THE SHALLOW-WATER MODELS WITH CUBIC NONLINEARITY

ROBIN MING CHEN, TIANQIAO HU, AND YUE LIU

Abstract. In the present study several integrable equations with cubic nonlinearity are derived as asymptotic models from the classical shallow water theory. The starting point in our derivation is the Euler equation for an incompressible fluid with the simplest bottom and surface conditions. The approximate equations are obtained by working under suitable scalings that allow for the modeling of water waves of relatively large amplitude, truncating the asymptotic expansions of the unknowns to appropriate order, and introducing a special Kodama transformation. The so obtained equations exhibit cubic order nonlinearities and can be related to the following integrable systems: the Novikov equation, the modified Camassa–Holm equation, and a Camassa–Holm type equation with cubic nonlinearity. Analytically, the formation of singularities of the solution to some of these quasi-linear model equations is also investigated, with an emphasis on the understanding of the effect of the nonlocal higher order nonlinearities. In particular it is shown that one of the models accommodates the phenomenon of curvature blow-up.

Keywords: Shallow-water models; Novikov equation; modified Camassa–Holm equation; curvature blow-up.

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1. Introduction

The theory of water waves embodies the Euler equations of fluid mechanics along with the crucial behavior of boundaries. Due to the complexity and the difficulties arising in the theoretical and numerical study for the full system, simpler model equations have been proposed as effective approximations in various specific physical regimes.

The present paper is along the same line of study. In particular we consider the shallow-water (or long-wave) approximation to the irrotational gravity water wave system. Such approximation is usually carried out formally from the governing equations via double asymptotic expansions in the following two fundamental dimensionless positive parameters (see, for example, [16]):

$$\text{the amplitude parameter } \varepsilon := \frac{a}{h_0}, \quad \text{and} \quad \text{the shallowness parameter } \mu := \frac{h_0^2}{\lambda^2}, \tag{1.1}$$

where a, h_0 and λ are the typical amplitude of the wave, the depth of the water, and the wavelength, respectively. The shallow-water/long-wave regime then corresponds to assuming μ to be small: $\mu \ll 1$. Further relating ε with μ then allows one to derive model equations in particular asymptotic regimes.

Arguably, one of the most famous and simplest long-wave asymptotic models which accommodates genuine nonlinear behavior is the Korteweg–de Vries (KdV) equation [35]. The nonlinear effect in the KdV modeling is reflected in that the wave amplitude is assumed to be small but finite: $\varepsilon = O(\mu)$. Such a scaling is later implemented to generate a family of asymptotically equivalent equations, namely the BBM-type equations [1]. Both the KdV equation and the BBM class posses smooth soliton solutions and global solutions for very general initial data, in particular all physically relevant waves; see, for example, [12, 44].

However some other fundamental nonlinear phenomena, such as *wave-breaking* and surface singularities, are prevented from the KdV model, due to its strong dispersive effect that regularizes the progressively nonlinear steepening. This promotes the need to seek model equations that incorporate stronger nonlinear effects to better describe singular wave phenomena for larger amplitude waves.

A natural approach is to consider regimes that bring higher-order nonlinear terms, characterized by larger values of ε, for instance, the so-called Camassa–Holm (CH) scaling for shallow water waves of *moderate amplitude* [18]

$$\mu \ll 1, \qquad \varepsilon = O(\mu^{\frac{1}{2}}). \tag{1.2}$$

With this scaling, a two-parameter family of approximation equations are derived [18] including the well-known Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx},$$

and the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx},$$

where u is the horizontal component of the velocity field at some specific depth, and m is the so-called momentum density. The CH equation was first considered in [26] as a bi-Hamiltonian equation, and the DP equation was first derived in [20] in the study of integrable equation. The CH equation was later proposed in [5, 18, 32] in the context of water waves. Similar to the KdV equation, the CH and DP equations are both completely integrable. In contrast to KdV, on the other hand, both CH and DP, and their multi-component generalizations (for example, [19, 31, 41]) accommodate solutions exhibiting certain degree of singularities, namely the breaking waves [7, 13, 14, 23, 39] and peaking waves [5, 20, 36, 37]. Note that for the full water wave problem, the traveling wave solutions of greatest height have a peak at their crest; see [10, 11, 15].

The discovery of the CH and DP equations motivates the search for various generalization models with interesting properties and applications. Since these two equations are both quadratic nonlinear, one may wonder as the nonlinearity becomes more pronounced, and hence the hyperbolic property tends to be more dominant, what kind of singularity can be triggered. In the context of asymptotic modeling, this amounts to considering larger amplitude waves.

For the CH and DP equations, the formation of singularities in the solution that develops from a localized and smooth initial data is in the form of blow-up of the slope, while the solution remains bounded [9, 14, 39]. One of the motivations of this paper comes from the recent works on a new type of singularity formation for cubic nonlinear models, namely the *curvature blow-up*, i.e. the second derivative u_{xx} of solution becomes unbounded in finite time while the solution u and its gradient u_x remain bounded. Examples can be found in the modified Camassa–Holm (mCH) equation [6, 25, 28, 29, 41, 42] and the generalized modified Camassa–Holm (gmCH) equation [7, 24, 41]. Indeed, these equations inherit certain energy conservation and momentum persistence property that allow the control of u and u_x. Yet the presence of the higher order nonlocal nonlinearity induces the blow-up of the higher derivative. On the other hand, such CH-type equations with cubic nonlinearity (also including the Novikov equation [40]) were only studied in the framework of integrable systems theory and, to the best of the authors' knowledge, there were limited attempts of the relations of these equations to the physically relevant models in the context of water waves. To this end, we would like to perform a modeling under a different scaling from (1.2), with the purpose of deriving cubic nonlinear equations (of CH-type) that may host the aforementioned curvature blow-up phenomenon.

It is worthwhile pointing out that most of the higher order nonlinear descendants of the CH equation (like the mCH, gmCH, Novikov, etc.) are derived in the context of integrable systems. Another goal of the present study is to propose a hydrodynamic approach to derive some of those cubic nonlinear models, including the mCH and Novikov equations.

Roughly speaking, since we expect the cubic nonlinearity to appear at the order of $O(\varepsilon^2\mu)$, leaving the $O(\mu^2)$ terms as higher order ones, this naturally leads to a scaling requirement $\varepsilon = o(\mu^{1/2})$.

Therefore we impose

$$\mu \ll 1, \qquad \varepsilon = O(\mu^{\frac{2}{5}}), \tag{1.3}$$

which also corresponds to a shallow-water regime for waves of moderate amplitude but larger than the one in the CH scaling regime (1.2). Proceeding analogously as for the CH equation, we first derive an equation for the scaled surface elevation η

$$\begin{aligned} 2(\eta_x+\eta_t) &+ \frac{1}{3}\mu\eta_{xxx} + 3\varepsilon\eta\eta_x - \frac{3}{4}\varepsilon^2\eta^2\eta_x + \frac{3}{8}\varepsilon^3\eta^3\eta_x + \varepsilon\mu\left(\frac{23}{12}\eta_x\eta_{xx} + \frac{5}{6}\eta\eta_{xxx}\right) \\ &+ \frac{115}{192}\varepsilon^4\eta^4\eta_x + \varepsilon^2\mu\left(\frac{23}{16}\eta_x^3 + \frac{29}{8}\eta\eta_x\eta_{xx} + \frac{3}{4}\eta^2\eta_{xxx}\right) = 0 + O(\varepsilon^5,\mu^2). \end{aligned} \tag{1.4}$$

A similar equation for the surface was also derived in [43] under a larger amplitude scaling $\varepsilon = O(\mu^{1/4})$. By relating the horizontal velocity u with η, a cubic nonlinear equation for u is obtained.

Here we adapt the idea of [2] to expand η in terms of u together with its derivatives using the so-called Kodama transformation [33]. In particular, the expansion takes the following form

$$\eta \sim u + \varepsilon A + \mu B + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \varepsilon^3 K + \varepsilon^2\mu G + \varepsilon\mu^2 H \tag{1.5}$$

where

$$\begin{aligned} &A := \lambda_1 u^2, \quad B := \lambda_2 u_{xx}, \quad E := \lambda_3 u^3, \quad K = \lambda_0 u^4, \quad C := \lambda_4 u_x^2 + \lambda_5 u u_{xx}, \\ &D := \lambda_6 u_{xxxx}, \quad G := \lambda_7 u u_x^2 + \lambda_8 u^2 u_{xx}, \quad H := \lambda_9 u_x u_{xxx} + \lambda_{10} u u_{xxxx} + \lambda_{11} u_{xx}^2. \end{aligned}$$

This type of transformation was first introduced by Kodama in [33], and was used by Dullin et al. [21] to derive a shallow water wave model under the influence of surface tension. A further splitting of u_{xxt} together with an equation for u_t generates one more degree of freedom ν, cf. (3.6)–(3.7). Then the expected specific form of the equations imposes exactly the same number of constraints on these parameters, leading to exact parameter values in the resulting model equations. In particular, this allows us to obtain the following types of equations.

