1,p_2)-\nabla_{p_2}\mathcal{G}(z_1,z_2,p_1,p_2))>\chi \quad \text{for all } y\in\Omega',\ p_1,p_2\in\mathbb{R}^n,\ z_1,z_2,\lambda\in\mathbb{R}.$$

The elliptic problem (3.5) can be rewritten as an operator equation $\mathscr{F}(u,\lambda)=0$, with $\mathscr{F}=(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3)$ corresponding to the first three equations and the homogeneous Dirichlet condition incorporated into the definition of the space. The main restriction is that the *linearized* problem is of transmission type, that is,

$$\begin{aligned} &\mathscr{F}_{1u}(0,0)v := \nabla\cdot(a(y)\nabla v)\,, \\ &\mathscr{F}_{2u}(0,0)v := -\llbracket N(y)\cdot a(y)\nabla v\rrbracket + g_1(y)v_1+g_2(y)v_2, \\ &\mathscr{F}_{3u}(0,0)v := \llbracket v\rrbracket\,. \end{aligned} \tag{3.6}$$

Here $\llbracket\,\cdot\,\rrbracket := (\,\cdot\,)_2-(\,\cdot\,)_1$ denotes the jump of a quantity over Γ, and the coefficients a, g_i are obtained from $\mathcal{A}_i$ and $\mathcal{G}$ in the obvious way.

In typical applications, one asks for solutions whose restriction to Ω_i is smooth up to the boundary. We therefore set

$$\mathscr{X} := \left\{u : u|_{\Omega_i}\in C^{2+\alpha}(\overline{\Omega_i}),\ u|_{y=\pm1}=0\right\}, \tag{3.7}$$

and take as the codomain for the corresponding nonlinear mapping $\mathscr{F}$ the space

$$\mathscr{Y} := \left\{w : w|_{\Omega_i}\in C^{0+\alpha}(\overline{\Omega_i})\right\}\times C^{1+\alpha}(\Gamma)\times C^{2+\alpha}(\Gamma). \tag{3.8}$$

While the jump conditions on Γ in (3.6) are somewhat exotic, there is a well-established literature regarding them, including the Schauder estimates [33] that we require. It is then quite straightforward to generalize Theorem 1.1 and Theorem 1.6 to the setting of (3.5). Indeed, Amick and Turner explicitly mention how their fixed point theory accommodates spaces similar to (3.7) and (3.8) (see, [3, Remark 2.2, Remark 3.2]), and in [2] they apply it to a transmission problem in hydrodynamics that is a special case of what we consider in Section 7.

Corollary 3.1. *Consider the quasilinear elliptic problem* (3.5)*. Assume that the corresponding linearized operator L is of transmission type* (3.6) *and the transversal linearized operator L' satisfies the spectral hypothesis* (1.7)*. Fix $\mu\in(0,\sqrt{|\nu_1|/2})$ and an integer $M\geq 2$. Then there exist neighborhoods $U\subset\mathscr{X}\times\mathbb{R}$ and $V\subset\mathbb{R}^3$ of the origin and a coordinate map $\Psi=\Psi(A,B,\lambda)$ exhibiting all the properties claimed in Theorem 1.1, 1.6, and 1.9.*

3.4. **Diagonal elliptic systems.** Another possibility is to study systems of quasilinear elliptic PDE. To do this in complete generality is beyond the scope of this paper, but, with just a minor modification, the above argument can treat a special class of systems that are important for the proof of Theorem 1.9.

Letting Ω again be any connected cylinder as in Section 1, we consider solutions $u=(u^1,u^2)$ to

$$\begin{cases} \nabla\cdot\mathcal{A}^i(y,\nabla u^i,\lambda)+\mathcal{B}^i(y,u,\nabla u,\lambda)=0 & \text{in } \Omega \\ -N(y)\cdot\mathcal{A}^i(y,\nabla u^i,\lambda)+\mathcal{G}^i(y,u,\lambda)=0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

for $i = 1, 2$. We assume that the coefficients $\mathcal{A}^i$, $\mathcal{B}^i$, $\mathcal{G}^i$ are C^{M+4} in their arguments, and also that uniform ellipticity (1.2) and obliqueness (1.3) hold. Suppose further that the linearized problem at $(u, \lambda) = (0, 0)$ is *diagonal* in the sense that (3.9) can be rewritten as

$$\begin{cases} Lu^1 = \mathcal{N}^1(u^1, u^2, \lambda) \\ Lu^2 = \mathcal{N}^2(u^1, u^2, \lambda), \end{cases} \tag{3.10}$$

where $L : \mathscr{X}_{\mathrm{b}} \to \mathscr{Y}_{\mathrm{b}}$ is a bounded linear operator, and each $\mathcal{N}^i$ is a divergence form nonlinear operator that is (i) C^{M+3} as a mapping $\mathscr{X}_{\mathrm{b}} \times \mathscr{X}_{\mathrm{b}} \times \mathbb{R} \to \mathscr{Y}_{\mathrm{b}}$, and (ii) satisfies $\mathcal{N}^i(0, \lambda) = 0$ and $\mathcal{N}^i_u(0, 0) = 0$. Arguing exactly as in Section 2.2, this problem can be reformulated as a fixed-point equation of the form (2.13) but with six components — three each for u^1 and u^2. While in Appendix A we only state Amick and Turner fixed-point theorem for a two-dimensional center manifold, the theory as originally formulated in [3, Section 3] applies to systems of arbitrary dimension. Using it as before allows us to recover Theorem 1.1 and Theorem 1.6, with the reduction function now taking values in the product space $\mathscr{X}_\mu \times \mathscr{X}_\mu$.

3.5. **Commuting linearization and reduction.** With the above center manifold theory for diagonal systems at our disposal, we are now prepared to prove Theorem 1.9.

Proof of Theorem 1.9. Suppose that we are in the setting of Theorem 1.1. Throughout the argument, we will work with a fixed value of λ that is taken sufficiently small. For convenience, it will therefore be suppressed.

Consider the following (truncated) *augmented problem*

$$\mathscr{G}^r(u, \dot{u}) := \Big(\mathscr{F}^r(u),\ D_u \mathscr{F}^r(u)\dot{u}\Big) = 0. \tag{3.11}$$

Recall that $\mathscr{F}^r$ denotes the truncated nonlinear mapping in the sense of Lemma 2.4. Naturally, (3.11) is a (truncated) diagonal system satisfying (3.9) and (3.10), and so we may apply the modified version of Theorem 1.1 to classify its small bounded solutions. In particular, there exists neighborhoods $U \subset \mathscr{X}_{\mathrm{b}} \times \mathscr{X}_{\mathrm{b}}$ and $V \subset \mathbb{R}^4$ of the origin, and a reduction function $(\Phi, \Upsilon) \in C^{M+1}(\mathbb{R}^4, \mathscr{X}_\mu \times \mathscr{X}_\mu)$ so that $(w, \dot{w}) \in U$ solves (3.11) if and only if

$$\begin{cases} w = (A + Bx)\varphi_0(y) + \Phi(A, B, \dot{A}, \dot{B}) \\ \dot{w} = (\dot{A} + \dot{B}x)\varphi_0(y) + \Upsilon(A, B, \dot{A}, \dot{B}), \end{cases} \tag{3.12}$$

for some $(A, B, \dot{A}, \dot{B}) \in V$. We are recycling notation here somewhat, as Φ above is not the same as the one occurring in Remark 1.2. Let us now *define*

$$\Psi(A, B) := \Phi(A, B, 0, 0).$$

It is easy to check that this function has all the properties of the reduction function furnished by Theorem 1.1 to the original (truncated) problem. In particular, this means that any sufficiently small $w \in \mathscr{X}_{\mathrm{b}}$ satisfying $\mathscr{F}^r(w) = 0$ can be written

$$w(x, y) = (A + Bx)\,\varphi_0(y) + \Psi(A, B)(x, y) \qquad \text{in } \Omega, \tag{3.13}$$

for some A, $B \in \mathbb{R}$. Moreover, $v := w(\,\cdot\,, 0)$ solves the reduced ODE (1.9), with f defined by (1.10). For simplicity, let us also normalize $\varphi_0(0) = 1$, which implies $A = v(0)$ and $B = v'(0)$. Note also that, by construction, the range of Φ and Υ lie in the kernel of the projection $\mathcal{Q}$ onto $\ker L$. Consequently, the coefficients A and B in (3.13) and (3.12) must indeed coincide.

Fix a small solution $u \in \mathscr{X}_{\mathrm{b}}$ to $\mathscr{F}(u) = \mathscr{F}^r(u) = 0$, and let $\dot{u} \in \mathscr{X}_{\mathrm{b}}$ be a solution of the linearized problem as in the statement of Theorem 1.9. Even though we do not assume that $\|\dot{u}\|_{\mathscr{X}}$ is small, the structure of the augmented problem problem ensures that, for all $\delta > 0$ sufficiently small, $(u, \delta\dot{u})$ lies in U and

$$\mathscr{G}(u, \delta\dot{u}) = \mathscr{G}^r(u, \delta\dot{u}) = 0.$$

In that case, we can use (3.12) with $(u, \delta\dot{u})$ in place of $(w, \dot{w})$ to see that

$$\delta\dot{u}(x, y) = \delta(\dot{A} + \dot{B}x)\varphi_0(y) + \Upsilon(A, B, \delta\dot{A}, \delta\dot{B}).$$

As Υ is C^{M+1}, an expansion of the right-hand side above in δ yields

$$\dot{u} = (\dot{A} + \dot{B}x)\varphi_0(y) + D_{(\dot{A},\dot{B})}\Upsilon(A, B, 0, 0) \cdot (\dot{A}, \dot{B}). \tag{3.14}$$

We claim further that

$$D_{(A,B)}\Psi(A, B) = D_{(\dot{A},\dot{B})}\Upsilon(A, B, 0, 0). \tag{3.15}$$

To see this, first observe that the construction of the reduction functions Ψ, and (Φ, Υ) ensure that

$$\mathscr{F}^r\left((A + Bx)\varphi_0 + \Phi(A, B, \dot{A}, \dot{B})\right) = 0,$$
$$\mathscr{G}^r\big((A + Bx)\varphi_0 + \Phi(A, B, \dot{A}, \dot{B}), (\dot{A} + \dot{B}x)\varphi_0 + \Upsilon(A, B, \dot{A}, \dot{B})\big) = 0,$$

for all $(A, B, \dot{A}, \dot{B}) \in V$. Note that $\mathscr{F}^r$ is C^2 as a mapping $\mathscr{X}_\mu \to \mathscr{Y}_{4\mu}$. This permits us to differentiate the first equation with respect to (A, B), and upon evaluating at $(A, B, 0, 0)$ we find that

$$D_u\mathscr{F}^r(u)\left[\varphi_0 + D_A\Psi(A, B)\right] = 0, \qquad D_u\mathscr{F}^r(u)\left[x\varphi_0 + D_B\Psi(A, B)\right] = 0.$$

Likewise, the second component of $\mathscr{G}^r$ is $(u, \dot{u}) \mapsto \mathscr{F}^r_u(u)\dot{u}$, which is C^1 as a mapping $\mathscr{X}_\mu \times \mathscr{X}_\mu \to \mathscr{Y}_{4\mu}$. Taking its derivative with respect to $(\dot{A}, \dot{B})$ and evaluating at $(A, B, 0, 0)$ leads to the identities

$$D_u\mathscr{F}^r(u)\left[\varphi_0 + D_{\dot{A}}\Upsilon(A, B, 0, 0)\right] = 0, \qquad D_u\mathscr{F}^r(u)\left[x\varphi_0 + D_{\dot{B}}\Upsilon(A, B, 0, 0)\right] = 0.$$

Combining the two identities above we conclude

$$D_u\mathscr{F}^r(u)\left[D_A\Psi(A, B) - D_{\dot{A}}\Upsilon(A, B, 0, 0)\right] = 0, \qquad D_u\mathscr{F}^r(u)\left[D_B\Psi(A, B) - D_{\dot{B}}\Upsilon(A, B, 0, 0)\right] = 0.$$

On the other hand, by construction

$$\mathcal{Q}\left[D_A\Psi(A, B) - D_{\dot{A}}\Upsilon(A, B, 0, 0)\right] = 0, \qquad \mathcal{Q}\left[D_B\Psi(A, B) - D_{\dot{B}}\Upsilon(A, B, 0, 0)\right] = 0.$$

We know from Lemma 2.3 that the bordered operator $w \mapsto (D_u\mathscr{F}^r(0)w, \mathcal{Q}w)$ is invertible $\mathscr{X}_\mu \to \mathscr{Y}_\mu \times \ker L$. Moreover, if $u \in \mathscr{X}_{\mathrm{b}}$ has $\|u\|_{\mathscr{X}}$ sufficiently small, the same is true for $w \mapsto (D_u\mathscr{F}^r(u)w, \mathcal{Q}w)$ by a perturbation argument. Hence we have proved the key identity (3.15), at least when $A = v(0)$ and $B = v'(0)$ correspond to a sufficiently small solution $u \in \mathscr{X}_{\mathrm{b}}$. The uniform smallness of u in particular means that, say by Lemma 2.4, we do not have to worry about the cut-off functions when performing this perturbative argument.

Theorem 1.9 follows almost immediately. From (1.9), (3.14), and (3.15) we see that $\dot{v} := \dot{u}(\,\cdot\,, 0)$ solves the reduced equation

$$\dot{v}'' = g(v, v', \dot{v}, \dot{v}'),$$

for

$$g(A, B, \dot{A}, \dot{B}) := \frac{d^2}{dx^2}\bigg|_{x=0}\left[(\dot{A}, \dot{B}) \cdot D_{(A,B)}\Psi(A, B)(x, 0)\right].$$

But, recalling the definition of f (1.10), this becomes exactly the claimed ODE (1.16). □

4. General strategy to apply the reduction procedure

In the course of proving Theorem 1.6, we have shown that each term Ψ_{ijk} can indeed be uniquely determined by iteratively solving a hierarchy of equations of the form (2.19), where the terms of the right-hand side involve information about various Fréchet derivatives of $\mathscr{F}$ at $(0, 0)$. In this section, we briefly illustrate how this process is carried out in practice, and also how the reduced equation (1.9) can be rescaled to obtain homoclinic or heteroclinic solutions. To simplify the presentation, we will assume that $a_{11} = \nabla_{p_1}\mathcal{A}(\,\cdot\,, 0, 0, 0) \equiv 1$, and hence $\underline{L} = L$.