Case 1. The CH-mCH-Novikov equation

$$\begin{aligned} m_t + u_x - \frac{\mu}{4}u_{xxx} + \frac{\varepsilon}{2}(2u_x m + u m_x) + \frac{k_1\varepsilon^2}{4}((u^2 - \beta\mu u_x^2)m)_x + \frac{k_2\varepsilon^2}{4}(u^2 m_x + 3u u_x m) \\ = 0 + O(\varepsilon^5,\mu^2), \end{aligned} \tag{1.6}$$

where $m = u - \beta\mu u_{xx}$, $\beta = \dfrac{5}{12}$, $k_2 = \dfrac{69}{5}$, and $k_1 \simeq -15.1765$ is the only real root of

$$2000k_1^3 + 106200k_1^2 + 1871550k_1 + 10934031 = 0.$$

Case 2. The CH-Novikov equation

$$m_t + u_x - \frac{\mu}{4}u_{xxx} + \frac{\varepsilon}{2}(2u_x m + u m_x) + \frac{k_2\varepsilon^2}{4}(u^2 m_x + 3u u_x m) = 0 + O(\varepsilon^5,\mu^2). \tag{1.7}$$

Case 3. A cubic CH-type equation

$$m_t + u_x - \frac{\mu}{4}u_{xxx} + \frac{\varepsilon}{2}(2u_x m + u m_x) + \frac{k_3\varepsilon^2}{4}\left((u^2 - \frac{1}{4}\beta\mu(u^2)_{xx})u\right)_x = 0 + O(\varepsilon^5,\mu^2), \tag{1.8}$$

where $k_3 = \dfrac{46}{5}$.

Mathematically, under suitable scaling limits the quadratic terms in (1.6) and (1.7) can be dropped out in a formal scaling limit, leaving (1.6) as the mCH-Novikov equation

$$m_t + k_1((u^2 - u_x^2)m)_x + k_2(u^2 m_x + 3u u_x m) = 0 \tag{1.9}$$

where k_1 and k_2 satisfy conditions given above as in Case 1, and (1.7) as the Novikov equation

$$m_t + k_2(u^2 m_x + 3u u_x m) = 0 \tag{1.10}$$

with parameter k_2 given as in Case 2.

As is explained earlier, with the cubic nonlinear models at hand, our second goal is to study the formation of singularities due to the higher order nonlinear effects and construct initial data that lead to the finite time curvature blow-up. To this end, we will at this moment only focus our attention on equation (1.9) where only cubic nonlinearities are present, and consider the following Cauchy problem

$$\begin{cases} m_t + k_1\left[(u^2 - u_x^2)m\right]_x + k_2\left(u^2 m_x + 3uu_x m\right) = 0, \\ u(0,x) = u_0(x), \end{cases} \qquad t > 0,\ x \in \mathbb{R}. \tag{1.11}$$

Moreover for mathematical speculation we will allow ourselves to consider a more general range of parameter values for k_1 and k_2 than that which is given in Case 1.

It turns out that the two groups of cubic nonlinearities in (1.11) play quite different roles in the blow-up analysis. In the case $k_1 = 0$, (1.11) becomes the Novikov equation (1.10), and when the initial momentum density m_0 does not change sign then the solution exists globally for all time [45]. On the other hand when the Novikov nonlinearity is not present ($k_2 = 0$ i.e., the mCH equation), it is shown [6, 29, 38] that the curvature could still blow up in finite time even if m_0 does not change sign. This leads to a natural question of understanding how the interaction between these two groups of cubic nonlinearities would affect the singularity formation mechanism.

It is also worthwhile pointing out that, as was discovered by Brandolese et al [3, 4], many quadratic nonlinear CH-type equations exhibit a very strong non-diffusive character that extremely "localized" information about the data is enough to lead to finite time blow-up of solutions. Such a phenomenon comes from the fact that the nonlinear nonlocal effects are over-dominated by the local nonlinearities of the equations. This hyperbolic feature seems to be slightly counter balanced by the stronger nonlocal effects due to higher nonlinearity of the equations, as was explored in [6, 7]. Thus it would be interesting to study how local structures of the initial data may affect the evolution of solutions to equation (1.11), and in particular, the formation of singularity. Since the equation involves both the mCH and Novikov types of nonlinearity, it is reasonable to expect some kind of relaxed local-in-space blow-up criterion in the spirit of [6, 7]. However as pointed out above, the two types of nonlinearities do not seem to cooperate in a good way to produce blow-ups, making the analysis rather subtle.

A refined Beale–Kato–Majda type blow-up criterion (cf. Lemma 5.2) singles out the right blow-up quantity to look at. Tracking the dynamics of such a quantity along the characteristics reveals explicit local and nonlocal interplays between the solution and its gradient, cf. Lemma 5.3. Using the two conservation laws provides a way to control the nonlocal convolution. This allows one to derive crucial monotonicity property of u, u_x and m along the characteristics, which in turn leads to a Riccati dynamics for m, cf. Theorem 5.3. This result covers a wide range of parameter values of k_1 and k_2, in particular the equation (1.6) in Case 1.

We also provide a different way of approach which does not rely on the use of the conservation laws. Instead, taking advantage of the sign preservation of the momentum density m, the nonlocal terms can be shown to have good signs provided that the initial momentum density does not change sign. Therefore it remains to examine the local terms. It turns out that a Riccati type inequality can be obtained as long as the "local oscillation" $|u_x/u|$ is reasonably mild. Note that the sign condition on m already rules out fast oscillations. A further refined analysis on the evolution of u_x/u can be performed to show that mild oscillations will persist along the characteristics with carefully chosen data, and therefore closes the argument.

The remainder of the paper is organized as follows. In Section 2, the model equation for the free surface with higher order terms is formally derived from shallow water in the Euler equation for an incompressible fluid, with the computational details provided in Appendix A. Section 3 together with Appendix B is devoted to the derivation of a family of asymptotically equivalent equations, namely the CH-mCH-Novikov equation (1.6). Some other equations with cubic nonlinearity will be derived in Section 4. Section 5 is focused on the mCH-Novikov equation (1.11). A blow-up criterion will be derived and special initial data will be constructed that lead to the curvature blow-up.

2. Derivation of the free surface equation

The main goal of this section is to formally derive of model equation (1.4) for the free surface from the Euler equations. Compared with the model equation derived in [32] which is truncated at the order $O(\varepsilon^3, \varepsilon\mu)$, the new model (1.4) contains more higher order terms which will be useful to derive a class of unidirectional wave equations including cubic nonlinear terms.

Consider the two-dimensional incompressible irrotational flows in the domain $\{(x, z) : 0 < z < h(x, t)\}$ with a parametrization of the free surface $h = h(x, t)$, where the horizontal and vertical directions are represented by x and z, respectively. The governing system is given by

$$\begin{cases} u_t + uu_x + wu_z = -\frac{1}{\rho}P_x, \\ w_t + uw_x + ww_z = -\frac{1}{\rho}P_z - g, \\ u_x + w_z = 0, \\ u_z - w_x = 0, \end{cases}$$

where the pressure is written as $P(t, x, z) = p_a + \rho g(h_0 - z) + p(t, x, z)$, where p_a is the constant atmospheric pressure, and p is the dynamic pressure. In addition, we pose the "no-flow" condition on the flat bed, i.e., $w|_{z=0} = 0$. On the surface $z = h_0 + \eta$, the dynamic condition $P = p_a$ and the kinematic condition yield $p = \rho g\eta$ and $w = \eta_t + u\eta_x$.

Next we perform the following standard nondimensionalization

$$x \to \lambda x,\ z \to h_0 z,\ \eta \to a\eta,\ t \to \frac{\lambda}{\sqrt{gh_0}}t,\ u \to \sqrt{gh_0}u,\ w \to \sqrt{\mu gh_0}w,\ p \to \rho gh_0 p.$$

Recalling (1.1), we further assume that u, w and p are proportional to the wave amplitude, that is, $u \to \varepsilon u, \quad w \to \varepsilon w, \quad p \to \varepsilon p$. To examine the problem in an appropriate far field, we follow the approach employing the far field variable with the right-going wave:

$$\xi = \varepsilon^{1/2}(x - t), \quad \tau = \varepsilon^{3/2}t. \tag{2.1}$$

We also transform $w \to \sqrt{\varepsilon}w$ to keep mass conservation. Therefore, the governing equations become

$$\begin{cases} -u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ \varepsilon\mu\{-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ u_\xi + w_z = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ u_z - \varepsilon\mu w_\xi = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\ p = \eta & \text{on} \quad z = 1 + \varepsilon\eta, \\ w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) & \text{on} \quad z = 1 + \varepsilon\eta, \\ w = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.2}$$

Before applying the asymptotic expansion, we Taylor expand the boundary terms: $f(1+\varepsilon\eta) = \sum_{n=0}^{\infty} \frac{(\varepsilon\eta)^n}{n!} f^{(n)}(1)$ to obtain

$$\begin{cases} -u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi & \text{in} \quad 0<z<1, \\ \varepsilon\mu\{-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\} = -p_z & \text{in} \quad 0<z<1, \\ u_\xi + w_z = 0 & \text{in} \quad 0<z<1, \\ u_z - \varepsilon\mu w_\xi = 0 & \text{in} \quad 0<z<1, \\ p + \varepsilon\eta p_z + \frac{\varepsilon^2\eta^2}{2} p_{zz} + \frac{\varepsilon^3\eta^3}{6} p_{zzz} = \eta & \text{on} \quad z=1, \\ w + \varepsilon\eta w_z + \frac{\varepsilon^2\eta^2}{2} w_{zz} + \frac{\varepsilon^3\eta^3}{6} w_{zzz} = -\eta_\xi + \varepsilon\eta_\tau + \varepsilon\eta_\xi(u + \varepsilon\eta u_z + \frac{\varepsilon^2\eta^2}{2} u_{zz} + \frac{\varepsilon^3\eta^3}{6} u_{zzz}) & \text{on} \quad z=1, \\ w = 0 & \text{on} \quad z=0. \end{cases} \tag{2.3}$$

A double asymptotic expansion is then introduced to seek a solution of the system formally,

$$q \sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \mu^m q_{nm} \quad \text{as} \quad \varepsilon \to 0,\ \mu \to 0,$$

where q will be taken to be the functions u, w, p and η, and all functions q_{nm} satisfy the far field conditions $q_{nm} \to 0$ as $|\xi| \to \infty$ for every n, $m = 0, 1, 2, 3, ...$

Substituting the asymptotic expansions of u, w, p, η into (2.3), we check all the coefficients at each order $O(\varepsilon^i\mu^j)$ $(i,\ j = 0, 1, 2, 3, ...)$. For example at $O(1)$ we obtain

$$\begin{cases} -u_{00,\xi} = -p_{00,\xi} & \text{in} \quad 0<z<1, \\ 0 = p_{00,z} & \text{in} \quad 0<z<1, \\ u_{00,\xi} + w_{00,z} = 0 & \text{in} \quad 0<z<1, \\ u_{00,z} = 0 & \text{in} \quad 0<z<1, \\ p_{00} = \eta_{00}, \quad w_{00} = -\eta_{00,\xi} & \text{on} \quad z=1, \\ w_{00} = 0 & \text{on} \quad z=0. \end{cases} \tag{2.4}$$

From the fourth equation in (2.4) it follows that u_{00} is independent of z. Thanks to the third equation in (2.4) and the boundary condition of w on $z = 0$, we get

$$w_{00} = w_{00}|_{z=0} + \int_0^z w_{00,z'}\,dz' = -zu_{00,\xi},$$

which along with the boundary condition on $z = 1$ implies $u_{00,\xi}(\tau,\xi) = \eta_{00,\xi}(\tau,\xi)$. Therefore

$$u_{00}(\tau,\xi) = \eta_{00}(\tau,\xi), \quad w_{00} = -z\eta_{00,\xi},$$

here use has been made of the far field conditions u_{00}, $\eta_{00} \to 0$ as $|\xi| \to \infty$. On the other hand, from the second equation in (2.4), it follows that

$$p_{00} = p_{00}|_{z=1} + \int_1^z p_{00,z'}\,dz' = \eta_{00}.$$

At $O(\varepsilon^1\mu^0) = O(\varepsilon)$ we obtain

$$\begin{cases} -u_{10,\xi} + u_{00,\tau} + u_{00}u_{00,\xi} = -p_{10,\xi} & \text{in} \quad 0<z<1, \\ 0 = p_{10,z} & \text{in} \quad 0<z<1, \\ u_{10,\xi} + w_{10,z} = 0 & \text{in} \quad 0<z<1, \\ u_{10,z} = 0 & \text{in} \quad 0<z<1, \\ p_{10} + p_{00,z}\eta_{00} = \eta_{10} & \text{on} \quad z=1, \\ w_{10} + \eta_{00}w_{00,z} = -\eta_{10,\xi} + \eta_{00,\tau} + u_{00}\eta_{00,\xi} & \text{on} \quad z=1, \\ w_{10} = 0 & \text{on} \quad z=0. \end{cases} \tag{2.5}$$

From the fourth equation in (2.5), we know that u_{10} is independent to z, that is, $u_{10} = u_{10}(\tau, \xi)$. Thanks to the third equation in (2.5) and the boundary conditions of w on $z = 0$, we get

$$w_{10} = w_{10}|_{z=0} + \int_0^z w_{10,z'} dz' = -z u_{10,\xi}. \tag{2.6}$$

Hence, from the third equation in (2.5) and (2.6) and the boundary conditions of w on $z = 1$, we obtain that

$$u_{10,\xi} = \eta_{10,\xi} - \eta_{00,\tau} - (u_{00}\eta_{00})_\xi \quad \text{and} \quad w_{10} = z(\eta_{00,\tau} + 2\eta_{00}\eta_{00,\xi} - \eta_{10,\xi}) \tag{2.7}$$

Thanks to the second equation in (2.5) , we deduce that

$$p_{10,\xi} = \eta_{10,\xi} = u_{10,\xi} + \eta_{00,\tau} + (u_{00}\eta_{00})_\xi. \tag{2.8}$$

Taking account of the first equation in (2.5) and (2.7), it must be

$$-p_{10,\xi} = -u_{10,\xi} + \eta_{00,\tau} + \eta_{00}\eta_{00,\xi},$$

which along with (2.8) and (2.7) implies

$$2\eta_{00,\tau} + 3\eta_{00}\eta_{00,\xi} = 0.$$

Similarly, at the orders $O(\varepsilon^0\mu^1)$, $O(\varepsilon^2\mu^0)$, $O(\varepsilon^1\mu^1)$, $O(\varepsilon^3\mu^0)$, $O(\varepsilon^4\mu^0)$ and $O(\varepsilon^2\mu^1)$, the relation between p_{ij}, η_{ij}, u_{ij}, w_{ij} and their τ-derivatives can be obtained; see, for example, [30].

As is discussed in the Introduction, the scaling relation (1.3) suggests us to seek terms up to the order of $O(\varepsilon^3\mu^1)$. Following the same procedure as above (please refer to Appendix A for details), we obtain the following equation for η

$$\begin{aligned} 2\eta_\tau + 3\eta\eta_\xi + \frac{1}{3}\mu\eta_{\xi\xi\xi} - \frac{3}{4}\varepsilon\eta^2\eta_\xi + \frac{3}{8}\varepsilon^2\eta^3\eta_\xi + \frac{115}{192}\varepsilon^3\eta^4\eta_\xi + \alpha\eta^5\eta_\xi + \varepsilon\mu\left(\frac{23}{12}\eta_\xi\eta_{\xi\xi} + \frac{5}{6}\eta\eta_{\xi\xi\xi}\right) \\ + \varepsilon^2\mu\left(\frac{23}{16}\eta_\xi^3 + \frac{29}{8}\eta\eta_\xi\eta_{\xi\xi} + \frac{3}{4}\eta^2\eta_{\xi\xi\xi}\right) = 0 + O(\varepsilon^5, \varepsilon^3\mu, \mu^2), \end{aligned} \tag{2.9}$$

where α is some constant we do not specify here.

Recall the original transformation $x = \varepsilon^{-\frac{1}{2}}\xi + \varepsilon^{-\frac{3}{2}}\tau,\ t = \varepsilon^{-\frac{3}{2}}\tau$, namely,

$$\frac{\partial}{\partial\xi} = \varepsilon^{-\frac{1}{2}}\partial_x, \quad \frac{\partial}{\partial\tau} = \varepsilon^{-\frac{3}{2}}(\partial_x + \partial_t). \tag{2.10}$$

The equation (2.9) transforms to

$$\begin{aligned} 2(\eta_x + \eta_t) + \frac{1}{3}\mu\eta_{xxx} + 3\varepsilon\eta\eta_x + \varepsilon^2 A_1\eta^2\eta_x + \varepsilon^3 A_2\eta^3\eta_x + \varepsilon\mu\big(A_3\eta_x\eta_{xx} + A_4\eta\eta_{xxx}\big) \\ + A_8\varepsilon^4\eta^4\eta_x + \varepsilon^2\mu\big(A_5\eta\eta_x\eta_{xx} + A_6\eta^2\eta_{xxx} + A_7\eta_x^3\big) = 0 + O(\varepsilon^5, \varepsilon^3\mu, \mu^2). \end{aligned} \tag{2.11}$$

where $A_1 = -\frac{3}{4}$, $A_2 = \frac{3}{8}$, $A_3 = \frac{23}{12}$, $A_4 = \frac{5}{6}$, $A_5 = \frac{29}{8}$, $A_6 = \frac{3}{4}$, $A_7 = \frac{23}{16}$, $A_8 = \frac{115}{192}$.