4.1. **Iteration.** The smoothness of $\mathscr{F}^r$ and Lemma 2.4 allow us to write

$$\mathscr{F}^r(u,\lambda) = \sum_{1\le \ell+k\le K} \lambda^k D_u^\ell D_\lambda^k \mathscr{F}^r(0,0)[u,u,\ldots,u] + O\bigg(\sum_{\ell+k=K+1} \|u\|_{\mathscr{X}_\mu}^\ell |\lambda|^k \bigg) \tag{4.1}$$

in $\mathscr{Y}_{(K+4)\mu}$ for any integer $K \le M$, where the $D_u^\ell D_\lambda^k \mathscr{F}^r(0,0)$ are symmetric bounded ℓ-linear mappings $(\mathscr{X}_\mu)^\ell \to \mathscr{Y}_{(K+4)\mu}$. For $i+j+k \le K$, the remainder terms in (4.1) do not contribute to $\mathscr{R}_{ijk}$ in (2.19). Therefore, when solving (2.19), it is sufficient to work with the truncated version of (4.1) that results from simply setting these remainder terms to zero.

For an integer $K \ge 1$ and a smooth function $g = g(A,B,\lambda)$, we define $\mathcal{T}_K g$ to be the K-th order Taylor expansion of g at 0, that is,

$$\mathcal{T}_K g(A,B,\lambda) := \sum_{i+j+k\le K} \partial_A^i \partial_B^j \partial_\lambda^k g(0,0,0) A^i B^j \lambda^k.$$

Plugging (4.1) and (1.13) into (2.17) we see that, for $1 \le K \le M$,

$$\mathcal{T}_K \sum_{1\le \ell+k\le K} \lambda^k D_u^\ell D_\lambda^k \mathscr{F}(0,0)[u^{(K)},\ldots,u^{(K)}] = 0, \tag{4.2}$$

where

$$\begin{aligned} u^{(K)}(x,y;A,B,\lambda) &:= \mathcal{T}_K\left[A\varphi_0(y) + Bx\varphi_0(y) + \Psi(A,B,\lambda)(x,y)\right] \\ &= A\varphi_0(y) + Bx\varphi_0(y) + \sum_{\substack{2\le i+j+k\le K \\ i+j\ge 1}} \Psi_{ijk}(x,y) A^i B^j \lambda^k. \end{aligned}$$

More explicitly, at $K=1$, the definition of $u^{(K)}$ reads simply $u^{(1)}(x,y) = A\varphi_0(y) + Bx\varphi_0(y)$. For $K \ge 2$, we may use the facts that

$$D_\lambda^k \mathscr{F}(0,0) = 0, \quad \text{and} \quad u^{(K)} = u^{(K-1)} + \sum_{\substack{i+j+k=K \\ i+j\ge 1}} \Psi_{ijk}(x,y) A^i B^j \lambda^k \tag{4.3}$$

to derive the equations for Ψ_{ijk} when $i+j+k=K$. Below we give two example calculations for $K=2,3$. As all of the derivatives of $\mathscr{F}$ are evaluated at $(0,0)$, the base point will be suppressed for notational convenience.

When $K=2$, (4.2) and (4.3) imply that

$$L\bigg(\sum_{\substack{i+j+k=2 \\ i+j\ge 1}} \Psi_{ijk} A^i B^j \lambda^k \bigg) = -\lambda D_u D_\lambda \mathscr{F}_{u\lambda} u^{(1)} - D_u^2 \mathscr{F}[u^{(1)}, u^{(1)}],$$

from which $\{\Psi_{ijk} : \; i+j+k=2\}$, and hence $u^{(2)}$, can be uniquely solved by applying Lemma 2.3. At $K=3$, a similar calculation gives

$$\begin{aligned} L\bigg(\sum_{\substack{i+j+k=3 \\ i+j\ge 1}} \Psi_{ijk} A^i B^j \lambda^k \bigg) &= -Lu^{(2)} - \lambda D_u D_\lambda \mathscr{F} u^{(2)} - D_u^2 \mathscr{F}[u^{(2)}, u^{(2)}] \\ &\quad - \lambda^2 D_u D_\lambda^2 \mathscr{F} u^{(1)} - \lambda D_u^2 D_\lambda \mathscr{F}[u^{(1)}, u^{(1)}] - D_u^3 \mathscr{F}[u^{(1)}, u^{(1)}, u^{(1)}]. \end{aligned}$$

The right-hand side is explicit. Grouping like terms and applying Lemma 2.3 we may determine $u^{(3)}$.

This process repeats at each stage: we have to iteratively solve linear equations of the form

$$L\Psi_{ijk} = F_{ijk}, \qquad \mathcal{Q}\Psi_{ijk} = 0, \tag{4.4}$$

where $i+j+k=K$ and F_{ijk} depends only on $u^{(K-1)}$.

In summary the procedure for calculating Ψ_{ijk} can be explained in the following way. First, one Taylor expands the terms in $\mathscr{F}$ to order M to obtain a Taylor-truncated operator that is naturally defined on weighted spaces. The composition of the K-th order Taylor-truncated operator with $u^{(K)}$ is a polynomial in A, B, λ whose $\mathscr{Y}_{K\mu}$ coefficients depend on the Ψ_{ijk}. Setting the coefficients of $A^iB^j\lambda^k$ for $i+j+k \le K$ equal to zero, we obtain a series of equations for the Ψ_{ijk}. Working in order of increasing $i+j+k$, this becomes a sequence of linear problems of the form (4.4) where F_{ijk} is known. Lemma 2.3 ensures that these equations can be solved uniquely.

4.2. **Anticipated scaling.** The reduced ODE (1.9) always admits two degrees of freedom: we may select a length scale for the x-variable as well as an amplitude scale for the unknown. Making intelligent choices can vastly simplify the expansion procedure. For example, if we hope to find a heteroclinic solution, the reduced ODE must have a certain structure, and this leads to an anticipated scaling.

By design, (1.9) always has an equilibrium at the origin. In applications we are interested in cases where the linearized problem there is nondegenerate in that $D_{(A,B)}f(0,0,\lambda)$ has no zero eigenvalue for $0 < |\lambda| \ll 1$. Treating λ as fixed and performing a double expansion in (A,B) we have

$$\begin{aligned} f(A,B,\lambda) &= \partial_A f(0,0,\lambda)A + \partial_B f(0,0,\lambda)B + \frac{1}{2}\partial_A^2 f(0,0,\lambda)A^2 + \partial_A\partial_B f(0,0,\lambda)AB \\ &\quad + \frac{1}{2}\partial_B^2 f(0,0,\lambda)B^2 + \frac{1}{6}\partial_A^3 f(0,0,\lambda)A^3 + O\left((|A|+|B|)^2|B|\right). \end{aligned} \tag{4.5}$$

Note that the nondegeneracy condition forces $\partial_A f(0,0,\lambda) \neq 0$. For nontrivial heteroclincic or homoclinic solutions, we need a second rest point, which in terms of the above expansion translates to the right-hand side of (4.5) being nonlinear in A. Therefore, let us assume that there is a least integer $m \ge 2$ such that $\partial_A^m f(0,0,\lambda) \neq 0$.

Now, we introduce a rescaling of the axial variable $X = \kappa x$ and amplitude $v = aV$. Given the above discussion, we want v'', v, and v^m to appear as $O(1)$ terms in the corresponding rescaled version of the reduced ODE (1.9). This balancing forces the inverse length scale κ and the amplitude scale a to satisfy

$$a\kappa^2 \sim a\partial_A f(0,0,\lambda) \sim a^m \partial_A^m f(0,0,\lambda) \quad \text{as} \quad \lambda \to 0. \tag{4.6}$$

Clearly, then, κ and a involve roots of f_A and $\partial_A^m f$. To avoid this inconvenience, we may reparameterize $\lambda = \lambda(\varepsilon)$, and consider

$$\lambda = \lambda_p \varepsilon^p, \qquad \kappa = \kappa_n \varepsilon^n, \qquad a = a_q \varepsilon^q \tag{4.7}$$

for some $p, n, q \in \mathbb{N}$. It then follows from (4.6) that

$$\varepsilon^{2n} \sim \partial_A f(0,0,\varepsilon^p) \sim \partial_A^m f(0,0,\varepsilon^p)\varepsilon^{(m-1)q} \qquad \text{as} \quad \varepsilon \to 0. \tag{4.8}$$

In particular, this implies that when we carry out the iteration procedure of Section 4.1, A, B, and λ have differing orders of magnitude. It therefore suffices to compute the Ψ_{ijk} for i,j,k in the index set

$$\mathcal{J} = \left\{(i,j,k) \in \mathbb{N}^3 : \ qi + (q+n)j + k \le q + 2n, \ i+j+k \ge 2, \ i+j \ge 1\right\}.$$

Notice that we have not taken into account the contribution of $\partial_B f(0,0,\lambda)B$ in the expansion (4.5). This can be justified, for instance, when the system has the reversal symmetry $(v(x), v'(x)) \mapsto (v(-x), -v'(-x))$. However, if $\partial_B f(0,0,\lambda) \neq 0$, the length scale will be over-determined since there is a linear term in B in (4.5) which also suggests a choice of κ. For this to be compatible with (4.6), we must therefore have

$$|\partial_A f(0,0,\lambda)| \sim |\partial_B f(0,0,\lambda)|^2 \quad \text{as} \quad \lambda \to 0. \tag{4.9}$$

With enough parameters in the problem, one can always arrange for this to hold; see, for example, Section 6.

5. Anti-plane shear

Consider a homogeneous, incompressible, isotropic elastic cylinder $\mathcal{D} = \Omega \times \mathbb{R}$ with generators parallel to z-axis and cross section $\Omega \subset \mathbb{R}^2$ in the (x,y)-plane. *Anti-plane shear* describes the situation where the deformation takes the form

$$\mathrm{id} + u(x,y)e_3, \tag{5.1}$$

where e_3 is the standard basis vector $(0,0,1)^T$. That is, the displacement of each particle is parallel to the generators of the cylinder and independent of its axial position.

For an isotropic elastic solid, the strain energy density $\mathcal{W}$ is a function of the three principal invariants I_1, I_2, I_3 of the Cauchy–Green tensor. In this section, we will consider a polynomial rubber elastic model, which corresponds to the case where $\mathcal{W}$ is a polynomial in I_1 and I_2 [47]. Thus we can write

$$\mathcal{W}(I_1, I_2) := \sum_{i+j=1}^{N} C_{ij}(I_1 - 3)^i (I_2 - 3)^j.$$

Note that when $N = 1$, $C_{01} = 0$, this reduces to the standard neo-Hookean solid model [49]. Values of $N > 2$ are rarely used in practice because it is difficult to fit such a large number of material properties to experimental data. Therefore, we restrict our attention to the quadratic case $N = 2$; this will result in a quasilinear PDE with a 4-Laplacian term, cf. (5.8).

Assuming incompressibility, it is shown in [30] that under an "ellipticity condition"

$$\frac{d}{dR}\left[R\left(\frac{\partial W}{\partial I_1}(I_1, I_2) + \frac{\partial W}{\partial I_2}(I_1, I_2) \right)\bigg|_{I_1 = I_2 = 3 + R^2} \right] > 0 \quad \text{for all} \quad R \geq 0,$$

the energy function W is admissible if and only if there exists some constant $k \in \mathbb{R}$ such that

$$k \frac{\partial W}{\partial I_1}(I_1, I_2) + (k-1)\frac{\partial W}{\partial I_2}(I_1, I_2) = 0.$$

For this reason, we will consider a class of quadratic neo-Hookean materials whose strain energy density takes the form

$$\mathcal{W}(I_1, I_2) := C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 \tag{5.2}$$

Imposing the anti-plane shear ansatz (5.1), we know that the principal invariants satisfy $I_1 = I_2 = 3 + |\nabla u|^2 =: I$, and $I_3 = 1$. Hence, we may identify $\mathcal{W}$ with the function $\mathcal{W}(I) = \mathcal{W}(3 + |\nabla u|^2)$; see, for example, [30, 25]. At infinitesimal deformations, the shear modulus is given by $2\mathcal{W}'(3)$ which is supposed to be positive. For simplicity, we normalize $\mathcal{W}'(3) = 1$. Then the quadratic neo-Hookean model (5.2) becomes

$$\mathcal{W}(I) = (I - 3) + w_1(I - 3)^2, \qquad \mathcal{W}'(3 + |\nabla u|^2) = 1 + 2w_1|\nabla u|^2, \tag{5.3}$$

where $w_1 := \mathcal{W}''(3)/2$ is a material constant.

While there has been an extensive literature on the sustainability of anti-plane shear deformation in various constitutive settings [26, 45], the analytical results concerning the existence of nontrivial equilibria are mostly restricted to the variational construction of Sobolev solutions [17, 52]. Our contribution in this section is the construction of a new class of solutions on an unbounded cylinder which limit to distinct limits as $x \to -\infty$ and $x \to +\infty$. We call these equilibria *anti-plane shear fronts*.

Following Healey and Simpson [22], we suppose that the body is subjected to a parameter-dependent "live" body force $b = b(u, \lambda)$. As in, e.g., [27, 25], we consider the geometrical setting where $\Omega = \mathbb{R} \times (-\pi/2, \pi/2)$ is an infinite strip and homogeneous Dirichlet boundary conditions are imposed on $\{y = \pm\pi/2\}$.

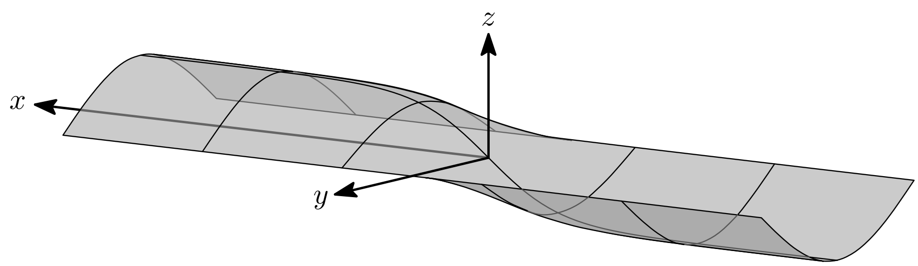

FIGURE 2. Leading-order approximation of the front-type solutions in Theorem 5.1(a). The graph $z = u(x, y)$ is the image of the strip $\{z = 0, |y| < \pi/2\}$ under the anti-plane deformation (5.1).