Remark 2.1. It is noted that the high-order terms $O(\varepsilon^5, \mu^2)$ in (2.9) only depend on the function η and its ξ derivatives. By the scaling invariance in (2.11), $O(\varepsilon^5, \mu^2)$ would not generate any low order terms in (2.11) under the transformations in (2.10).

3. Derivation of model equations with cubic nonlinear terms

Having derived the equation of the free surface η in Section 2, the focus of the development in this section is the derivation of the model equations that incorporate cubic nonlinearities of various kinds including the CH, mCH and Novikov types, as given in (1.6).

3.1. **Asymptotic expansion using Kodama transformation.** Recall that we assume $\mu \ll 1$ and work in the regime where $\varepsilon = O(\mu^{\frac{2}{5}})$. Since we expect our final model equations to be cubic nonlinear, a higher-order approximation (in ε and μ) is needed. Thus it is natural to post the Kodama transformation of the form

$$\eta = u + \varepsilon A + \mu B + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \varepsilon^3 K + \varepsilon^2\mu G + \varepsilon\mu^2 H, \tag{3.1}$$

where A, B, C, D, E, H, K and G are the parameters which are related to u and its derivatives but independent of ε and μ. Doing so allows enough degree of freedom in the expansion that may later be optimized. For example, to obtain the CH-type terms, as described in [32], one can choose $A = \lambda_1 u^2$, $B = \lambda_2 u_{xx}$ and $K = \lambda_0 u^4$, where $\lambda_0,\ \lambda_1$ and λ_2 are some constants to be determined later. With such a choice, (3.1) becomes

$$\eta = u + \lambda_1\varepsilon u^2 + \lambda_2\mu u_{xx} + \varepsilon\mu C + \mu^2 D + \varepsilon^2 E + \lambda_0\varepsilon^3 u^4 + \varepsilon^2\mu G + \varepsilon\mu^2 H. \tag{3.2}$$

To proceed, we will substitute the Kodama transformation (3.2) into (2.11). The resulting equation will purely consists of u-terms. Collecting at each order we have

$$\begin{aligned}
O_0(1) &:= 2(u_x + u_t), \qquad\qquad O_0(\varepsilon) := 4\lambda_1\varepsilon(uu_x + uu_t) + 3\varepsilon uu_x,\\
O_0(\varepsilon^2) &:= 2\varepsilon^2(E_x + E_t) + 9\lambda_1\varepsilon^2 u^2 u_x + A_1\varepsilon^2 u^2 u_x,\\
O_0(\varepsilon^3) &:= 3\varepsilon^3(uE)_x + 6\lambda_1^2\varepsilon^3 u^3 u_x + A_1\lambda_1\varepsilon^3(u^4)_x + A_2\varepsilon^3 u^3 u_x + 2\lambda_0\varepsilon^3((u^4)_x + (u^4)_t),\\
O_0(\varepsilon^4) &:= \varepsilon^4\left(\lambda_0 + \lambda_1 + A_1\lambda_1^2 + A_2\lambda_1 + \frac{A_8}{5}\right)(u^5)_x + A_1\varepsilon^4(u^2E)_x\\
O_0(\mu) &:= 2\lambda_2\mu(u_{xxx} + u_{xxt}) + \frac{1}{3}\mu u_{xxx}, \qquad O_0(\mu^2) := 2\mu^2(D_x + D_t) + \frac{\lambda_2}{3}\mu^2 u_{xxxxx},\\
O_0(\varepsilon\mu) &:= 2\varepsilon\mu(C_x + C_t) + (2\lambda_1 + 3\lambda_2 + A_3)\varepsilon\mu u_x u_{xx} + \left(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4\right)\varepsilon\mu uu_{xxx},\\
O_0(\varepsilon^2\mu) &:= \frac{1}{3}\varepsilon^2\mu E_{xxx} + 2\varepsilon^2\mu(G_x + G_t) + 3\varepsilon^2\mu(uC)_x + 3\lambda_2\lambda_1\varepsilon^2\mu(u^2u_{xx})_x + \lambda_2 A_1\varepsilon^2\mu(u^2u_{xx})_x\\
&\quad + 2\lambda_1 A_3\varepsilon^2\mu(uu_x^2)_x + A_4\lambda_1\varepsilon^2\mu u^2 u_{xxx} + \lambda_1 A_4\varepsilon^2\mu u(u^2)_{xxx}\\
&\quad + A_5\varepsilon^2\mu uu_x u_{xx} + A_6\varepsilon^2\mu u^2 u_{xxx} + A_7\varepsilon^2\mu u_x^3,\\
O_0(\varepsilon\mu^2) &:= \frac{1}{3}\varepsilon\mu^2 C_{xxx} + 2\varepsilon\mu^2(H_x + H_t) + 3\varepsilon\mu^2(uD)_x + 3\lambda_2^2\varepsilon\mu^2 u_{xx}u_{xxx}\\
&\quad + A_3\lambda_2\varepsilon\mu^2(u_x u_{xxx})_x + A_4\lambda_2\varepsilon\mu^2 u_{xx}u_{xxx} + A_4\lambda_2\varepsilon\mu^2 uu_{xxxxx},\\
O_0(\varepsilon^2\mu^2) &:= \frac{1}{3}\varepsilon^2\mu^2 G_{xxx} + 3\lambda_1\varepsilon^2\mu^2(u^2D)_x + 3\lambda_2\varepsilon^2\mu^2(u_{xx}C)_x + 3\varepsilon^2\mu^2(Hu)_x\\
&\quad + \lambda_2^2 A_1\varepsilon^2\mu^2(uu_{xx}^2)_x + A_1\varepsilon^2\mu^2(u^2D)_x + A_3\varepsilon^2\mu^2(u_xC_x)_x + 2A_3\lambda_1\lambda_2\varepsilon^2\mu^2(uu_xu_{xxx})_x\\
&\quad + A_4\varepsilon^2\mu^2 Cu_{xxx} + A_4\lambda_1\lambda_2\varepsilon^2\mu^2u^2u_{xxxxx} + A_4\lambda_1\lambda_2\varepsilon^2\mu^2u_{xx}(u^2)_{xxx} + A_4\varepsilon^2\mu^2 uC_{xxx}\\
&\quad + A_5\lambda_2\varepsilon^2\mu^2 uu_x u_{xxxx} + A_5\lambda_2\varepsilon^2\mu^2 u_x u_{xx}^2 + A_5\lambda_2\varepsilon^2\mu^2 uu_{xx}u_{xxx}\\
&\quad + A_6\lambda_2\varepsilon^2\mu^2 u^2 u_{xxxxx} + 2A_6\lambda_2\varepsilon^2\mu^2 uu_{xx}u_{xxx} + 3A_7\lambda_2\varepsilon^2\mu^2 u_x^2 u_{xxx}.
\end{aligned}$$

And this yields the following equation

$$u_t + u_x + \frac{1}{2}\left[O_0(\varepsilon) + O_0(\varepsilon^2) + O_0(\varepsilon^3) + O_0(\mu^2) + O_0(\mu) + O_0(\varepsilon\mu) + O_0(\varepsilon\mu^2)\right] = 0 + O(\varepsilon^3\mu, \varepsilon^2\mu^2, \mu^3).$$

Here the subscript in O_0 is just to emphasize that the terms may change at each step.

The next step is to eliminate the t derivatives using the equation itself. As before, we expand the time derivatives, namely

$$\begin{aligned} u_t =& - u_x - 2\lambda_1\varepsilon(uu_x + uu_t) - \frac{3}{2}\varepsilon uu_x \\ & - \frac{1}{2}\big[O_0(\varepsilon^2) + O_0(\varepsilon^3) + O_0(\mu^2) + O_0(\mu) + O_0(\varepsilon\mu) + O_0(\varepsilon\mu^2)\big] + O(\varepsilon^3\mu, \varepsilon^2\mu^2, \mu^3). \end{aligned} \tag{3.3}$$

In order to have the whole $\varepsilon^2\mu^2$-order terms, we need to bring μ^2 and $\varepsilon\mu^2$-order terms back even though they will be ignored as high-order at the end.