A static equilibrium then satisfies

$$\begin{cases} \nabla \cdot \big(\mathcal{W}'(3 + |\nabla u|^2)\nabla u\big) - b(u, \lambda) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

The system (5.4) carries a variational structure with the energy

$$E(u) := \int_\Omega \big[\mathcal{W}(3 + |\nabla u|^2) + B(u, \lambda)\big]\, dx\, dy,$$

where $B_u = \frac{1}{2}b$. Note that (5.4) is invariant under the "reversibility" reflection $u(x, y) \mapsto u(-x, y)$ about the (y, z)-plane. We will assume in addition that it is invariant under the reflection $u \mapsto -u$, which forces

$$b(\,\cdot\,, \lambda) \text{ is odd, and hence } B(\,\cdot\,, \lambda) \text{ is even.} \tag{5.5}$$

The eigenvalue problem for the linearized transversal operator corresponding to (5.4) is simple to compute:

$$\begin{cases} w_{yy} - b_u(0, 0)w = \nu w & \text{in } (-\pi/2, \pi/2) \\ w = 0 & \text{on } \{y = -\pi/2, \pi/2\}. \end{cases}$$

If the body force b satisfies

$$b_u(0, 0) = -1, \tag{5.6}$$

then $\nu = 0$ is a simple eigenvalue, and the rest of the spectrum is negative. The kernel of the linearized operator is generated by

$$\varphi_0(y) := \cos y.$$

To make things concrete, we introduce a specific ansatz for the body force:

$$b(u, \lambda) = -u + \lambda b_1 u + b_2 u^3. \tag{5.7}$$

Note that this satisfies both (5.5) and (5.6). One can of course add higher-order terms in u if desired; see Appendix B. Following (4.7), we reparametrize $\lambda = \lambda_2\varepsilon^2$ (see Section 5.1 for more details). The model (5.4) then becomes

$$\begin{cases} \Delta u + 2w_1 \nabla \cdot \big(|\nabla u|^2 \nabla u\big) + u - b_1\lambda_2\varepsilon^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

Theorem 5.1 (Fronts in anti-plane shear deformation)**.** *Consider the anti-plane shear problem* (5.8) *with strain energy W given by* (5.3) *and a live body force b of the form* (5.7) *with $b_2 = 0$.*

(a) *When $b_1\lambda_2 < 0$, $w_1 > 0$, there exists $\varepsilon_0 > 0$ and a family of front-type solutions*
$$\left\{(u^\varepsilon, \varepsilon) \in C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R} : -\varepsilon_0 < \varepsilon < \varepsilon_0\right\}$$
bifurcating from the unforced state $(u,\varepsilon) = (0,0)$. It exhibits the asymptotics:
$$u^\varepsilon(x,y) = a_1\varepsilon \tanh(\kappa_1 \varepsilon x)\cos(y) + O(\varepsilon^2) \quad \text{in } C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}), \tag{5.9}$$
where $a_1 = \sqrt{\dfrac{-2b_1\lambda_2}{3w_1}}$, $\kappa_1 = \sqrt{\dfrac{-b_1\lambda_2}{2}}$.

(b) *When $b_1\lambda_2 > 0$ and $w_1 < 0$, there exists $\varepsilon_0 > 0$ and a family of homoclinic-type solutions*
$$\left\{(u^\varepsilon, \varepsilon) \in C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}) \times \mathbb{R} : -\varepsilon_0 < \varepsilon < \varepsilon_0\right\}$$
bifurcating from the unforced state $(u,\varepsilon) = (0,0)$. It exhibits the asymptotics:
$$u^\varepsilon(x,y) = a_1\varepsilon \operatorname{sech}(\kappa_1 \varepsilon x)\cos(y) + O(\varepsilon^2) \quad \text{in } C^{2+\alpha}_{\mathrm{b}}(\overline{\Omega}), \tag{5.10}$$
where $a_1 = \sqrt{\dfrac{-4b_1\lambda_2}{3w_1}}$, $\kappa_1 = \sqrt{b_1\lambda_2}$.

See Figure 2 for an illustration of the solutions in case (a).

Remark 5.2. It is worth emphasizing that more detailed information about u^ε can be obtained by combining Remark 1.12 and the form of the reduced ODE (1.9) found in Section 5.2 below. For instance, it is possible to check that u^ε inherits the monotonicity properties (in the axial variable x) of its leading-order approximation in (5.9) or (5.10).

Remark 5.3. Including the cubic term in (5.7) for the body force allows one to treat more general rubber elastic material. In that setting, there exist families of front-type solutions (5.9) when $b_1\lambda_2 < 0$ and $b_2 + 2w_1 > 0$, and homoclinic solutions of the form (5.10) when $b_1\lambda_2 > 0$ and $b_2 + 2w_1 < 0$; see Appendix B.

5.1. **Center manifold reduction.** The linearized operator of (5.4) at $(u,\varepsilon) = (0,0)$ with assumptions (5.3) and (5.6) is simply
$$L := 1 + \Delta \colon \mathscr{X}_\mu \to \mathscr{Y}_\mu,$$
where
$$\mathscr{X}_\mu := \left\{u \in C^{2+\alpha}_\mu(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}, \qquad \mathscr{Y}_\mu := C^{0+\alpha}_\mu(\overline{\Omega}).$$
Here, we are exploiting the fact that the boundary conditions are linear by including them in the definition of $\mathscr{X}_\mu$. The kernel of L is the two-dimensional space
$$\ker L = \{u(x,y) = (A + Bx)\varphi_0(y) : \ A, B \in \mathbb{R}\}.$$
The bounds for the partial Green's function follow exactly from Proposition 2.2. As for the projection $\mathcal{Q}$ onto the kernel in Remark 1.8, we choose it to be
$$\mathcal{Q}u := \big(v(0) + v'(0)x\big)\varphi_0(y) \qquad \text{where } v(x) := u(x,0).$$
Applying Theorem 1.1, we find that all small solutions (u,ε) of (5.8) are of the form
$$u(x,y) = v(0)\varphi_0 + v'(0)x\varphi_0 + \Psi(v(0), v'(0), \varepsilon)(x,y)$$
for a C^{M+1} coordinate map $\Psi \colon \mathbb{R}^3 \to C^{2+\alpha}_\mu$. The function v then satisfies the reduced ODE
$$v'' = f(v, v', \varepsilon), \qquad \text{where} \quad f(A,B,\varepsilon) := \frac{d^2}{dx^2}\bigg|_{x=0} \Psi(A,B,\varepsilon)(x,0). \tag{5.11}$$
From Section 3.1 we see that the reversibility symmetry $u(x,y) \mapsto u(-x,y)$ of (5.8) implies
$$\Psi(A,-B,\varepsilon)(-x,y) = \Psi(A,B,\varepsilon)(x,y), \tag{5.12a}$$

while the additional symmetry $u \mapsto -u$ implies that

$$\Psi(-A,-B,\varepsilon)(x,y) = -\Psi(A,B,\varepsilon)(x,y). \tag{5.12b}$$

Plugging (5.12) into (5.11), we find that f has the symmetries

$$f(A,-B,\varepsilon) = f(A,B,\varepsilon), \qquad f(-A,-B,\varepsilon) = -f(A,B,\varepsilon). \tag{5.13}$$

We now use Theorem 1.6 to expand the coordinate map Ψ and hence the function f. That is, we seek solutions $u \in \mathscr{X}_\mu$ with the Faye–Scheel ansatz

$$u(x,y) = (A+Bx)\varphi_0(y) + \sum_{\mathcal{J}} \Psi_{ijk}(x,y)A^iB^j\varepsilon^k + \mathcal{R}, \tag{5.14}$$

where the index set $\mathcal{J}$ can be determined from the anticipated scaling described in Section 4.2.

In fact from (5.8) we have

$$Lu = b_1\lambda u - 2w_1\nabla\cdot(|\nabla u|^2\nabla u).$$

Expanding f as in (4.5), we see that $f_A(0,0,\lambda) \sim \lambda$. Since $w_1 \neq 0$, then the right-hand side of the above is indeed cubic in u so that $\partial_A^3 f(0,0,\lambda) \sim 1$. Recalling (4.8), this predicts a balancing $\varepsilon^{2n} \sim \varepsilon^p \sim \varepsilon^{2q}$. We therefore take $n = q = 1$ and $p = 2$, which explains the reparametrization $\lambda = \lambda_2\varepsilon^2$ in (5.8), and leads to the index set $\mathcal{J}$ given by

$$\mathcal{J} := \left\{(i,j,k) \in \mathbb{N}^3 : \ i+2j+k \le 3,\ i+j+k \ge 2,\ i+j \ge 1\right\}, \tag{5.15}$$

and the error term $\mathcal{R}$ is of the order $O\left((|A|+|B|^{1/2}+|\varepsilon|)^4\right)$ in $\mathscr{X}_\mu$. This truncation anticipates a scaling where $A \sim \varepsilon,\ B \sim \varepsilon^2$. Recall from Theorem 1.6 that $\Psi_{ijk}(0,0) = \partial_x\Psi_{ijk}(0,0) = 0$.

Plugging (5.14) into the nonlinear term in (5.8), we obtain

$$\nabla\cdot(|\nabla u|^2\nabla u) = -A^3\nabla\cdot\begin{pmatrix}0\\ \sin^3 y\end{pmatrix} + O\left((|A|+|B|^{1/2}+|\varepsilon|)^4\right) \qquad \text{in } \mathscr{X}_\mu.$$

Therefore, for each $(i,j,k) \in \mathcal{J}$, the equation for Ψ_{ijk} is

$$\begin{cases} \displaystyle\sum_{\mathcal{J}} L(\Psi_{ijk})A^iB^j\varepsilon^k = b_1\lambda_2\varepsilon^2(A+Bx)\cos y + b_1\sum_{\mathcal{J}}\Psi_{ijk}A^iB^j\varepsilon^{k+2} + 6w_1A^3\sin^2(y)\cos y \\ \mathcal{Q}\Psi_{ijk} = 0. \end{cases}$$

By Lemma 2.3, the above problem has a unique solution, and indeed we find:

$$\begin{aligned} \Psi_{101} &= \Psi_{011} = \Psi_{110} = \Psi_{200} = 0, \\ \Psi_{102} &= \frac{b_1\lambda_2}{2}x^2\cos y, \\ \Psi_{300} &= \frac{3w_1}{16}\left(4x^2\cos y - \cos y + \cos(3y)\right). \end{aligned}$$

5.2. **Reduced ODE and truncation.** From Theorem 1.1 we know that a small solution u of (5.8) solves the reduced ODE of the form (1.9) where $v(x) = u(x,0)$. Using the computed values of Ψ_{ijk} we see that

$$f(A,B,\varepsilon) = \sum_{\mathcal{J}} \frac{d^2}{dx^2}\Big|_{x=0}\Psi_{ijk}(x,0)A^iB^j\varepsilon^k + r(A,B,\varepsilon) = b_1\lambda_2A\varepsilon^2 + \frac{3w_1}{2}A^3 + r(A,B,\varepsilon)$$

where the error term $r \in C^{M+1}$ and

$$r(A,B,\varepsilon) = O\left(|A|(|A|+|B|^{1/2}+|\varepsilon|)^3 + |B|(|A|+|B|^{1/2}+|\varepsilon|)^2\right).$$

Setting $r = 0$, we obtain the truncated reduced ODE

$$v^0_{xx} = b_1\lambda_2\varepsilon^2v^0 + \frac{3w_1}{2}(v^0)^3.$$

When $b_1\lambda_2 < 0$ and $w_1 > 0$, this has an explicit heteroclinic orbit,

$$v^0(x) = a_1\varepsilon \tanh(\kappa_1\varepsilon x), \quad \text{where} \quad a_1 := \sqrt{\frac{-2b_1\lambda_2}{3w_1}}, \quad \kappa_1 := \sqrt{\frac{-b_1\lambda_2}{2}}.$$

On the other hand, when $b_1\lambda_2 > 0$ and $w_1 < 0$, there is a homoclinic solution

$$v^0(x) = a_1\varepsilon \operatorname{sech}(\kappa_1\varepsilon x), \quad \text{where} \quad a_1 := \sqrt{\frac{-4b_1\lambda_2}{3w_1}}, \quad \kappa_1 := \sqrt{b_1\lambda_2}.$$

5.3. **Proof of existence.** It remains now to confirm that the homoclinic and heteroclinic orbits above persist for the full reduced ODE (that is, when r is reintroduced). For the heteroclinic case, it is often useful to examine invariant quantities. Here, however, the symmetry properties in (5.13) are strong enough that a simpler argument is possible.

Proof of Theorem 5.1. Introducing the scaled variables

$$x = \varepsilon^{-1}X, \qquad v(x) = \varepsilon V(X),$$

the reduced equation (5.11) can be written as the planar system

$$\begin{cases} V_X = W \\ W_X = b_1\lambda_2 V + \dfrac{3w_1}{2}V^3 + R(V,W,\varepsilon), \end{cases}$$

where the rescaled error term $R(V,W,\varepsilon) = O(|\varepsilon|(|V|+|W|))$. At $\varepsilon = 0$, this corresponds to a rescaling of the truncated equation.

Consider the situation in part (a), where $b_1\lambda_2 < 0$, $w_1 > 0$. At $\varepsilon = 0$, the explicit solution $V = a_1 \tanh(\kappa_1 X)$ crosses the W-axis transversely. As usual, this implies that for small nonzero ε, the unstable manifold of the negative equilibrium will transversely intersect the W-axis. Combining the reversibility symmetry $(V(X), W(X)) \mapsto (V(-X), -W(-X))$ with the reflection symmetry $(V(X), W(X)) \mapsto (-V(X), -W(X))$, we obtain existence of a (reversible) heteroclinic orbit connecting the two nontrivial equilibria.

A similar argument works for part (b), where $b_1\lambda_2 > 0$, $w_1 < 0$. When $\varepsilon = 0$, the explicit solution $V = a_1 \operatorname{sech}(\kappa_1 X)$ crosses the V-axis transversely. This intersection persists for small ε, and reversibility then guarantees the existence of a (reversible) homoclinic orbit to the origin. ☐

6. Fronts in 2D Fisher–KPP

As a second application of our general theory, we consider a reaction diffusion equation arising in mathematical biology. The classical Fisher–KPP equation [16, 31] is the one-dimensional problem

$$v_t = v_{xx} + \sigma v(\rho^2 - v), \tag{6.1}$$

where $v = v(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. This models the propagation of an allele within a population; $\sigma > 0$ measures the advantageousness of the mutant gene, while $\rho^2 > 0$ describes the carrying capacity. It is well known that Fisher–KPP supports traveling fronts moving at any wave speed greater than $2\rho\sqrt{\sigma}$. However, it has been observed experimentally by Möbius, Murray, and Nelson [43] that, in the presence of obstacles, invasion fronts may slow down and display two-dimensional characteristics. Recently, Minors and Dawes [42] proposed a two-dimensional version of Fisher–KPP with certain "reactive" boundary conditions as a possible explanation for this phenomenon. For traveling waves, it takes the form

$$\begin{cases} \Delta u + \lambda u_x + u(\rho^2 - u) = 0 & \text{in } \mathbb{R} \times (0,1) \\ u_y = 0 & \text{on } \{y = 0\} \\ u_y + \beta u = 0 & \text{on } \{y = 1\}. \end{cases} \tag{6.2}$$

Here the unknown $u = u(x, y)$, $\beta > 0$ is an absorption constant, λ is the wave speed, and $\rho^2 > 0$ is the carrying capacity of the allele. Note that Minors and Dawes discuss a slightly more general problem. For instance, we scaled the domain to be the infinite strip of unit height $\Omega := \mathbb{R} \times (0, 1)$. Also, they allow Robin or Neumann conditions to be imposed on either boundary.

In [42], numerical evidence is given that the two-dimensional Fisher–KPP equation (6.2) does indeed have fronts that move arbitrarily slowly in certain regimes. As the main contribution of this section, we give the first rigorous proof of the existence of these waves.

Theorem 6.1 (2D Fisher–KPP fronts). *Fix $\beta > 0$, let $\rho_0 > 0$ be the unique solution to $\rho_0 \tan(\rho_0) = \beta$ on $(0, \pi/2)$, and choose a positive constant $\lambda_1 > 2$. There exists $0 < \varepsilon_0 \ll 1$, and a family of front solutions (u, λ, ρ^2) to the two-dimensional Fisher–KPP equation,*

$$\left\{(u, \lambda, \rho^2) = (u^\varepsilon,\, \lambda_1\varepsilon,\, \rho_0^2 + \varepsilon^2) \in C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R} : -\varepsilon_0 < \varepsilon < \varepsilon_0\right\}$$

with

$$u^\varepsilon(x, y) = \varepsilon^2 V^\varepsilon(\varepsilon x)\cos(\rho_0 y) + O(\varepsilon^3) \qquad \text{in } C_{\mathrm{b}}^{2+\alpha}(\overline{\Omega}).$$

Here, V^ε is to leading order a front for the one-dimensional Fisher–KPP equation (6.1) *with carrying capacity $1/\sigma$ and σ given by* (6.9).

6.1. **Center manifold reduction.** The first step is to choose parameters so that the spectral condition (1.7) is satisfied. The eigenvalue problem for the transversal linearized operator at $(u, \lambda) = (0, 0)$ is simply

$$\begin{cases} w_{yy} + \rho^2 w = \nu w & \text{in } (0,1) \\ w_y = 0 & \text{on } \{y = 0\} \\ w_y + \beta w = 0 & \text{on } \{y = 1\}. \end{cases}$$

An elementary calculation shows that there are no eigenvalues $\nu \geq \rho^2$, and $\nu < \rho^2$ is in the spectrum if and only if

$$\tan(\sqrt{\rho^2 - \nu}) = \frac{\beta\sqrt{\rho^2 - \nu}}{\rho^2 - \nu}. \tag{6.3}$$

Taking $\beta > 0$ to be fixed, the critical value for the parameter ρ is defined to be the unique ρ_0 so that the only nonnegative solution of (6.3) is $\nu = 0$. Clearly, this occurs precisely when $\tan(\rho_0) = \beta/\rho_0$, and in that case the kernel is generated by

$$\varphi_0(y) := \cos(\rho_0 y).$$

Now, we reconsider the full problem posed on Ω. As in the previous application, we take advantage of the linearity of the boundary conditions by encoding them directly into the definition of the space: let

$$\mathscr{X} := \left\{u \in C^{2+\alpha}(\overline{\Omega}) : u_y|_{y=0} = 0,\ (\beta u + u_y)\,|_{y=1} = 0\right\}, \qquad \mathscr{Y} := C^{0+\alpha}(\overline{\Omega}).$$

with the exponentially weighted counterparts $\mathscr{X}_\mu$ and $\mathscr{Y}_\mu$, respectively. The linearized operator at $(u, \lambda) = (0, 0)$ is thus

$$L := \Delta + \rho_0^2 \colon \mathscr{X}_\mu \to \mathscr{Y}_\mu, \tag{6.4}$$

and its kernel is the two-dimensional subspace

$$\ker L = \{u(x, y) = (A + Bx)\varphi_0(y) : A, B \in \mathbb{R}\}.$$

We have some freedom to choose a projection $\mathcal{Q}$ onto $\ker L$. As the boundary condition the bottom of the strip is simplest, a reasonable option is to take

$$\mathcal{Q}u := \big(v(0) + v'(0)x\big)\,\varphi_0(y) \qquad \text{where } v(x) := u(x, 0).$$

Applying Theorem 1.1, we infer the existence of a center manifold that must contain any sufficiently small solution to (6.2). To find the corresponding reduced equation, we will use Theorem 1.6 and follow the general procedure outlined in Section 4.

As in Section 5.1, we write the PDE as

$$Lu = -\lambda u_x - (\rho^2 - \rho_0^2)u + u^2,$$

where L is the linearized operator (6.4). From this we do not immediately see a length scale unless we assume certain parameter dependence on $\rho^2-\rho_0^2$. The u_x term imposes a compatibility condition (4.9), which, in the FKPP case, reads $f_A(0,0,\lambda) \sim |\rho^2 - \rho_0^2| \sim \lambda^2$. The quadratic term in the PDE suggests that $m = 2$ and $f_{AA}(0,0,\lambda) \sim 1$. Plugging this in (4.8) we see that $\varepsilon^{2n} \sim \varepsilon^{2p} \sim \varepsilon^q$, and hence one can pick $n = p = 1$ and $q = 2$. This choice corresponds to the reparametrization

$$\lambda = \lambda_1\varepsilon, \quad \rho^2 = \rho_0^2 + \rho_2\varepsilon^2,$$

and the index set $\mathscr{J}$ given by

$$\mathscr{J} := \big\{(i,j,k) \in \mathbb{N}^3 : 2i + 3j + k \le 4,\ i + j + k \ge 2,\ i + j \ge 1\big\}. \tag{6.5}$$

Expressed in the new parameter regime, the PDE becomes

$$Lu = -\rho_2\varepsilon^2 u - \lambda_1\varepsilon u + u^2. \tag{6.6}$$

Seek solutions $u \in \mathscr{X}_\mu$ with the Faye–Scheel ansatz

$$u(x,y) = (A + Bx)\varphi_0(y) + \sum_{\mathscr{J}} \Psi_{ijk}(x,y)A^iB^j\varepsilon^k + \mathcal{R}(x,y), \tag{6.7}$$

where the error term

$$\mathcal{R} = O\Big((|A|^{1/2} + |B|^{1/3} + |\varepsilon|)^5\Big) \qquad \text{in } \mathscr{X}_\mu.$$

Note that, in contrast to the previous section, the truncation condition anticipates an eventual scaling where $A \sim \varepsilon^2$, $B \sim \varepsilon^3$. As in the previous section, computing the coefficients Ψ_{ijk} can be performed according to the general strategy. Substituting (6.7) to (6.6) it follows that

$$\begin{aligned} L\bigg(\sum_{\mathscr{J}} \Psi_{ijk}A^iB^j\varepsilon^k\bigg) &= -\rho_2\varphi_0 A\varepsilon^2 - \lambda_1\varphi_0 B\varepsilon - \rho_2 x\varphi_0 B\varepsilon^2 - \rho_2\sum_{\mathscr{J}} \Psi_{ijk}A^iB^j\varepsilon^{k+2} \\ &\quad - \lambda_1\sum_{\mathscr{J}} \partial_x\Psi_{ijk}A^iB^j\varepsilon^{k+1} + \bigg((A+Bx)\varphi_0 + \sum_{\mathscr{J}} \Psi_{ijk}A^iB^j\varepsilon^k\bigg)^2, \end{aligned}$$

which results in four equations

$$L\Psi_{101} = 0, \qquad L\Psi_{011} = -\lambda_1\varphi_0, \qquad L\Psi_{102} = -\rho_2\varphi_0 - \lambda_1\partial_x\Psi_{101}, \qquad L\Psi_{200} = \varphi_0^2$$

augmented with $\mathcal{Q}\Psi_{ijk} = 0$. The unique solvability of each of these problems is ensured by Lemma 2.3. In particular, one can verify immediately that $\Psi_{101} = 0$ and hence $L\Psi_{102} = -\rho_2\varphi_0$. Therefore

$$\Psi_{011} = -\frac{\lambda_1}{2}x^2\cos(\rho_0 y), \qquad \Psi_{102} = -\frac{\rho_2}{2}x^2\cos(\rho_0 y).$$

Solving for Ψ_{200} is much more complicated. Differentiating the equations for Ψ_{200} with respect to x, we know that $L(\partial_x\Psi_{200}) = 0$. Hence $\partial_x\Psi_{200} = (c_1 + c_2 x)\varphi_0(y)$ for some constants c_1 and c_2. Antidifferentiating, this means that

$$\Psi_{200} = \left(c_1 x + \frac{1}{2}c_2 x^2\right)\cos(\rho_0 y) + g(y),$$

for some function g. The constants c_1, c_2 and the function g will be determined from the projection condition and the PDE. Combining these, we obtain

$$\begin{cases} g'' + \rho_0^2 g + c_2\cos(\rho_0 y) = \cos^2(\rho_0 y) & \text{in } (0,1) \\ g'(0) = g'(1) + \beta g(1) = 0, \quad g(0) = c_1 = 0. \end{cases}$$

This is an elementary ODE that can be solved explicitly, resulting in a somewhat complicated expression for Ψ_{200}. However in the reduced ODE we only need to find

$$\frac{d^2}{dx^2}\Big|_{x=0}\Psi_{200}(x,0) = c_2 = \frac{4\sin(\rho_0)(3-\sin^2(\rho_0))}{3(\sin(2\rho_0)+2\rho_0)}.$$

Finally, this gives the expansion

$$f(A,B,\varepsilon) = -\rho_2 A\varepsilon^2 - \lambda_1 B\varepsilon + \frac{4\sin(\rho_0)(3-\sin^2(\rho_0))}{3(\sin(2\rho_0)+2\rho_0)}A^2 + r(A,B,\varepsilon).$$

6.2. **Reduced ODE and truncation.** Having the coefficients Ψ_{ijk} in hand, we may then apply Theorem 1.1(i) to calculate the reduced ODE. Letting $v := u(\,\cdot\,,0)$, we see it is given by (1.9) with

$$f(A,B,\varepsilon) = \sum_{\mathcal{J}} \frac{d^2}{dx^2}\Big|_{x=0}\Psi_{ijk}(x,0)A^iB^j\varepsilon^k + r(A,B,\varepsilon),$$

where the remainder term $r \in C^{M+1}$ satisfies

$$r(A,B,\varepsilon) = O\left(|A|(|A|^{1/2}+|B|^{1/3}+|\varepsilon|)^3 + |B|(|A|^{1/2}+|B|^{1/3}+|\varepsilon|)^2\right)$$

in some neighborhood of $(0,0,0)$. Inserting the computed values of Ψ_{ijk}, reveals that

$$v'' = \sigma v^2 - \varepsilon^2 v - \lambda_1\varepsilon v' + r(v,v',\varepsilon), \tag{6.8}$$

where

$$\sigma := \frac{4}{3}\frac{\sin(\rho_0)(3-\sin^2(\rho_0))}{2\rho_0+\sin(2\rho_0)} > 0, \tag{6.9}$$

because $\rho_0 \in (0,\pi/2)$. Rearranging (6.8) slightly and truncating the remainder term, this becomes the following one-dimensional Fisher–KPP equation:

$$v^0_{xx} + \lambda_1\varepsilon v^0_x + \sigma v^0\left(\frac{\varepsilon^2}{\sigma} - v^0\right) = 0.$$

6.3. **Proof of existence.** In contrast to the elasticity problem in Section 5, the 2D Fisher–KPP system (6.2) lacks reversibility and reflection symmetry. In their place, we make use of the robustness of the heteroclinic solutions to the 1D Fisher–KPP equation.

Proof of Theorem 6.1. Working in the scaled variables,

$$x = \varepsilon^{-1}X, \qquad v^0(x) = \varepsilon^2 V^0(X),$$

we see that V^0 solves

$$-\lambda_1 V^0_X = V^0_{XX} + \sigma V^0\left(\frac{1}{\sigma} - V^0\right).$$

In the usual way, this can be converted to a first-order planar system

$$\begin{cases} V^0_X = W^0 \\ W^0_X = -\sigma V^0\left(\dfrac{1}{\sigma} - V^0\right) - \lambda_1 W^0, \end{cases} \tag{6.10}$$

which has rest points $(V^0_+, W^0_+) := (0,0)$ and $(V^0_-, W^0_-) := (1/\sigma, 0)$. A quick calculation shows that, for any $\lambda_1 > 2$, (V^0_+, W^0_+) is a sink while (V^0_-, W^0_-) is a saddle. Following the classical argument of Kolmogorov, Petrovsky, and Piskunov [31], one can show that there exists a triangular region

$$\mathscr{T}^0 = \left\{(V,W) \in \mathbb{R}^2 : W < 0,\ W + c_1 V > 0,\ W - c_2(V - V^0_-) > 0\right\},$$

for some explicit $c_1, c_2 > 0$, so that (i) the vector field for (6.10) enters $\mathscr{T}^0$ transversally along each of the boundary components, and (ii) the unstable manifold at (V^0_-, W^0_-) enters $\mathscr{T}^0$ non-tangentially

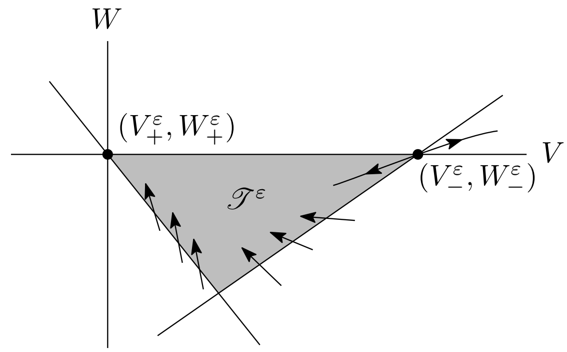

FIGURE 3. The positively invariant triangular region $\mathscr{T}^\varepsilon$

there. As a result, $\mathscr{T}^0$ is positively invariant, and one can conclude that there exists a heteroclinic orbit (V^0, W^0) contained in $\mathscr{T}^0$ and satisfying $V^0(X) \to V_\pm^0$ as $X \to \pm\infty$.

Finally, we must show that this orbit persists for the full reduced equation (6.8). Applying the same rescaling $x \mapsto X$ and $v \mapsto V$ gives the planar system

$$\begin{cases} V_X = W \\ W_X = -\sigma V \left(\dfrac{1}{\sigma} - V\right) - \lambda_1 W + R(V, W, \varepsilon), \end{cases} \tag{6.11}$$

where the remainder term $R(V, W, \varepsilon) = O(\varepsilon(|V| + |W|))$. At $\varepsilon = 0$, this is precisely the truncated problem (6.10). Moreover, for each $\varepsilon \geq 0$ sufficiently small, (6.11) has two rest points, $(V_\pm^\varepsilon, W_\pm^\varepsilon)$, with $(V_+^\varepsilon, W_+^\varepsilon) = (0, 0)$, and $(V_-^\varepsilon, W_-^\varepsilon) = (V_-^0 + O(\varepsilon), 0)$. It follows from the robustness of transversal intersections that there is a positively invariant triangular region $\mathscr{T}^\varepsilon$ for (6.11) that limits to $\mathscr{T}^0$ as $\varepsilon \to 0$; see Figure 3. By the same reasoning as above, we have that $\mathscr{T}^\varepsilon$ contains a heteroclinic orbit $(V^\varepsilon, W^\varepsilon)$ satisfying $V^\varepsilon \to V_\pm^\varepsilon$ as $X \to \pm\infty$. The theorem now follows by undoing the scaling. □

7. ROTATIONAL BORES IN A CHANNEL

Our final application, and our initial motivation for writing this paper, pertains to water waves. Like the anti-plane shear problem in Section 5, it has a reflection symmetry in x, and so we expect to have to expand $f(A, B, \varepsilon)$ to third order in A to obtain fronts. Unlike the anti-plane shear problem, however, there is no additional reflection symmetry in u. Thus the existence and persistence of heteroclinic orbits can no longer be described in terms of a transverse intersection in the plane, and we must instead introduce a second physical parameter. To solve for this auxiliary parameter in terms of ε, we will make heavy use of a conserved quantity called the *flow force* [5]. In particular, we will investigate the so-called *conjugate flow* equations which give a necessary condition for the existence of a front connecting two x-independent solutions [4]. This analysis is quite involved, so much so, in fact, that the expressions for the Taylor coefficients of the coordinate map Ψ in Theorems 1.1 and 1.6 are too large to reproduce here. For this reason we will also highlight several important special cases where the formulas simplify drastically.