Step 1. At order ε, we substitute (3.3) into $\frac{1}{2}O_0(\varepsilon)$, and it gives

$$\begin{aligned} 2\lambda_1\varepsilon(uu_x + uu_t) + \frac{3}{2}\varepsilon uu_x =& \frac{3}{2}\varepsilon uu_x - \frac{4}{3}\lambda_1^2\varepsilon^2((u^3)_x + (u^3)_t) - \lambda_1\varepsilon^2(u^3)_x \\ & - \lambda_1\varepsilon u\big[O_0(\varepsilon^2) + O_0(\varepsilon^3) + O_0(\mu) + O_0(\varepsilon\mu) + O_0(\mu^2) + O_0(\varepsilon\mu^2)\big]. \end{aligned} \tag{3.4}$$

This expansion generates higher order terms. It leads to the following terms in asymptotic order:

$$\begin{aligned} O_1(\varepsilon) &:= \frac{3}{2}\varepsilon uu_x, & O_1(\varepsilon^2) &:= \frac{1}{2}O_0(\varepsilon^2) - \frac{4}{3}\lambda_1^2\varepsilon^2((u^3)_x + (u^3)_t) - \lambda_1\varepsilon^2(u^3)_x, \\ O_1(\varepsilon^3) &:= \frac{1}{2}O_0(\varepsilon^3) - \lambda_1\varepsilon uO_0(\varepsilon^2), & O_1(\varepsilon^4) &:= \frac{1}{2}O_0(\varepsilon^4) - \lambda_1\varepsilon uO_0(\varepsilon^3) \\ O_1(\varepsilon\mu) &:= \frac{1}{2}O_0(\varepsilon\mu) - \lambda_1\varepsilon uO_0(\mu), & O_1(\mu) &:= \frac{1}{2}O_0(\mu), \\ O_1(\mu^2) &:= \frac{1}{2}O_0(\mu^2), & O_1(\varepsilon^2\mu) &:= \frac{1}{2}O_0(\varepsilon^2\mu) - \lambda_1\varepsilon uO_0(\varepsilon\mu), \\ O_1(\varepsilon\mu^2) &:= \frac{1}{2}O_0(\varepsilon\mu^2) - \lambda_1\varepsilon uO_0(\mu^2), & O_1(\varepsilon^2\mu^2) &:= \frac{1}{2}O_0(\varepsilon^2\mu^2) - \lambda_1\varepsilon uO_0(\varepsilon\mu^2). \end{aligned}$$

Step 2. For $O_1(\varepsilon^2)$ term, we can choose $E = \lambda_3 u^3$. Then we expand the time derivatives as

$$u_t = -u_x - 2\lambda_1\varepsilon(uu_x + uu_t) - \frac{3}{2}\varepsilon uu_x - \frac{1}{2}\big[O_0(\mu) + O_0(\mu^2) + O_0(\varepsilon^2)\big] + O(\varepsilon^2, \varepsilon\mu).$$

Hence the $O_1(\varepsilon^2)$-order term takes the following form,

$$\begin{aligned} O_1(\varepsilon^2) = & \left(\frac{1}{2}\lambda_1 + \frac{A_1}{6}\right)\varepsilon^2(u^3)_x - (6\lambda_3 - 8\lambda_1^2)\lambda_1\varepsilon^3u^2(uu_x + uu_t) - \left(\frac{9}{2}\lambda_3 - 6\lambda_1^2\right)\varepsilon^3u^3u_x \\ & - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2u^2\big[O_0(\varepsilon^2) + O_0(\mu) + O_0(\mu^2)\big]. \end{aligned}$$

We now denote

$$I_{f(u)} := \text{ the coefficient of } f(u).$$

Then coefficient of u^2u_x is given by

$$I_{u^2u_x} := \frac{3}{2}\lambda_1 + \frac{A_1}{2}. \tag{3.5}$$

And the following terms in asymptotic order take the form

$$O_2(\mu) := O_1(\mu), \qquad O_2(\varepsilon^2) := \left(\frac{3}{2}\lambda_1 + \frac{A_1}{2}\right)\varepsilon^2 u^2 u_x, \qquad O_2(\varepsilon\mu) := O_1(\varepsilon\mu)$$
$$O_2(\varepsilon^3) := O_1(\varepsilon^3) - (6\lambda_3 - 8\lambda_1^2)\lambda_1\varepsilon^3 u^2(uu_x + uu_t) - \left(\frac{9}{2}\lambda_3 - 6\lambda_1^2\right)\varepsilon^3 u^3 u_x,$$
$$O_2(\varepsilon^4) := O_1(\varepsilon^4) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\varepsilon^2),$$
$$O_2(\mu^2) := O_1(\mu^2), \qquad O_2(\varepsilon^2\mu) := O_1(\varepsilon^2\mu) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\mu),$$
$$O_2(\varepsilon\mu^2) := O_1(\varepsilon\mu^2), \qquad O_2(\varepsilon^2\mu^2) := O_1(\varepsilon^2\mu^2) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\mu^2).$$

Now the equation has the form of
$$u_t + u_x + O_1(\varepsilon) + O_2(\varepsilon^2) + O_2(\mu) + O_2(\varepsilon\mu) + O_2(\varepsilon^2\mu) + O_2(\varepsilon^4) = 0 + O(\varepsilon^3\mu, \mu^3),$$
and the expression for u_t is given by
$$\begin{aligned} u_t = &- u_x - \frac{3}{2}\varepsilon uu_x - \left(\frac{1}{2}\lambda_1 + \frac{A_1}{6}\right)\varepsilon^2(u^3)_x - \frac{1}{2}O_0(\mu) - \frac{1}{2}O_0(\varepsilon\mu) + \lambda_1\varepsilon u O_0(\mu) \\ &- \frac{1}{2}O_0(\varepsilon^2\mu) + \lambda_1\varepsilon u O_0(\varepsilon\mu) + \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\mu) + O(\varepsilon^2, \mu^2). \end{aligned} \tag{3.6}$$

Step 3. We now consider $O_2(\mu)$ term. Here, another parameter is required. To this end, splitting the time derivative $\lambda_2\mu u_{xxt}$, it appears that
$$\lambda_2\mu u_{xxt} = \lambda_2(1-\nu)\mu u_{xxt} + \lambda_2\nu\mu u_{xxt}, \tag{3.7}$$
where ν is the new parameter which will be determined later. We remove the u_{xxt} term by eliminating the t derivatives using (3.6). Thereby, it yields
$$\lambda_2\nu\mu u_{xxt} = -\lambda_2\nu\mu u_{xxx} - \frac{3}{2}\lambda_2\nu\varepsilon\mu(uu_x)_{xx} + \lambda_2\nu\mu(F_{\varepsilon^2} + F_\mu + F_{\varepsilon\mu} + F_{\varepsilon^2\mu})_{xx} + O(\varepsilon^3\mu, \varepsilon\mu^3),$$
where we define
$$F_{\varepsilon^2} := -\left(\frac{1}{2}\lambda_1 + \frac{A_1}{6}\right)\varepsilon^2(u^3)_x, \qquad F_\mu := -\lambda_2\mu(u_{xxx} + u_{xxt}) - \frac{1}{6}\mu u_{xxx},$$
$$F_{\varepsilon\mu} := -\frac{1}{2}O_0(\varepsilon\mu) + \lambda_1 u\varepsilon O_0(\mu), \qquad F_{\varepsilon^2\mu} := -\frac{1}{2}O_0(\varepsilon^2\mu) + \lambda_1 u\varepsilon O_0(\varepsilon\mu) + \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\mu).$$

This way $O_2(\mu)$ takes the form
$$\begin{aligned} \lambda_2\mu(u_{xxx} + u_{xxt}) + \frac{1}{6}\mu u_{xxx} = &\left(\lambda_2(1-\nu) + \frac{1}{6}\right)\mu u_{xxx} + \lambda_2(1-\nu)\mu u_{xxt} - \frac{3}{2}\nu\lambda_2\varepsilon\mu(3u_x u_{xx} + uu_{xxx}) \\ &+ \lambda_2\nu\mu(F_{\varepsilon^2} + F_\mu + F_{\varepsilon\mu} + F_{\varepsilon^2\mu})_{xx}. \end{aligned}$$

The coefficient of u_{xxt} can be written as
$$I_{u_{xxt}} := \lambda_2(1-\nu).$$

This procedure leads to the following terms in asymptotic order:
$$O_3(\mu) := (\lambda_2(1-\nu) + \frac{1}{6})\mu u_{xxx} + \lambda_2(1-\nu)\mu u_{xxt}, \quad O_3(\varepsilon^3) := O_2(\varepsilon^3),$$
$$O_3(\varepsilon\mu) := O_2(\varepsilon\mu) - \frac{3}{2}\nu\lambda_2\varepsilon\mu(3u_x u_{xx} + uu_{xxx}), \qquad O_3(\varepsilon^4) := O_2(\varepsilon^4)$$
$$O_3(\varepsilon^2\mu) := O_2(\varepsilon^2\mu) + \lambda_2\nu\mu(F_{\varepsilon^2})_{xx}, \qquad O_3(\mu^2) := O_2(\mu^2) + \lambda_2\nu\mu(F_\mu)_{xx}$$

$$O_3(\varepsilon\mu^2) := O_2(\varepsilon\mu^2) + \lambda_2\nu\mu(F_{\varepsilon\mu})_{xx}, \qquad O_3(\varepsilon^2\mu^2) := O_2(\varepsilon^2\mu^2) + \lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx}.$$

Proceeding systematically, we continue to compute the $O_3(\varepsilon\mu)$ terms, $O_4(\varepsilon^3)$, $O_5(\varepsilon^4)$, and finally all the $\varepsilon^2\mu$-order terms generated in the asymptotic expansions. For the purpose of keeping the presentation simple, the details of the computation are provided in Appendix B.