7.1. **Statement of the problem.** Working in dimensionless variables, we consider an infinite channel bounded by horizontal walls at $Y = 0$ and $Y = 1$. Inside the channel there is a lower layer of fluid with density normalized to 1, and an upper layer of lighter fluid with density $0 < \rho \leq 1$. There is a sharp interface between the two layers at the height $Y = h + \eta(x)$ where $h \in (0, 1)$ is a reference height to be chosen later. See Figure 4 for an illustration.

This physical setting is sometimes called channel flow. It is widely used as a model in geophysics, for example, where the dynamics in the upper atmosphere are not expected to have much relevance for the motion of the interface. It is interesting to note, however, that if we instead allow the

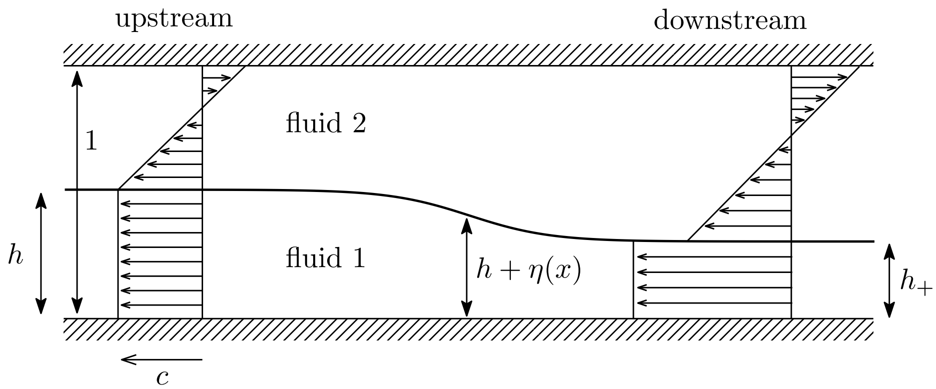

FIGURE 4. The class of bores under consideration. There are superposed fluid layers bounded by rigid plates at $Y = 0$ and $Y = 1$. The upper layer has constant density ρ and constant vorticity ω, while the bottom layer has unit density and zero vorticity. In the "upstream limit" $x \to -\infty$, the lower layer has thickness h, while in the downstream limit this thickness is h_+. At intermediate values of x, the layers are separated by a sharp interface a height $Y = h + \eta(x)$. In the moving frame, the upstream velocity in the lower layer is $-c$. Finally, the upstream velocity is continuous across the interface, but the downstream velocity may not be.

upper boundary to be free, then monotone bores of the type we construct here do not exist; see [7, Corollary 4.12]. A survey of the literature on both channel flow and the two free boundary case can be found in [21, Section 7], for example.

Suppose that there is no surface tension along the interface and hence that the pressure is continuous across it. For water, it is reasonable to assume that the particle velocity field is incompressible (that is, divergence free) in each fluid region. Thus there are so-called *stream functions*, ψ_1 in the lower fluid and ψ_2 in the upper fluid, so that the velocity field in the i-th fluid is $\nabla^\perp \psi_i := (-\partial_Y \psi_i, \partial_x \psi_i)$. Lastly, we suppose that the curl of the velocity field is some constant $\omega \in \mathbb{R}$ in the upper layer, but 0 in the lower layer. Standard arguments involving Bernoulli's law then lead to the following free boundary problem for the functions ψ_1, ψ_2, η:

$$
\begin{aligned}
-\Delta_{x,Y}\psi_1 &= 0 && \text{for } 0 < Y < h+\eta, && \text{(7.1a)}\\
-\Delta_{x,Y}\psi_2 &= \omega && \text{for } h+\eta < Y < 1, && \text{(7.1b)}\\
\psi_2 &= -m_2 && \text{on } Y = 1, && \text{(7.1c)}\\
\psi_1 = \psi_2 &= 0 && \text{on } Y = h+\eta, && \text{(7.1d)}\\
\psi_1 &= m_1 && \text{on } Y = 0, && \text{(7.1e)}\\
\tfrac{1}{2}\rho|\nabla_{x,Y}\psi_2|^2 - \tfrac{1}{2}|\nabla_{x,Y}\psi_1|^2 + (\rho-1)\eta &= Q && \text{on } Y = h+\eta. && \text{(7.1f)}
\end{aligned}
$$

The boundary conditions (7.1c)–(7.1d) are called *kinematic boundary conditions*, while (7.1f) is called the *dynamic boundary condition*. The constants m_1, m_2 are the mass fluxes in each layer, while Q is a Bernoulli constant. We will always consider classical solutions where the functions ψ_1, ψ_2, η are all $C^{2+\alpha}_{\mathrm{b}}$ on (the closures of) their respective domains.

While our methods can also be used to construct solitary wave solutions of (7.1), we will focus on the much more difficult case of fronts, sometimes called *bores* in the literature. That is, we will seek solutions where ψ_1, ψ_2, η have well-defined limits as $x \to -\infty$ ("upstream") and as $x \to +\infty$ ("downstream") that do not coincide. For simplicity, and because this is the case of most interest in applications, we assume that the velocity in the upstream state is continuous. The upstream limit is then uniquely determined by requiring

$$\psi_{1Y}(x,0),\, \psi_{2Y}(x,0) \to -c, \quad \eta(x) \to 0 \qquad \text{as } x \to -\infty. \tag{7.1g}$$

Here the Froude number c is a dimensionless wavespeed as measured in a reference frame where the fluid particles on the bed are stationary in the upstream limit; this is in keeping with typical conventions for periodic and solitary waves without vorticity. The second requirement that $\eta \to 0$ as $x \to -\infty$ means that h is the upstream thickness of the lower fluid region.

Throughout this section we will view ρ, ω as fixed and treat c, h as parameters. This is in part motivated by the fact that ρ and ω are both constants of motion for the time-dependent problem.

7.2. **Main results.** Our main existence result is informally described in Theorem 7.1 below. A crucial part of the proof is an understanding of the so-called *conjugate flow equations* which constrain the upstream and downstream depths h, h_+ of the lower layer and the Froude number c. To streamline the presentation, we defer a detailed discussion of these equations to Section 7.3 below. There, we also prove Lemma 7.7, which gives sufficient conditions for the conjugate flow equations to be locally solvable for c and h_+ in terms of h.

Theorem 7.1 (Existence of rotational bores)**.** *Consider the water wave problem* (7.1) *with fixed density ratio* $0 < \rho \leq 1$ *and (constant) vorticity* ω*, and suppose that the height* $h_0 \in (0,1)$ *and Froude number* c_0 *satisfy the hypotheses* (7.12) *of Lemma 7.7 as well as* (7.19) *below. Then, for* $0 < |\varepsilon| \ll 1$*, there is a family of bore-type solutions of* (7.1) *with upstream depths* $h^\varepsilon = h_0 + \varepsilon$*, Froude numbers* $c^\varepsilon = c_0 + O(\varepsilon)$*, and*

$$\begin{aligned} \eta^\varepsilon(x) &= a_1 \varepsilon \frac{1 + \tanh(\kappa_1 |\varepsilon| x)}{2} + O(\varepsilon^2), \\ \psi_1^\varepsilon(x,Y) &= -c^\varepsilon (Y - h^\varepsilon) + c^\varepsilon \eta^\varepsilon(x)(1-Y) + O(\varepsilon^2), \\ \psi_2^\varepsilon(x,Y) &= -c^\varepsilon (Y - h^\varepsilon) - \tfrac{1}{2}\omega (Y - h^\varepsilon)^2 + c^\varepsilon \eta^\varepsilon(x)(1+Y) + O(\varepsilon^2), \end{aligned} \tag{7.2}$$

in $C_{\mathrm{b}}^{2+\alpha}$ *of their respective domains, for some constants* $a_1 \neq 0$ *and* $\kappa_1 > 0$*.*

Remark 7.2. The characterization of η^ε as a solution of a second-order ODE actually furnishes much more detailed information. In particular, we can check that η^ε inherits the strict monotonicity properties of its leading order approximation. Combining this with a maximum principle argument yields monotonicity of the full solutions; see Theorem 7.8.

The various assumptions in Theorem 7.1, as well as the explicit formulas for the parameters a_1, κ_1 in (7.2), can all be stated explicitly in terms of h_0, c_0, ρ, ω. Sadly, the formulas are quite lengthy, and so it is perhaps more instructive to look at special cases. The most classical and well-studied of these is the irrotational regime where $\omega = 0$.

Corollary 7.3 (Irrotational bores)**.** *The hypotheses of Theorem 7.1 are satisfied if we set*

$$\omega = 0, \qquad h_0 = \frac{1}{1+\sqrt{\rho}}, \qquad c_0 = \pm \frac{\sqrt{1-\rho}}{1+\sqrt{\rho}}.$$

The relevant family of conjugate flows $(h^\varepsilon, h_+^\varepsilon, c^\varepsilon)$ *and constants* a_1, κ_1 *in* (7.2) *are given by*

$$c^\varepsilon = c_0, \qquad h_+^\varepsilon = h_0, \qquad a_1 = -1, \qquad \kappa_1^2 = \frac{3(\sqrt{\rho}+1)^4}{4\sqrt{\rho}(\rho - \sqrt{\rho} + 1)}.$$

This is the case treated by Mielke [41]. Notice that, in particular, the solutions $(h^\varepsilon, h_+^\varepsilon, c^\varepsilon)$ have *exact* formulas and that h_+^ε and c^ε are actually constants [34]. This simplifies the analysis enourmously.

When $\omega \neq 0$, interesting new phenomena can occur. For example, the upper fluid may contain *critical layers*, curves along which $\psi_{2Y} = 0$. In the setting of Theorem 7.1, there will always be such a critical layer provided c_0, h_0, and ω satisfy the inequality

$$c_0(c_0 + (1 - h_0)\omega) < 0. \tag{7.3}$$

The upstream height of the critical layer is then $h^\varepsilon - c^\varepsilon/\omega$. Perhaps the simplest situation where this arises is when $\rho = 1$ so that the fluid density is homogeneous.

Corollary 7.4 (Homogeneous-density bores). *The hypotheses of Theorem 7.1 are satisfied if we set*

$$\rho = 1, \qquad h_0 = \frac{2}{3} \qquad c_0 = -\frac{2\omega}{9} \neq 0.$$

The relevant family of conjugate flows $(h^\varepsilon, h_+^\varepsilon, c^\varepsilon)$ *and constants* a_1, κ_1 *in* (7.2) *are given by*

$$c^\varepsilon = c_0 + \frac{\omega}{3}\varepsilon, \qquad h_+^\varepsilon = h_0 - \varepsilon, \qquad a_1 = -2 \qquad \kappa_1^2 = \frac{243}{16}.$$

In particular, there is an upstream critical layer at height $8/9 + 2\varepsilon/3$.

As with the irrotational case, we can solve the conjugate flow equations explicitly, this time with $h_+^\varepsilon, c^\varepsilon$ both linear functions of ε.

For general but fixed $\rho < 1$ and $\omega \neq 0$, even the necessary condition (7.12a) for h_0, c_0 cannot be solved explicitly, let alone the full conjugate flow equations (7.11) for h, h_+, c, and a comprehensive analysis of these systems of polynomial equations is beyond the scope of the present paper. On the other hand, for fixed h_0, c_0, one can solve (7.12a) for ρ and ω. The resulting formulas are long and not particularly informative, and one must of course additionally check that ρ lies in the physical range $(0, 1]$. Provided the remaining nondegeneracy hypothesis (7.12b) in Lemma 7.7 holds, one then obtains the existence of a family of conjugate flows which can be expanded in the small parameter ε. We content ourselves with two concrete examples obtained in this way, whose parameter values were carefully chosen so as to avoid a profusion of nested radicals.

Example 7.5 (An example without critical layers). The hypotheses of Theorem 7.1 are satisfied if we set

$$\rho = \frac{25}{52}, \qquad \omega = -\frac{9}{10}, \qquad h_0 = \frac{2}{3}, \qquad c_0 = \frac{1}{2}.$$

The relevant family of conjugate flows $(h^\varepsilon, h_+^\varepsilon, c^\varepsilon)$ and constants a_1, κ_1 in (7.2) are given by

$$c^\varepsilon = c_0 - \frac{6}{29}\varepsilon + O(\varepsilon^2), \qquad h_+^\varepsilon = h_0 - \frac{179}{725}\varepsilon + O(\varepsilon^2), \qquad a_1 = -\frac{904}{725}, \qquad \kappa_1^2 = \left(\frac{226}{145}\right)^2 \frac{243}{43}.$$

None of these solutions have critical layers.

Example 7.6 (An example with critical layers). The hypotheses of Theorem 7.1 are satisfied if we set

$$\rho = \frac{1}{28}, \qquad \omega = -18, \qquad h_0 = \frac{2}{3}, \qquad c_0 = 1.$$

The relevant family of conjugate flows $(h^\varepsilon, h_+^\varepsilon, c^\varepsilon)$ and constants a_1, κ_1 in (7.2) are given by

$$c^\varepsilon = c_0 + \frac{3}{4}\varepsilon + O(\varepsilon^2), \qquad h_+^\varepsilon = h_0 + \frac{11}{10}\varepsilon + O(\varepsilon^2), \qquad a_1 = \frac{1}{10}, \qquad \kappa_1^2 = \frac{243}{3040}.$$

In particular, there is a critical layer upstream at height $13/18 + (25/24)\varepsilon + O(\varepsilon^2)$.

7.3. **Conjugate flows.** This subsection is devoted to the statement and proof of Lemma 7.7 on the existence of conjugate flows. Interesting in its own right, it is also one of main tools in proving Theorem 7.1.

Upstream limit and downstream limits. Under mild regularity assumptions, the existence of the downstream and upstream limits

$$\psi_1^\pm(Y) := \lim_{x\to\pm\infty} \psi_1(x,Y), \qquad \psi_2^\pm(Y) := \lim_{x\to\pm\infty} \psi_2(x,Y), \qquad \eta^\pm := \lim_{x\to\pm\infty} \eta(x)$$

forces $(\psi_1^\pm, \psi_2^\pm, \eta^\pm)$ to each be x-independent solutions of (7.1). In particular, $\psi_1^\pm$ are linear in Y while $\psi_2^\pm$ are quadratic. We will generally eliminate $\eta^\pm$ in favor of the upstream thickness h and downstream thickness $h_+ := h + \eta^+$ of the lower fluid.

Upstream, we have the additional restrictions (7.1g), as well as the continuity assumption $\psi_{1Y}^-(h) = \psi_{2Y}^-(h)$ at the interface. Thus the upstream state is completely determined by the parameters c, h, ω:

$$\psi_1^- = -c(Y-h), \qquad \psi_2^- = -c(Y-h) - \tfrac{1}{2}\omega(Y-h)^2. \tag{7.4}$$

Sending $x \to -\infty$ in (7.1) we recover similarly explicit formulas for the fluxes m_1, m_2 and Bernoulli constant Q:

$$m_1 = ch, \qquad m_2 = c(1-h) + \tfrac{1}{2}\omega(1-h)^2, \qquad Q = (\rho-1)\tfrac{1}{2}c^2. \tag{7.5}$$

Now we turn to the downstream limit. In general, we cannot require it to also have a continuous velocity field, and hence the two constants

$$c_1^+ := \psi_{1y}^+(h_+), \qquad c_2^+ := \psi_{2y}^+(h_+)$$

may differ. In terms of c_1^+ and c_2^+, the analogues of (7.4) and (7.5) are

$$\begin{aligned} \psi_1^+ &= -c_1^+(Y-h_+), \qquad \psi_2^+ = -c_2^+(Y-h_+) - \tfrac{1}{2}\omega(Y-h_+)^2, \qquad m_1 = c_1^+ h, \\ m_2 &= c_2^+(1-h_+) + \tfrac{1}{2}\omega(1-h_+)^2, \qquad Q = \tfrac{1}{2}\big(\rho(c_2^+)^2 - (c_1^+)^2\big) + (\rho-1)(h_+ - h). \end{aligned} \tag{7.6}$$

Eliminating m_1 and m_2 between (7.5) and (7.4), we can easily solve for c_1^+ and c_2^+ in terms of the other parameters. Further eliminating Q we obtain a single equation relating the remaining parameters h, h_+, c, ρ, ω. Eventually, this equation simplifies to

$$\frac{h_+ - h}{(1-h_+)^2 h_+^2} p(h, h_+, c) = 0, \tag{7.7}$$

where $p = p(h, h_+, c)$ is a polynomial its arguments (as well as ρ, ω),

$$\begin{aligned} p := \omega^2 h_+^2 (h_+ - h)(2 - h_+ - h)^2 \rho + 4h_+^2(2h_+^2 - c^2 h_+ - 4h_+ - c^2 h + 2c^2 + 2)\rho \\ + 4c\omega(1-h)h_+^2(2 - h_+ - h)\rho - 4(1-h_+)^2(2h_+^2 - c^2 h_+ - c^2 h). \end{aligned} \tag{7.8}$$

Since we are only interested in configurations where $h_+ \neq h$ and neither h nor h_+ is 0 or 1, (7.7) reduces to the polynomial equation $p(h, h_+, c) = 0$.

Flow force. To obtain a second constraint on the parameters h, h_+, c, ω, ρ, we introduce a quantity called the *flow force*, which is related to the conservation of momentum [5]. In our variables, it takes the form

$$\begin{aligned} \mathcal{S}(x) := &\int_0^{h+\eta} \left(\frac{1}{2}(\psi_{1Y}^2 - \psi_{1X}^2) - Y + \frac{1}{2}c^2 + h \right) dY \\ &+ \rho \int_{h+\eta}^1 \left(\frac{1}{2}(\psi_{2Y}^2 - \psi_{2X}^2) - Y - \omega\psi_2 + \frac{1}{2}c^2 + h \right) dY. \end{aligned} \tag{7.9}$$

For solutions of (7.1), one can check that this quantity is independent of x. In particular, sending $x \to \pm\infty$ and simplifying we eventually obtain the polynomial equation

$$\begin{aligned} 0 = \tilde{q}(h, h_+, c) := \omega^2 h_+(h_+ - h)(h_+ + 3h - 4)\rho + 12h_+(h_+ - c^2 - 1)\rho \\ + 12c\omega(h-1)h_+\rho - 12(h_+ - 1)(h_+ - c^2). \end{aligned}$$

Here as above we have used our assumptions that $h_+ \neq h$ and $h, h_+ \neq 0, 1$ to drop some nonzero factors.

Constructing conjugate flows. The equations $p = \tilde{q} = 0$ are called the *conjugate flow equations* for our problem [4]. Because of a degeneracy in this system when $h_+ = h$, it will be easier to work with an equivalent system where the polynomial $\tilde{q}$ is replaced by

$$q(h, h_+, c) := \frac{2(h-1)h}{h_+ - h}\left(\tilde{q}(h, h_+, c) - \frac{\tilde{q}(h, h, c)}{p(h, h, c)} p(h, h_+, c)\right), \tag{7.10}$$

which one can verify is also a polynomial in its arguments (as well as ω, ρ). We denote this "desingularized" set of conjugate flow equations by

$$\mathcal{P}(h, h_+, c) := \big(p(h, h_+, c), q(h, h_+, c)\big) = 0, \tag{7.11}$$

where p and q are defined in (7.8) and (7.10) above.

Using the implicit function theorem, it is now straightforward to give conditions guaranteeing the existence of a one-parameter families of conjugate flows, that is, solutions (h, h_+, c) of (7.11). We record one such result in the following lemma.

Lemma 7.7 (Existence of conjugate flows)**.** *For a fixed density ρ and vorticity ω, suppose that the depth $h_0 \in (0, 1)$ and Froude number $c_0 \neq 0$ satisfy*

$$\mathcal{P}(h_0, h_0, c_0) = 0 \tag{7.12a}$$

as well as the nondegeneracy conditions

$$\det \mathcal{P}_{(c,h_+)}(h_0, h_0, c_0) \neq 0, \quad \det \big(\mathcal{P}_h + \mathcal{P}_{h_+},\ \mathcal{P}_c\big)(h_0, h_0, c_0) \neq 0. \tag{7.12b}$$

Then there exists a family of solutions $\{(h^\varepsilon, h_+^\varepsilon, c^\varepsilon)\}$ to the conjugate flow equations (7.11) for $|\varepsilon| < \varepsilon_0 \ll 1$ that depends analytically on ε and satisfies

$$\begin{aligned} h^\varepsilon &= h_0 + \varepsilon, \\ h_+^\varepsilon &= h_0 + h_{+,1}\,\varepsilon + h_{+,2}\,\varepsilon^2 + O(\varepsilon^3), \\ c^\varepsilon &= c_0 + c_1\varepsilon + c_2\varepsilon^2 + O(\varepsilon^3). \end{aligned}$$

Moreover, $h_{+,1} \neq 1$ so that, perhaps after shrinking ε_0, $h^\varepsilon \neq h_+^\varepsilon$ for $\varepsilon \neq 0$. Thus these conjugate flows are nontrivial in that the upstream and downstream states are distinct.

7.4. **Reformulating the problem.** In this subsection we reformulate (7.1) as the elliptic transmission problem (7.16) in a *fixed* domain. From now on we assume that the hypotheses of Lemma 7.7 are satisfied so that $h^\varepsilon, h_+^\varepsilon, c^\varepsilon$ are all defined.

Flattening the interface. Our problem (7.1) is a free boundary problem in that the interface $Y = h^\varepsilon + \eta$ between the two regions is itself an unknown. As usual, it is helpful to switch to new coordinates where this boundary is fixed. In the absence of critical layers, one can use an elegant partial hodograph transformation in which ψ_1, ψ_2 are thought of as independent variables and Y the dependent variable [13]. We are interested in bores with critical layers, and therefore must allow for ψ_2 to be a multivalued function of Y. This leads us to instead make a simple piecewise-linear change of coordinates in the vertical variable Y:

$$y := \begin{cases} -1 + \dfrac{1}{h^\varepsilon + \eta} Y & \text{for } 0 < Y < h^\varepsilon + \eta \\[2ex] 1 - \dfrac{1}{1 - h^\varepsilon - \eta} + \dfrac{1}{1 - h^\varepsilon - \eta} Y & \text{for } h^\varepsilon + \eta < Y < 1. \end{cases} \tag{7.13}$$

Thus the lower layer $0 < Y < h^\varepsilon + \eta$ is mapped onto the strip $-1 < y < 0$ while the upper layer $h^\varepsilon + \eta < Y < 1$ is mapped onto the strip $0 < y < 1$. Using subscripts Y_1, Y_2, y_1, y_2 to denote the vertical variables in the two layers, we have the chain rules

$$\begin{aligned} \partial_{x_1} &= \partial_x - \frac{1+y}{h^\varepsilon + \eta}\eta_x \partial_{y_1}, & \partial_{x_2} &= \partial_x - \frac{1-y}{1-h^\varepsilon - \eta}\eta_x \partial_{y_2}, \\ \partial_{Y_1} &= \frac{1}{h^\varepsilon + \eta}\partial_{y_1}, & \partial_{Y_2} &= \frac{1}{1-h^\varepsilon + \eta}\partial_{y_2}, \end{aligned} \tag{7.14}$$

where the partials on the left hand side are with respect to the original (x, Y) variables and those on the right are with respect to the transformed (x, y) variables.

Subtracting off the trivial solution. The upstream flow itself obviously solves (7.1), and so we work with normalized differences u_1, u_2 between our stream functions and these “trivial” ones:

$$u_1(x,y) := \frac{\psi_1(x,Y) - \psi_1^-(Y)}{c^\varepsilon}, \qquad u_2(x,y) := \frac{\psi_2(x,Y) - \psi_2^-(Y)}{c^\varepsilon}. \tag{7.15}$$

Note that the ψ_i^- terms on the right hand side of (7.15) are functions of the original variable Y and not the transformed variable y. It is straightforward to obtain the corresponding functions of y by first solving (7.13) for Y and then plugging into the explicit formulas (7.4). Neither this choice nor the normalizing factor of c^ε are essential, but both are convenient in later calculations.

Final form of the equations. We now plug (7.15) into (7.1) and use (7.14) to obtain a system of the form (3.5) for $u := (u_1, u_2)$ alone. We use one of the kinematic boundary conditions, (7.1d), in order to write η as the trace of u_1,

$$\eta(x) = u_1(x, 0),$$

thus eliminating it from the problem. Abusing notation slightly, we will nevertheless continue to write η instead of $u_1|_\Gamma$ whenever convenient. The transformed problem is then

$$\begin{aligned} \nabla \cdot \mathcal{A}_1(y, u_1, \nabla u_1, u_1|_\Gamma, u_{1x}|_\Gamma, \varepsilon) &= 0 && \text{in } \Omega_1 := \mathbb{R} \times (-1, 0), && (7.16a)\\ \nabla \cdot \mathcal{A}_2(y, u_2, \nabla u_2, u_1|_\Gamma, u_{1x}|_\Gamma, \varepsilon) &= 0 && \text{in } \Omega_2 := \mathbb{R} \times (0, 1), && (7.16b)\\ \mathcal{G}(u_1, u_2, \nabla u_1, \nabla u_2, \varepsilon) &= 0 && \text{on } \Gamma := \mathbb{R} \times \{0\}, && (7.16c)\\ \mathcal{K}(u_1, u_2, \varepsilon) &= 0 && \text{on } \Gamma, && (7.16d)\\ u_1 &= 0 && \text{on } \mathbb{R} \times \{-1\}, && (7.16e)\\ u_2 &= 0 && \text{on } \mathbb{R} \times \{1\}, && (7.16f) \end{aligned}$$

where the functions $\mathcal{A}, \mathcal{G}, \mathcal{K}$ are given by

$$\begin{aligned} \mathcal{A}_1(y, u_1, \nabla u_1, u_1|_\Gamma, u_{1x}|_\Gamma, \varepsilon) &:= \begin{pmatrix} (h^\varepsilon + \eta)u_{1x} - (y+1)\eta_x u_{1y} \\ (h^\varepsilon + \eta)^{-1}((1+y)^2\eta_x^2 + 1)u_{1y} - \eta_x(y+1)u_{1x} \end{pmatrix} \\ \mathcal{A}_2(y, u_2, \nabla u_2, u_1|_\Gamma, u_{1x}|_\Gamma, \varepsilon) &:= \begin{pmatrix} (1-h^\varepsilon - \eta)u_{2x} - (1-y)\eta_x u_{2y} \\ (1-h^\varepsilon-\eta)^{-1}((1+y)^2\eta_x^2 + 1)u_{2y} - (1-y)\eta_x u_{2x} \end{pmatrix} \\ \mathcal{K}(y, u_1, u_2, \varepsilon) &:= u_2 - u_1 - \frac{\omega}{2c^\varepsilon}u_1^2 \\ \mathcal{G}(u_1, u_2, \nabla u_1, \nabla u_2, \varepsilon) &:= \frac{\rho}{2}\left(u_{2x}^2 - \frac{2\eta_x u_{2x} u_{2y}}{1-h^\varepsilon-\eta} + \frac{(1+\eta_x^2)u_{2y}^2}{(1-h^\varepsilon-\eta)^2} - \frac{2(c^\varepsilon + \omega\eta)u_{2y}}{1-h^\varepsilon-\eta} \right) \\ &\quad - \frac{1}{2}\left(u_{1x}^2 - \frac{2\eta_x u_{1x}u_{1y}}{h^\varepsilon+\eta} + \frac{(1+\eta_x^2)u_{1y}^2}{(h^\varepsilon+\eta)^2} - \frac{2c^\varepsilon u_{1y}}{h^\varepsilon+\eta} \right) \\ &\quad + \frac{c^\varepsilon\omega\rho + \rho - 1}{(c^\varepsilon)^2}\eta + \frac{\omega^2\rho}{2(c^\varepsilon)^2}\eta^2. \end{aligned} \tag{7.17}$$

We write (7.16) as $\mathscr{F}(u_1, u_2, \varepsilon) = 0$ where $\mathscr{F}\colon \mathscr{X} \times \mathbb{R} \to \mathscr{Y}$ with $\mathscr{X}, \mathscr{Y}$ defined in (3.7) and (3.8).