3.2. **The special form of the CH-mCH-Novikov equation.** Having obtained the asymptotic expansion up to sufficiently high order, we are ready to turn to the procedure of deriving equation (1.6). Notice that this requires choosing specific values of the parameters in the Kodama transformation, which can be determined through the following procedure.

Note that the CH-type equation requires

$$I_{u_{xxt}} = -\beta, \qquad I_{uu_{xxx}} = -\frac{\beta}{2}, \qquad I_{u_x u_{xx}} = -\beta$$

for some parameter β. It is determined that $\beta = \frac{5}{12}$ and $\lambda_1, \lambda_2, \nu$ are given by

$$\begin{cases} \lambda_2(1-\nu) = -\beta, \\ \lambda_1 + \frac{3}{2}(1-3\nu)\lambda_2 + \frac{A_3}{2} = -\beta. \end{cases} \tag{3.8}$$

On the other hand, equation (1.6) requires that

$$I_{u^2u_x} = \frac{1}{4}(3k_1 + 4k_2), \quad I_{u^2u_{xxx}} = -\frac{1}{4}\beta(k_1 + k_2).$$

Therefore from (3.5) and (B.1)

$$\begin{cases} \frac{3}{2}\lambda_1 + \frac{A_1}{2} = \frac{1}{4}(3k_1 + 4k_2), \\ \frac{3}{2}(1-\nu)\lambda_1\lambda_2 + \frac{A_1}{2}(1-\nu)\lambda_2 + \frac{1}{2}A_4\lambda_1 + \frac{1}{2}A_6 = -\frac{1}{4}\beta(k_1 + k_2). \end{cases} \tag{3.9}$$

where $A_1 = -3/4$, $A_3 = 23/12$, $A_4 = 5/6$,$A_6 = 3/4$. Combining this with (3.8) we have

$$\lambda_1 = \frac{k_1}{2} + \frac{189}{20}, \quad \lambda_2 = \frac{k_1}{6} + \frac{179}{60}, \quad k_2 = \frac{69}{5}, \quad \nu = \frac{10k_1 + 204}{10k_1 + 179}, \tag{3.10}$$

where $k_1 \in \mathbb{R}$ is arbitrarily. The coefficients of $(u^4)_x$ and $(u^5)_x$ must vanish for equation (1.6) to emerge, and hence from (B.2) and (B.3)

$$I_{(u^4)_x} = \frac{3}{8}\lambda_3 + \frac{1}{8}A_2 + \frac{1}{4}\lambda_1 A_1 = 0,$$
$$I_{(u^5)_x} = \frac{1}{10}\left(-19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3\right) = 0,$$

where $A_2 = 3/8$ and $A_8 = 115/192$. Then it gives

$$\lambda_3 = \frac{k_1}{4} + \frac{23}{5} \quad \text{and} \quad \lambda_0 = \frac{3}{19}k_1^3 + \frac{13083}{1520}k_1^2 + \frac{1189081}{7600}k_1 + \frac{108125767}{114000}. \tag{3.11}$$

Also, for other terms, we require from (B.5) that

$$I_{u_x^3} = -\frac{1}{4}\beta k_1, \qquad I_{uu_xu_{xx}} = -\frac{1}{4}\beta(4k_1 + 3k_2).$$

With this choice, it in turn implies that

$$\begin{cases} \lambda_3 - \frac{3}{2}\lambda_4 + \frac{1}{2}A_7 - \lambda_2\nu(3\lambda_1 + A_1) + A_3\lambda_1 = -\frac{1}{4}\beta k_1, \\ 3\lambda_3 - 3\lambda_5 - 9(1-\nu)\lambda_2\lambda_1 + A_1(1-3\nu)\lambda_2 + (A_3 + 3A_4)\lambda_1 + \frac{1}{2}A_5 - 2\lambda_1^2 = -\frac{1}{4}\beta(4k_1 + 3k_2), \end{cases}$$

where $A_5 = 29/8$, $A_7 = 23/16$. Then we obtain

$$\lambda_4 = -\frac{1}{6}k_1^2 - \frac{671}{120}k_1 - \frac{56327}{1200} \quad \text{and} \quad \lambda_5 = -\frac{1}{6}k_1^2 - \frac{67}{15}k_1 - \frac{30437}{1200}. \tag{3.12}$$

This way ν and λ_i $(i = 1, \dots, 5)$ are obtained in terms of $k_2 = \frac{69}{5}$ and any k_1.

Lastly, the coefficients of $\varepsilon^2\mu^2$-order terms should satisfy that:

$$I_{u^2 u_{xxxxx}} = I_{uu_x u_{xxxx}} = I_{uu_{xx}u_{xxx}} = 0, \quad I_{u_x^2 u_{xxx}} = \frac{k_1}{4}\beta^2, \quad I_{u_x u_{xx}^2} = k_1\frac{1}{2}\beta^2.$$

Since the coefficient of the term $u^2 u_{xxxxx}$ needs to be zero, from (B.6) it follows that

$$-\frac{A_4}{2}\lambda_1\beta - \frac{A_6}{2}\beta - \lambda_2\lambda_3 + \frac{4}{3}\lambda_1^2\lambda_2 - \frac{1}{6}\lambda_1\lambda_5 = 0,$$

where $\beta = \frac{5}{12}$ and λ_i $(i = 1, 2, 3)$ only depend on k_1. This way the parameter k_1 should be a real root of the following equation

$$2000k_1^3 + 106200k_1^2 + 1871550k_1 + 10934031 = 0. \tag{3.13}$$

And then $k_1 \simeq -15.1765$. Notice that since the determinant of the matrix in (B.7) is nonzero, we can obtain $\lambda_7,\ \lambda_8,\ \lambda_9,\ \lambda_{10}$ for any parameters λ_6 and λ_{11}.

In summary, if we take the Kodama transformation to be

$$\begin{aligned} \eta = u &+ \lambda_1\varepsilon u^2 + \lambda_2\mu u_{xx} + \varepsilon\mu(\lambda_4 u_x^2 + \lambda_5 uu_{xx}) + \varepsilon^2\lambda_3 u^3 + \varepsilon^3\lambda_0 u^4 + \mu^2(\lambda_6 u_{xxxx}) \\ &+ \varepsilon^2\mu(\lambda_7 uu_x^2 + \lambda_8 u^2 u_{xx}) + \varepsilon\mu^2(\lambda_9 u_x u_{xxx} + \lambda_{10} uu_{xxxx} + \lambda_{11} u_{xx}^2), \end{aligned} \tag{3.14}$$

where the parameters satisfy conditions (3.10)–(3.13) and λ_6, λ_{11} can be any real number, then we arrive at (1.6).

4. Other related nonlinear equations with cubic nonlinearity

Using the method as in Section 3, other shallow-water models can be derived when we choose suitable parameters in the Kodama transformation. In particular, the CH-Novikov equation and a new cubic nonlinear peakon equation will be derived in this section. Moreover, after certain rescaling, the mCH-Novikov and Novikov equations can also be obtained.

The CH-Novikov equation. Consider the same form of Kodama transformation as before. Now we impose $I_{u^2 u_x} = k_2$, $I_{u^2 u_{xxx}} = -\dfrac{1}{4}\beta k_2$, recalling (3.5) and (B.1), that is,

$$\begin{cases} \frac{3}{2}\lambda_1 + \frac{A_1}{2} = k_2 \\ \frac{3}{2}(1-\nu)\lambda_1\lambda_2 + \frac{A_1}{2}(1-\nu)\lambda_2 + \frac{1}{2}A_4\lambda_1 + \frac{1}{2}A_6 = -\frac{1}{4}\beta k_2, \end{cases} \tag{4.1}$$

where $A_1 = -3/4$, $A_3 = 23/12$, $A_4 = 5/6$, $A_6 = 3/4$. Together with (3.8), it follows that

$$\lambda_1 = \frac{189}{20}, \quad \lambda_2 = \frac{179}{60}, \quad k_2 = \frac{69}{5}, \quad \nu = \frac{204}{179}.$$

Setting $I_{(u^4)_x} = I_{(u^5)_x} = I_{u_x^3} = 0$ and $I_{uu_x u_{xx}} = -\frac{3}{4}k_2\beta$ with $\beta = \frac{5}{12}$, and using (B.2), (B.3) and (B.5), we know that

$$\lambda_3 = \frac{23}{5}, \qquad \lambda_0 = \frac{108125767}{114000}, \qquad \lambda_4 = -\frac{56327}{1200}, \qquad \lambda_5 = -\frac{30437}{1200}.$$