7.5. **Center manifold reduction.** The linearized operator at $(u, \varepsilon) = (0, 0)$ is

$$Lu = \begin{pmatrix} \nabla \cdot \Big(h_0 u_{1x}, h_0^{-1} u_{1y}\Big) \\ \nabla \cdot \Big((1-h_0)u_{2x}, (1-h_0)^{-1} u_{2y}\Big) \\ h_0^{-1} u_{1y} - \rho(1-h_0)^{-1} u_{2y} + c_0^{-2}(c_0\omega\rho + \rho - 1)u_1 \\ u_2 - u_1 \end{pmatrix}, \qquad L\colon \mathscr{X}_\mu \to \mathscr{Y}_\mu$$

which has the desired form (3.6). Moreover, straightforward calculations using the assumption $p(h_0, h_0, c_0) = 0$ in Lemma 7.7 show that the spectral hypothesis (1.7) is satisfied, and that

$$\ker L = \{u(x, y) = (A + Bx)\varphi_0(y) : A, B \in \mathbb{R}\},$$

where

$$\varphi_0(y) := \begin{cases} 1 + y & -1 \le y \le 0 \\ 1 - y & 0 \le y \le 1. \end{cases}$$

For the projection $\mathcal{Q}$ we choose

$$\mathcal{Q}u := (v(0) + v'(0)x)\varphi_0(y), \qquad \text{where } v(x) := u_1(x, 0) = \eta(x).$$

Applying Corollary 3.1, we obtain that all small solutions $(u, \varepsilon) \in \mathscr{X}_{\mathrm{b}} \times \mathbb{R}$ of (7.16) are of the form

$$u(x, y) = v(0)\varphi_0 + v'(0)x\varphi_0 + \Psi(v(0), v'(0), \varepsilon)(x, y)$$

for a C^M coordinate map $\Psi\colon \mathbb{R}^3 \to \mathscr{X}_\mu$. In this case the function v satisfies the reduced ODE

$$v'' = f(v, v', \varepsilon), \qquad \text{where} \quad f(A, B, \varepsilon) := \frac{d^2}{dx^2}\Big|_{x=0} \Psi(A, B, \varepsilon)(x, 0). \tag{7.18}$$

Remarkably, this is an ODE for the free surface elevation η alone. From the analysis in Section 3.1 — specifically (3.2) and (3.3) — we see that the reversibility symmetry $u(x, y) \mapsto u(-x, y)$ of (7.16) implies that $\Psi(A, -B, \varepsilon)(x, y) = \Psi(A, B, \varepsilon)(-x, y)$ and hence that f is even in B.

Corollary 3.1 moreover allows to expand Ψ (and thereby f) as in Theorem 1.6, leading to the ansatz

$$u(x, y) = (A + Bx)\varphi_0(y) + \sum_{\mathcal{J}} \Psi_{ijk}(x, y) A^i B^j \varepsilon^k + \mathcal{R}.$$

Anticipating a scaling where $A \sim \varepsilon$ and $B \sim \varepsilon^2$, we work with the index set

$$\mathcal{J} := \left\{(i, j, k) \in \mathbb{N}^3 : i + 2j + k \le 3,\ i + j + k \ge 2,\ i + j \ge 1\right\},$$

so that $\mathcal{R}$ is $O\big((|A| + |B|^{1/2} + |\varepsilon|)^4\big)$ in $\mathscr{X}_\mu$.

In principle, it is now straightforward to expand (7.16) and find the relevant Ψ_{ijk} by collecting like terms and solving a sequence of linear equations of the form (4.4). In practice, however, these calculations are extremely tedious, partly due to the unwieldy form of the water wave problem (7.17) but more seriously because of the lengthy expressions for the coefficients c_1, c_2 in the expansion of c^ε in Lemma 7.7. Lastly, in order to check if complicated rational functions of h_0, c_0 are in fact zero, we must appeal to the highly nonlinear system of polynomial conjugate-flow equations $\mathcal{P}(h_0, h_0, c_0) = 0$. We accomplished this latter task by transforming $\mathcal{P}$ into a Gröbner basis and performing reductions using a computer algebra system. In certain situations, for instance the irrotational regime treated in Corollary 7.3 and the homogeneous-density case considered in Corollary 7.4, the conjugate flow equations have simple exact solutions, and so the analysis is substantially easier.

7.6. **Reduced ODE and truncation.** In the general case, we eventually find that $f(A,B,\varepsilon)$ has the form

$$\begin{aligned} f(A,B,\varepsilon) &= \sum_{\mathcal{J}} \frac{d^2}{dx^2}\Big|_{x=0} \Psi_{ijk}(x,0) A^i B^j \varepsilon^k + r(A,B,\varepsilon) \\ &= f_{102}\varepsilon^2 A + f_{201}\varepsilon A^2 + f_{300}A^3 + r(A,B,\varepsilon) \end{aligned}$$

where the error term $r \in C^M$ and

$$r(A,B,\varepsilon) = O\left(|A|(|A|+|B|^{1/2}+|\varepsilon|)^3 + |B|(|A|+|B|^{1/2}+|\varepsilon|)^2\right).$$

The coefficients are given by

$$\begin{aligned} f_{300} &= \frac{3}{2}\frac{(1-\rho)h_0^3 + c_0^2(4-5h_0)}{c_0^2 h_0^3 (1-h_0)^2(\rho+(1-\rho)h_0)}, \\ f_{201} &= \frac{9}{2}\Big(c_0^2\big(1-h_0-2\rho+h_0^3(1-\rho)^2+h_0^2(4\rho-1)\big) - (1-h_0)^2(3h_0^2-3h_0+2)(1-\rho)\Big) \\ &\qquad \cdot \Big(c_0h_0(1-h_0)^2(\rho+(1-\rho)h_0)\big(c_0(c_0^2h_0^2+(1-h_0)(c_0^2h_0+2h_0-3c_0^2)) \\ &\qquad\qquad - \omega(1-h_0)^2h_0(h_0-c_0^2)\big)\Big)^{-1}, \\ f_{102} &= \frac{2f_{201}^2}{9f_{300}}. \end{aligned}$$

Using the assumptions (7.12) of Lemma 7.7, one can show that none of the above denominators vanish, and that $f_{300}, f_{201}, f_{102}$ are all nonzero. We additionally *assume* that $f_{300} > 0$, which is equivalent to requiring that

$$h_0^3(1-\rho) + 4c_0^2(1-h_0) > c_0^2 h_0. \tag{7.19}$$

The truncated version of (7.18) is then

$$v^0_{xx} = f_{102}\varepsilon^2 v^0 + f_{201}\varepsilon(v^0)^2 + f_{300}(v^0)^3, \tag{7.20}$$

which has the explicit solution

$$v^0(x) = a_1\varepsilon\frac{1+\tanh(\kappa_1\varepsilon x)}{2}, \tag{7.21}$$

where

$$a_1 = -\frac{f_{201}}{3f_{300}}, \qquad \kappa_1^2 = \frac{f_{201}^2}{18f_{300}}.$$

We note that (7.20) is the extended Korteweg–de Vries equation, more widely known as the Gardner equation, specialized to traveling waves. This is a common model for long internal waves [23].

7.7. **Flow force on the center manifold.** Arguing as in Section 5, we can show that many features of the phase portrait of the truncated ODE (7.20) persist in the full equation (7.18). In particular, there are three equilibria: saddles at 0 and $a_1\varepsilon + O(\varepsilon^2)$ and a center in between. Unfortunately, this is not enough information for the persistence of the heteroclinic orbit connecting the two saddles. For this we take advantage of the flow force $\mathcal{S}$ defined in (7.9).

Performing the various changes of variable, we can think of the flow force at a fixed x as a functional of (u,ε): $\mathcal{S} = \mathcal{S}(u,\varepsilon;x)$. Subtracting off its (constant) value at the trivial solution $u=0$ and setting $x=0$, we consider the difference

$$\tilde{\mathcal{S}}(u,\varepsilon) = \mathcal{S}(u,\varepsilon;0) - \mathcal{S}(0,\varepsilon;0).$$

Since $\tilde{\mathcal{S}}$ only involves the values of u and ∇u at $x = 0$, it is a smooth function both $\mathscr{X}_{\mathrm{b}} \times \mathbb{R} \to \mathbb{R}$ and $\mathscr{X}_\mu \times \mathbb{R} \to \mathbb{R}$. We record the useful formula

$$\tilde{\mathcal{S}}_u(0,\varepsilon)\dot{u} = \rho(c^\varepsilon)^2(\dot{u}_2 - \dot{u}_1)(0,0) \tag{7.22}$$

for its Fréchet derivative at $u = 0$.

When (u,ε) corresponds to a solution on the center manifold, we can write

$$\tilde{\mathcal{S}}(u,\varepsilon) = s(v(0), v'(0), \varepsilon), \qquad \text{where} \quad s(A,B,\varepsilon) := \tilde{\mathcal{S}}\big((A+Bx)\varphi_0 + \Psi(A,B,\varepsilon),\, \varepsilon\big).$$

Moreover, $s(v, v', \varepsilon)$ will be constant for solutions of (7.18). We now claim that s has the expansion

$$\begin{aligned} s(A,B,\varepsilon) &= s_{400}A^4 + s_{301}A^3\varepsilon + s_{202}A^2\varepsilon^2 + \tilde{r}(A,B,\varepsilon) \\ &= 2s_{020}\left(\frac{1}{2}B^2 - \frac{f_{300}}{4}A^4 - \frac{f_{201}}{3}A^3\varepsilon - \frac{f_{102}}{2}A^2\varepsilon^2\right) + \tilde{r}(A,B,\varepsilon), \end{aligned} \tag{7.23}$$

where

$$s_{020} = -\tfrac{1}{6}c_0^2(\rho + (1-\rho)h_0) < 0 \tag{7.24}$$

and the C^M error term $\tilde{r}$ satisfies

$$\tilde{r}(A,B,\varepsilon) = O\left(|A|(|A| + |B|^{1/2} + |\varepsilon|)^4 + |B|(|A| + |B|^{1/2} + |\varepsilon|)^3\right).$$

In particular, up to the nonzero factor $2s_{020}$, the truncation of s is precisely the Hamiltonian for the truncated ODE (7.20).

Using the reversibility symmetry, we check that s is even in B, and so the smoothness of s implies

$$\frac{s_B(A,B,\varepsilon)}{B} = 2s_{020} + O(|A| + |B| + \varepsilon), \tag{7.25}$$

where $s_{020} = \frac{1}{2}s_{BB}(0,0,0)$. Determining s_{020} in principle requires the coefficient Ψ_{020} in the expansion of Ψ. When we actually go about calculating this coefficient and plugging it into (7.22), however, we see that it actually does not contribute, and that (7.24) holds. It is then straightforward to obtain (7.23) by combining $s_{020} \neq 0$, (7.25), and the fact that s is a conserved quantity.

7.8. **Proof of existence.** Combining the previous subsection with Section 7.3, we are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. Introducing the scaled variables

$$x = |\varepsilon|^{-1}X, \quad v(x) = \varepsilon V(X), \quad v_x(x) = \varepsilon|\varepsilon|W(X), \quad s(v, v_x, \varepsilon) = 2\varepsilon^4 s_{020}S(V,W,\varepsilon),$$

the reduced ODE (7.18) can be written as the planar system

$$\begin{cases} V_X = W \\ W_X = f_{102}V + f_{201}V^2 + f_{300}V^3 + R(V,W,\varepsilon) \end{cases}$$

with conserved quantity

$$S(V,W,\varepsilon) = \frac{1}{2}W^2 - \frac{f_{102}}{2}V^2 - \frac{f_{201}}{3}V^3 - \frac{f_{300}}{4}V^4 + \tilde{R}(V,W,\varepsilon),$$

and where the error terms satisfy

$$R(V,W,\varepsilon) = O(|\varepsilon|(|V| + |W|)), \qquad \tilde{R}(V,W,\varepsilon) = O(|\varepsilon|(|V| + |W|)).$$

When $\varepsilon = 0$, we have the explicit heteroclinic solution $V = a_1(1 + \tanh(\kappa_1 X))/2$ connecting $(V,W) = (0,0)$ with $(V,W) = (a_1, 0)$. This is the scaled version of v^0 in (7.21). Both of these equilibrium have same value, namely 0, of the conserved quantity S. For $\varepsilon \neq 0$, the equilibria at $(0,0)$ remains fixed while the equilibrium at $(a_1, 0)$ persists but is perturbed. From Lemma 7.7 on conjugate flows, we in fact know that its *exact* location is $(\varepsilon^{-1}h_+^\varepsilon - 1, 0)$ and moreover that it continues to have $S = 0$. The persistence of the full heteroclinic orbit then follows from its characterization as a nondegenerate level curve of the conserved quantity S. □

7.9. **Critical layers and streamline pattern.** Finally, in this section, we explore some qualitative features of the waves constructed above. As we have seen, there are certain parameter regimes for which a streamline in the unperturbed flow is a critical layer. For small bores, that streamline will split either upstream or downstream, opening into a "half cat's eye" with its pupil at infinity.

Theorem 7.8 (Streamlines)**.** *In the setting of Theorem 7.1, suppose that* (7.3) *holds so that there is a critical layer, and suppose for concreteness that $\omega < 0$ (so that $c_0 > 0$) and $a_1\varepsilon < 0$. Perhaps shrinking ε further, the streamlines of the corresponding solution $(\psi_1^\varepsilon, \psi_2^\varepsilon, \eta^\varepsilon)$ have the qualitative features of Figure 1. Specifically,*

(a) (Monotonicity) *The interface is strictly monotone with $\eta_x^\varepsilon < 0$. Moreover, $\psi_x^\varepsilon < 0$ for $Y \neq 0, 1$ so that the vertical velocity is positive.*
(b) (Critical layer) *There is a unique C^1 curve $\mathcal{C}^\varepsilon$ in the interior of the upper fluid where $\psi_{2Y}^\varepsilon = 0$. Above this curve, $\psi_{2Y}^\varepsilon > 0$, and below it $\psi_{2Y}^\varepsilon < 0$. There are two streamlines, one above $\mathcal{C}^\varepsilon$ and one below, that both limit to $\mathcal{C}^\varepsilon$ upstream. In the region they enclose (the eye), every streamline is a horizontally unbounded curve that opens to the right and has a unique turning point which is located on $\mathcal{C}^\varepsilon$. Outside the eye region, all streamlines extend from upstream to downstream.*

Remark 7.9. In (7.26) below we will see that the vertical extent of the eye is $O(|\varepsilon|^{1/2})$. In particular, for the specific parameter values from Example 7.6, we see that the width of the eye is $\sqrt{330|\varepsilon|}/90 + O(\varepsilon)$, while the downstream width of the upper layer is $1/3 + O(\varepsilon)$. Changing the sign of ω (and hence c_0) changes the sign of the horizontal velocity throughout the fluid but preserves the streamline pattern. Changing the sign of $a_1\varepsilon$ changes the signs of η_x and ψ_x, reflecting the streamline pattern in Figure 1 but preserving the sign of the horizontal velocity.

Proof. We start by confirming monotonicity (a). From the proof of Theorem 7.1, our assumption that $a_1\varepsilon < 0$, and Remark 7.2, we know that $v' = \eta_x^\varepsilon < 0$. The asymptotics (7.2) also give $\psi_{2Y}^\varepsilon < 0$ along $Y = h^\varepsilon + \eta^\varepsilon$. Differentiating the kinematic condition (7.1d), we see that this implies $\psi_{2x}^\varepsilon = -\eta_x^\varepsilon \psi_{2Y}^\varepsilon < 0$ there as well. But, ψ_{2x}^ε is harmonic and vanishes on the upper boundary $\{y = 1\}$ and at infinity. The maximum principle therefore implies that $\psi_{2x}^\varepsilon < 0$ in the interior of the upper fluid. Similarly, we find that $\psi_{1x}^\varepsilon < 0$ in the interior of the lower fluid.

Now we turn to the more detailed claims in (b). Setting $\varepsilon = 0$, we have

$$\psi_2^0(x, Y) = \psi_2^0(Y) = -c_0(Y - h_0) - \tfrac{1}{2}\omega(Y - h_0)^2.$$

Differentiating, we find that $\psi_{2Y}^0 = 0$ at the unique height $Y_c^0 := h_0 - c_0/\omega$, which lies in $(h_0, 1)$ by (7.3). Since $\psi_{2YY}^0 = -\omega > 0$ and $\psi_2^\varepsilon = \psi_2^0 + O(\varepsilon)$ in $C_{\mathrm{b}}^{2+\alpha}$ by (7.2), the existence of $\mathcal{C}^\varepsilon$ now follows from the implicit function theorem. Indeed, it is the graph of a single-valued function of x. Moreover, $\psi_{2YY}^\varepsilon > 0$ for $0 < \varepsilon \ll 1$ so that $\psi_{2Y}^\varepsilon > 0$ above $\mathcal{C}^\varepsilon$ and $\psi_{2Y}^\varepsilon < 0$ below. From (7.2) we also have $\psi_{1Y}^\varepsilon < 0$ in the lower fluid.

Examining the explicit formula for the upstream state (7.4), we see that for $0 < -\varepsilon a_1 \ll 1$, the critical upstream height perturbs to $Y_c^\varepsilon := h^\varepsilon - c^\varepsilon/\omega$ with the stream function value $(c^\varepsilon)^2/2\omega$. There are exactly two heights downstream at which the stream function takes on this value; the corresponding streamlines bound the eye region. Looking at (7.6), we see that these heights are given by

$$Y_c^0 \pm \frac{1}{\omega}\sqrt{\frac{2c_0(c_0 + \omega(1 - h_0))}{1 - h_0} a_1\varepsilon} + O(|\varepsilon|). \tag{7.26}$$

From the assumptions $a_1\varepsilon < 0$ and (7.3), we have that the radicand is strictly positive and $O(|\varepsilon|)$.

Pick any point inside the eye region. Applying the implicit function theorem, we see that the streamline through this point is globally parameterized as a graph $\{x = \xi(Y)\}$ for some C^1 function ξ. Moreover, the discussion above shows that $\xi_Y = 0$ only on $\mathcal{C}^\varepsilon$, and $\xi_{YY} > 0$ there. The desired qualitative features of the streamline pattern inside the eye now follow. On the other hand, outside this region, $\psi^\varepsilon_Y \neq 0$, so all streamlines must extend from $x = -\infty$ to $x = +\infty$. □

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Appendix A. Amick–Turner fixed point theory

In this section, we present a highly abbreviated version of Amick and Turner's fixed point theorems in [2, 3]. Rather than state those results in full generality, we have specialized to the case most relevant to our needs. An effort has also been made to simplify and standardize the notation.

Following the procedure in Section 2.2 leads us to study equations of the general form

$$\begin{cases} U_1 = \xi_1 + F_1(U_1, U_2, R; \lambda, \beta)|_0^x \\ U_2 = \xi_2 + F_2(U_1, U_2, R; \lambda, \beta)|_0^x \\ \ R = F_3(U_1, U_2, R; \lambda, \beta). \end{cases} \tag{A.1}$$

Here, (U_1, U_2, R) are the unknowns. Motivated by (2.13), where U_2 arises as a scaled derivative of U_1, we work in the space

$$W := (U_1, U_2, R) \in C^{k+\alpha}_\mu(\mathbb{R}) \times C^{k-1+\alpha}_\mu(\mathbb{R}) \times C^{k+\alpha}_\mu(\overline{\Omega}) =: \mathbb{X}_\mu$$

for some $\mu \in [0, \overline{\mu})$, integer $k \geq 1$, and $\alpha \in (0, 1)$. As before, let $\mathring{\mathbb{X}}_\mu$ denote the corresponding homogeneous space, and $\mathbb{X}_{\mathrm{b}} := \mathbb{X}_0$.

In (A.1), there are three types of parameters: $\xi = (\xi_1, \xi_2)$ is "initial data" for $U = (U_1, U_2)$; $\lambda \in \mathbb{R}$ is the main parameter of bifurcation; and $\beta \in (0, 1]$ is, essentially, a rescaling of time needed to obtain a fixed point.

Next, we impose some conditions on the nonlinear mappings in (A.1). Assume that

$$F(W; \lambda, \beta) = \beta \mathfrak{L} W + \mathcal{H}(W; \lambda, \beta), \tag{A.2}$$

where $\mathfrak{L} = (\mathfrak{L}_1, \mathfrak{L}_2, 0)$ is a zeroth order linear mapping in the sense that

$$\mathfrak{L} \text{ is linear and bounded } \mathbb{X}_\mu \to \mathbb{X}_\mu \text{ and } \mathbb{X}_{\mathrm{b}} \to \mathring{\mathbb{X}}_{\mathrm{b}} \tag{A.3}$$

with bounds uniform in μ on compact subsets of $(0, \overline{\mu})$.

Finally, suppose that each component of the nonlinear function $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ takes the general form

$$\mathcal{H}_i(W; \lambda, \beta) = \frac{1}{\beta^p} \mathcal{I} S_g(\mathcal{D} W; \lambda, \beta). \tag{A.4}$$

Here, $p > 0$ corresponds roughly to the homogeneity of the nonlinearity created by S_g. Intuitively, we think of $\mathcal{D}$ as losing some number of derivatives, while $\mathcal{I}$ is smoothing. Between them is the mapping S_g, a general (parameter dependent) superposition operator. Note that p, $\mathcal{D}$, S_g, and $\mathcal{I}$ can vary for each component, but we will suppress this dependence to simplify notation. Also, one can assume more generally that $\mathcal{H}_i$ consists of a finite sum of terms of the form (A.4).

To state things more precisely, we introduce two (lower regularity) spaces:

$$\mathbb{Y}_\mu := C^{j+\alpha}_\mu(\mathbb{R};\mathbb{R}^\ell) \times C^{j+\alpha}_\mu(\overline{\Omega};\mathbb{R}^m), \qquad \mathbb{Z}_\mu := C^{j+\alpha}_\mu(\overline{\Omega}),$$

for some integers $j \geq 0$, $\ell, m \geq 1$ (again, each of these will in principle vary in i). Now, assume that

$$\mathcal{D} \text{ is linear and bounded } \mathbb{X}_\mu \to \mathbb{Y}_\mu \text{ and } \mathbb{X}_\mathrm{b} \to \mathbb{Y}_\mathrm{b} \tag{A.5}$$

with bounds uniform in μ on compact subsets of $(0,\overline{\mu})$. The superposition map S_g is defined by

$$S_g(Y;\lambda,\beta)(x,y) := g(x,y,Y_1(x),\ldots,Y_\ell(x),Y_{\ell+1}(x,y),\ldots,Y_{\ell+m}(x,y);\lambda,\beta), \tag{A.6}$$

for all $Y \in \mathbb{Y}_\mu$ and $(x,y) \in \overline{\Omega}$. Here, the function g is assumed to satisfy

$$\begin{gathered} g = g(x,y,w;\lambda,\beta) \in C^{M+3}(\overline{\Omega} \times \mathbb{R}^{\ell+m} \times \mathbb{R} \times (0,1];\mathbb{R}), \\ g(x,y,0;0,\beta) = 0, \quad g_w(x,y,0;0,\beta) = 0, \quad g_\lambda(x,y,0;0,\beta) = 0. \end{gathered} \tag{A.7}$$

One can show that (A.5)–(A.7) together ensure that

$$W \mapsto S_g(\mathcal{D}W;\lambda,\beta) \qquad \text{is bounded } \mathbb{X}_\mu \to \mathbb{Z}_\mu \text{ and } \mathbb{X}_\mathrm{b} \to \mathbb{Z}_\mathrm{b}.$$

Finally, $\mathcal{I}$ is supposed to be smoothing in that it satisfies

$$\mathcal{I} \text{ is linear and bounded } \mathbb{Z}_\mu \to \mathbb{X}_{i,\mu} \text{ and } \mathbb{Z}_\mathrm{b} \to \mathring{\mathbb{X}}_{i,\mathrm{b}}, \tag{A.8}$$

with bounds uniform in μ on compact subsets of $(0,\overline{\mu})$.

As is always the case with center-manifold constructions, Amick and Turner do not treat (A.1) directly but rather a truncated problem where each function g in (A.7) is replaced by

$$g^r(x,y,w_1,\ldots,w_{\ell+m};\lambda,\beta) := g(x,y,\eta_r(w_1),\ldots,\eta_r(w_{\ell+m});\lambda,\beta) \tag{A.9}$$

for an appropriate cutoff function η_r, which we will always take to be even. We write the resulting fixed-point equations as

$$\begin{cases} U_1 = \xi_1 + F_1^r(U_1,U_2,R;\lambda,\beta)|_0^x \\ U_2 = \xi_2 + F_2^r(U_1,U_2,R;\lambda,\beta)|_0^x \\ \ R = F_3^r(U_1,U_2,R;\lambda,\beta). \end{cases} \tag{A.10}$$

From [3, Lemma 4.1,Theorem 4.1] we know that, for each $M \in \mathbb{N}$, we can choose $\beta, r, \mu > 0$ sufficiently small so that F^r has $M+1$ Lipschitz-continuous derivatives acting from $\mathbb{X}_\mu \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{X}_{(k+M+3)\mu}$.

Theorem A.1 (Fixed point)**.** *Consider the truncated fixed-point equation* (A.10) *under the structural assumptions* (A.2)–(A.8) *enumerated above. Then, for any integer M, there exists $\mu \in (0,\overline{\mu})$, $r > 0$, and $\beta \in (0,1]$ so that the unique solution to* (A.10) *is given by*

$$W = (U_1,U_2,R) =: \mathscr{W}(\xi_1,\xi_2,\lambda) \in \mathbb{X}_\mu$$

where the mapping $\mathscr{W} \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{X}_\mu$ is C^{M+1}.

Proof. This result is found by combining Theorem 3.1, Theorem 3.3, Remark 3.2, and Theorem 4.1 of [3]. □

The coordinate mapping $\mathscr{W}$ has flatness properties analogous to (1.8). A particular instance of this which we will need is the following.

Lemma A.2. *Under the assumptions of Theorem A.1, we have*

$$|\mathscr{W}_3(\xi,\lambda)(0,0)| + |\partial_x \mathscr{W}_3(\xi,\lambda)(0,0)| \lesssim |\xi|(|\xi| + |\lambda|). \tag{A.11}$$

Proof. From the uniqueness of $\mathscr{W}$ we have $\mathscr{W}(0,\lambda)=0$ for all λ. Moreover, differentiating the third equation in (A.10) with respect to ξ we discover

$$D_\xi \mathscr{W}_3 = D_W F_3^r(\mathscr{W}) D_\xi \mathscr{W}.$$

At $(\xi,\lambda)=(0,0)$, this becomes simply

$$D_\xi \mathscr{W}_3(0,0) = \beta \mathfrak{L}_3 D_\xi \mathscr{W}(0,0),$$

where $\mathfrak{L}$ is the operator in (A.3). But we have assumed that the third component $\mathfrak{L}_3$ of this operator vanishes, and so we simply obtain $D_\xi \mathscr{W}_3(0,0)=0$. Thus $\|\mathscr{W}_3(\xi,\lambda)\|_{\mathbb{X}_\mu} \lesssim |\xi|(|\xi|+|\lambda|)$, which in particular implies (A.11). □

Appendix B. Iteration for anti-plane shear with a general body force

In this section, we revisit the center manifold reduction of the anti-plane shear problem where the live body force b takes a more general form. Recall the problem (5.4) with the original parameter λ. Plugging in the ansatz (5.3) for the strain energy we can write the problem as

$$\Delta u + 2w_1 \nabla\cdot\left(|\nabla u|^2 \nabla u\right) - b(u,\lambda) = 0. \tag{B.1}$$

Taylor expanding b and using (5.5) and (5.6) we obtain

$$b(z,\lambda) = -z + b_1\lambda z + \frac{1}{2} b_{z\lambda\lambda}(0,0)\lambda^2 z + b_2 z^3 + O\left((|\lambda|+|z|)^4\right),$$

where

$$b_1 := b_{z\lambda}(0,0), \quad b_2 := \frac{1}{6} b_{zzz}(0,0).$$

Because of the cubic term in (B.1), we would like to expand the reduced ODE (1.9) to third order. Following the general strategy in Section 4.1, we can replace (B.1) by its truncation at order $K=3$,

$$Lu = b_1\lambda u + \frac{1}{2} b_{u\lambda\lambda}(0,0)\lambda^2 u + b_2 u^3 - 2w_1\nabla\cdot\left(|\nabla u|^2\nabla u\right). \tag{B.2}$$

From a similar argument as in Section 5.1 we find the appropriate reparametrization $\lambda=\lambda_2\varepsilon^2$ and the index set $\mathcal{J}$ given by (5.15) as before.

With the scaling settled, we make the ansatz $(A+Bx)\varphi_0 + \sum_{\mathcal{J}} \Psi_{ijk} A^i B^j \varepsilon^k$ for u in (B.2), obtaining

$$L\left(\sum_{\mathcal{J}} \Psi_{ijk} A^i B^j \varepsilon^k\right) = b_1\lambda_2 A\varepsilon^2 \cos y + b_2 A^3 \cos^3 y + 2w_1 A^3 (\sin^3 y)_y.$$

Grouping like terms yields

$$L\Psi_{101} = L\Psi_{200} = L\Psi_{110} = L\Psi_{011} = 0,$$
$$L\Psi_{102} = b_1\lambda_2\cos y, \quad L\Psi_{201}=0, \quad L\Psi_{300} = b_2\cos^3 y + 6w_1\sin^2(y)\cos(y).$$

Applying Lemma 2.3 allows us to iteratively solve these equations, and ultimately we find that

$$\Psi_{101} = \Psi_{200} = 0, \qquad \Psi_{110} = \Psi_{200} = \Psi_{201} = 0,$$
$$\Psi_{102} = \frac{1}{2} b_1\lambda_2 x^2 \cos y, \quad \Psi_{300} = \frac{3b_2+6w_1}{8} x^2\cos y + \frac{b_2-6w_1}{32}(\cos y - \cos(3y)).$$

Thus,

$$f(A,B,\varepsilon) = b_1\lambda_2 A\varepsilon^2 + \frac{3(b_2+2w_1)}{4} A^3 + r(A,B,\varepsilon).$$

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