Then

$$\eta = u + \frac{189}{20}\varepsilon u^2 + \frac{179}{60}\mu u_{xx} - \varepsilon\mu\left(\frac{56327}{1200}u_x^2 + \frac{30437}{1200}uu_{xx}\right) + \frac{23}{5}\varepsilon^2 u^3 + \frac{108125767}{114000}\varepsilon^3 u^4.$$

These choices give the so-called CH-Novikov equation which takes the form of

$$\begin{aligned} u_t + u_x - \beta\mu u_{xxt} - \frac{1}{4}\mu u_{xxx} + \frac{3}{2}\varepsilon uu_x - \varepsilon\mu\frac{1}{2}\beta(2u_x u_{xx} + uu_{xxx}) + k_2\varepsilon^2 u^2 u_x \\ - \frac{3}{4}k_2\beta\varepsilon^2\mu uu_x u_{xx} - \frac{1}{4}k_2\beta\varepsilon^2\mu u^2 u_{xxx} = 0 + O(\varepsilon^5, \mu^2). \end{aligned} \tag{4.2}$$

The mCH-Novikov and Novikov equations. Applying the scaling transformation

$$u \to 2\varepsilon^{-1}u, \quad t \to (\beta\mu)^{-\frac{1}{2}}t, \quad x \to (\beta\mu)^{-\frac{1}{2}}x,$$

to equation (1.6) leads to the equation

$$m_t + u_x - \frac{3}{5}u_{xxx} + 2u_x m + u m_x + k_1((u^2 - u_x^2)m)_x + k_2(u^2 m_x + 3uu_x m) = 0. \tag{4.3}$$

If we further scale $t \to \delta^{-2}t$ and $u \to \delta^{-1}u$, then (4.3) takes the form of

$$\delta^{-2}m_t + u_x - \frac{3}{5}u_{xxx} + \delta^{-1}(2u_x m + u m_x) + k_1\delta^{-2}((u^2 - u_x^2)m)_x + k_2\delta^{-2}(u^2 m_x + 3uu_x m) = 0.$$

Rewriting it as

$$m_t + \delta^2 u_x - \delta^2\frac{3}{5}u_{xxx} + \delta(2u_x m + u m_x) + k_1((u^2 - u_x^2)m)_x + k_2(u^2 m_x + 3uu_x m) = 0 \tag{4.4}$$

and taking $\delta \to 0$, then formally in the limit function $u(t,x)$ satisfies the mCH-Novikov equation

$$m_t + k_1[(u^2 - u_x^2)m]_x + k_2(u^2 m_x + 3uu_x m) = 0. \tag{4.5}$$

Similarly, the Novikov equation can be obtained, viz.,

$$m_t + k_2(u^2 m_x + 3uu_x m) = 0. \tag{4.6}$$

Indeed under a further scaling in time $t \mapsto k_2 t$ the above equation becomes exactly the Novikov equation [40].

A cubic CH-type equation. Choose $I_{u^2u_x} = \frac{3}{4}k_3,\ I_{u^2u_{xxx}} = -\frac{1}{8}\beta k_3$, namely,

$$\begin{cases} \frac{3}{2}\lambda_1 + \frac{A_1}{2} = \frac{3}{4}k_3, \\ \frac{3}{2}(1-\nu)\lambda_1\lambda_2 + \frac{A_1}{2}(1-\nu)\lambda_2 + \frac{1}{2}A_4\lambda_1 + \frac{1}{2}A_6 = -\frac{1}{8}k_3\beta. \end{cases} \tag{4.7}$$

As a CH-type model, it should satisfy (3.8). Then we have

$$\lambda_1 = \frac{97}{20}, \quad \lambda_2 = \frac{29}{20}, \quad k_3 = \frac{46}{5}, \quad \nu = \frac{112}{87}.$$

For other terms, we choose $I_{(u^4)_x} = I_{(u^5)_x} = 0,\ I_{u_x^3} = -\frac{1}{8}\beta k_3,\ I_{uu_xu_{xx}} = -\frac{1}{2}\beta k_3$. It then gives that

$$\begin{cases} \frac{3}{8}\lambda_3 + \frac{1}{8}A_2 + \frac{1}{4}\lambda_1 A_1 = 0, \\ -19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3 = 0 \\ \lambda_3 - \frac{3}{2}\lambda_4 + \frac{1}{2}A_7 - \lambda_2\nu(3\lambda_1 + A_1) + A_3\lambda_1 = -\frac{1}{8}k_3\beta, \\ 3\lambda_3 - 3\lambda_5 - 9(1-\nu)\lambda_2\lambda_1 + A_1(1-3\nu)\lambda_2 + (A_3 + 3A_4)\lambda_1 + \frac{1}{2}A_5 - 2\lambda_1^2 = -\frac{1}{2}\beta k_3, \end{cases} \tag{4.8}$$

where $A_2 = 3/8$, $A_5 = 29/8$, $A_7 = 23/16$. Hence, we obtain

$$\lambda_0 = \frac{13067089}{114000},\ \lambda_3 = \frac{23}{10}, \quad \lambda_4 = -\frac{10373}{1200}, \quad \text{and } \lambda_5 = \frac{1261}{600}.$$

Then

$$\eta = u + \frac{97}{20}\varepsilon u^2 + \frac{29}{20}\mu u_{xx} + \varepsilon\mu\left(\frac{1261}{600}uu_{xx} - \frac{10373}{1200}u_x^2\right) + \frac{23}{10}\varepsilon^2 u^3 + \frac{13067089}{114000}\varepsilon^3 u^4.$$

It then follows that

$$m_t + u_x - \frac{1}{4}\mu u_{xxx} + \frac{1}{2}\varepsilon(2u_x m + u m_x) + \frac{1}{4}k_3\varepsilon^2\left((u^2 - \frac{1}{4}\beta\mu(u^2)_{xx})u\right)_x = 0 + O(\varepsilon^5, \mu^2). \tag{4.9}$$

Applying the scaling transformation

$$u \to 2\varepsilon^{-1}u, \quad t \to (\beta\mu)^{-\frac{1}{2}}t, \quad x \to (\beta\mu)^{-\frac{1}{2}}x,$$

the equation becomes

$$m_t + u_x - \frac{3}{5}u_{xxx} + (2u_x m + u m_x) + k_3\left((u^2 - \frac{1}{4}(u^2)_{xx})u\right)_x = 0. \tag{4.10}$$

5. Curvature blow-up

Having derived the model equations in Section 3 and Section 4, our attention is now turned to the blow-up analysis. In particular, as explained in the Introduction, we will consider the Cauchy problem for the mCH-Novikov equation (1.11), with $k_1, k_2 \in \mathbb{R}$.

It can be shown that the following two functionals are conserved quantities for (1.11)

$$H_1[u] = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx, \qquad H_2[u] = \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) dx. \tag{5.1}$$

The local well-posedness theory can be obtained following the standard argument of [27] with a slight modification.

Theorem 5.1. *Let $u_0 \in H^s$ with $s > \frac{5}{2}$. Then there exists a time $T > 0$ such that the Cauchy problem* (1.11) *has a unique strong solution $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$.*

It is also shown in [8] that equation (1.11) posseses the single and multi-peakon solutions. Moreover the single peakons are indeed orbitally stable in H^1.

5.1. Blow-up criterion.

Similar to the other CH-type equations, (1.11) can be reformulated into a nonlocal transport form. Therefore from standard transport theory, a Beale–Kato–Majda type of blow-up criterion can be obtained. A further refined analysis leads to the following lemma. The proof of this result follows a similar idea as in [29], and hence we will omit it for the brevity of the presentation.

Lemma 5.1. *Let $u_0 \in H^s$ with $s > \frac{5}{2}$ and u be the corresponding solution to* (1.11)*. Assume that $T^*_{u_0} > 0$ is the maximum time of existence. Then*

$$T^*_{u_0} < \infty \;\Rightarrow\; \int_0^{T^*_{u_0}} \|k_1 m u_x(\tau) + 2k_2 u u_x(\tau)\|_{L^\infty}\, d\tau = \infty. \tag{5.2}$$

Remark 5.1. The blow-up criterion (5.2) implies that the lifespan $T^*_{u_0}$ does not depend on the regularity index s of the initial data u_0.

As usual, now we proceed to obtain an improved blow-up criterion which is in some sense "pointwise".

Lemma 5.2. *Suppose that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then the corresponding solution u to the Cauchy problem* (1.11) *blows up in finite time $T^* > 0$ if and only if*

$$\liminf_{t\to T^*} \inf_{x\in\mathbb{R}} \left\{k_1 m(t,x)u_x(t,x) + 2k_2 u(t,x)u_x(t,x)\right\} = -\infty. \tag{5.3}$$

Proof. In view of Remark 5.1, it suffices to consider the case $s = 3$. Suppose that if $k_1 m u_x + k_2 u u_x$ is bounded from below on $[0,\, T^*_{u_0}) \times \mathbb{R}$, i.e., there exists a constant $K > 0$ such that

$$\left(k_1 m u_x + 2k_2 u u_x\right)(t,x) \geq -K \quad \text{on} \quad [0,\, T^*_{u_0}) \times \mathbb{R}. \tag{5.4}$$

Multiplying (1.11) by m and integrating over $\mathbb{R}$, and then integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m^2\, dx + \int_{\mathbb{R}} \left(k_1 u_x m + 2k_2 u u_x\right) m^2\, dx = 0. \tag{5.5}$$

The initial condition implies that $m_0 \in H^{s-2} \subset L^q$ for any $2 \leq q \leq \infty$. From (5.5) we see that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m^2\, dx \leq K \int_{\mathbb{R}} m^2\, dx.$$

Applying Gronwall's inequality yields that

$$\|m(t)\|_{L^2}^2 \le e^{2Kt}\|m_0\|_{L^2}^2 \qquad \text{for} \quad t \in [0, T_{u_0}^*). \tag{5.6}$$

Moreover using integration by parts and Sobolev embedding,

$$\|m(t)\|_{L^2}^2 = \int_{\mathbb{R}} \left(u^2 + u_{xx}^2 + 2u_x^2\right) \, dx \ge \|u\|_{H^2}^2 \ge \|u_x\|_{L^\infty}.$$

Similarly we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m_x^2 \, dx + k_1 \int_{\mathbb{R}} \left((u^2 - u_x^2)m\right)_{xx} m_x \, dx + k_2 \int_{\mathbb{R}} \left(u^2 m_x + 3uu_x m\right)_x m_x \, dx = 0.$$

Integrating by parts the second term yields

$$k_1 \int_{\mathbb{R}} \left[(u^2 - u_x^2)m\right]_{xx} m_x \, dx = \int_{\mathbb{R}} (5k_1 u_x m)\, m_x^2 \, dx - \int_{\mathbb{R}} \left(\frac{2}{3}k_1 u_x m\right) m^2 dx.$$

Integrating by parts the third term can be computed as

$$k_2 \int_{\mathbb{R}} \left(u^2 m_x + 3uu_x m\right)_x m_x = \int_{\mathbb{R}} (4k_2 uu_x)\, m_x^2 \, dx - \int_{\mathbb{R}} (6k_2 uu_x)\, m^2 dx - \int_{\mathbb{R}} 12k_2 u m_x m^2 dx.$$

This way we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} m_x^2 \, dx + \int_{\mathbb{R}} (5k_1 u_x m + 4k_2 uu_x)\, m_x^2 \, dx - \int_{\mathbb{R}} \left(\frac{2}{3}k_1 u_x m + 6k_2 uu_x\right) m^2 dx - \int_{\mathbb{R}} 12k_2 u m_x m^2 dx = 0.$$

So together with (5.5), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (m^2 + m_x^2) \, dx &= -\int_{\mathbb{R}} (k_1 u_x m + 2k_2 uu_x)\, m^2 \, dx - \int_{\mathbb{R}} (5k_1 u_x m + 4k_2 uu_x)\, m_x^2 \, dx \\
&\quad + \int_{\mathbb{R}} \left(\frac{2}{3}k_1 u_x m + 6k_2 uu_x\right) m^2 dx + \int_{\mathbb{R}} 12k_2 u m_x m^2 dx \\
&= -\int_{\mathbb{R}} (k_1 u_x m + 2k_2 uu_x) \left(\frac{1}{3}m^2 + 5m_x^2\right) \, dx + \int_{\mathbb{R}} k_2 uu_x \left(6m_x^2 - \frac{14}{3}m^2\right) \, dx \\
&\quad + 4\int_{\mathbb{R}} k_2 u_x m^3 dx \\
&\le \left(5K + 6|k_2|\|uu_x\|_{L^\infty} + 4|k_2|\|u\|_{H^1}\|m\|_{L^2}\right) \|m\|_{H^1}^2.
\end{aligned}$$

Applying Gronwall's inequality and (5.6) it follows that

$$\|m(t)\|_{H^1}^2 \le \exp\left(5Kt + e^{6|k_2|\|u_0\|_{H^1}\|m_0\|_{L^2}(e^{Kt}-1)/K}\right) \|m_0\|_{H^1}^2$$

for $t \in [0,\, T_{u_0}^*)$. From Theorem 5.1 this implies that the solution does not blow up in finite time.

On the other hand, if

$$\liminf_{t \uparrow T_{u_0}^*} \left[\inf_{x \in \mathbb{R}} \left(k_1 m(t,x) u_x(t,x) + 2k_2 u(t,x) u_x(t,x)\right)\right] = -\infty,$$

then either u_x or m blows up in finite time. The proof of Lemma 5.2 is hence completed. □

5.2. **Dynamics along the characteristics.** We are going to perform our blow-up analysis along the characteristics of equation (1.11). So let us define the characteristics associated to the mCH-Novikov equation (1.11) as

$$\begin{cases} q_t(t,x) = \left[k_1\left(u^2 - u_x^2\right) + k_2 u^2\right](t, q(t,x)), \\ q(0,x) = x, \end{cases} \qquad x \in \mathbb{R}, \quad t \in [0,T). \tag{5.7}$$

One can easily verify that

Proposition 5.1. *Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$, and let $T > 0$ be the maximal existence time of the strong solution u to the corresponding initial value problem* (1.11)*. Then* (5.7) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ such that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(2\int_0^t (k_1 m u_x + k_2 u u_x)(s, q(s,x))\, ds\right) > 0, \quad \forall\ (t,x) \in [0,T) \times \mathbb{R}. \tag{5.8}$$

Moreover, for all $(t,x) \in [0,T) \times \mathbb{R}$ it holds that

$$m(t, q(t,x)) = m_0(x) \exp\left(-\int_0^t (2k_1 m u_x + 3k_2 u u_x)(s, q(s,x))\, ds\right), \tag{5.9}$$

where $m_0(x) = m(0,x)$.

A direct consequence of Proposition 5.1 is that the momentum density satisfies the sign-persistence property as in the following corollary. We want to point out that such a feature proved to be the key to several qualitative results about the CH and DP equation. In that context, this invariance is related to a geometric interpretation of these model equations (see the discussion in [9, 22]), but we are not aware of such an interpretation in the general case considered in this paper. Note that the geometric structure is quite restrictive [17, 34].

Corollary 5.2. *Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Let $T > 0$ be the maximal existence time of the strong solution u to the corresponding initial value problem* (1.11)*. If $m_0(x) > 0$ for all $x \in \mathbb{R}$, then $m(t,x) > 0$ for all $(t,x) \in [0,T) \times \mathbb{R}$.*

Denote $p(x) = \frac{1}{2}e^{-|x|}$ the fundamental solution of $1 - \partial_x^2$ on $\mathbb{R}$, and define the two convolution operators p_+, p_- as

$$p_+ * f(x) = \frac{e^{-x}}{2}\int_{-\infty}^x e^y f(y) dy, \quad p_- * f(x) = \frac{e^x}{2}\int_x^\infty e^{-y} f(y) dy. \tag{5.10}$$

Then we have the relation

$$p = p_+ + p_-, \qquad p_x = p_- - p_+.$$

Now we compute the dynamics of a few important quantities along the characteristics $q(t,x_0)$. Denote $'$ the derivative $\partial_t + \left(k_1(u^2 - u_x^2) + k_2 u^2\right)\partial_x$ along the characteristics, and

$$\widehat{u}(t) := u(t, q(t,x_0)), \quad \widehat{u_x}(t) := u_x(t, q(t,x_0)), \quad \widehat{m}(t) := m(t, q(t,x_0)), \quad \widehat{M}(t) := (m u_x)(t, q(t,x_0)).$$

Lemma 5.3. *Let $u_0 \in H^s(\mathbb{R})$, $s > 5/2$. Then $u(t,x)$, $u_x(t,x)$, $m(t,x)$ and $(mu_x)(t,x)$ satisfy the following integro-differential equations*

$$\widehat{u}'(t) = -\frac{2}{3}k_1 \widehat{u_x}^3 + \left(\frac{k_1}{3} + \frac{k_2}{2}\right)\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right](t, q(t,x_0)), \tag{5.11}$$

$$\begin{aligned} \widehat{u_x}'(t) &= k_1\left(\frac{1}{3}\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right) + \frac{k_2 \widehat{u}}{2}(\widehat{u}^2 - \widehat{u_x}^2) \\ &\quad - \left(\frac{k_1}{3} + \frac{k_2}{2}\right)\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right](t, q(t,x_0)), \end{aligned} \tag{5.12}$$

$$\widehat{m}'(t) = -(2k_1 \widehat{m}\widehat{u_x} + 3k_2 \widehat{u}\widehat{u_x})\widehat{m}, \tag{5.13}$$

$$\begin{aligned} \widehat{M}'(t) &= -2k_1 \widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1 + 3k_2)\widehat{u}^2 - (6k_1 + 21k_2)\widehat{u_x}^2\right] \\ &\quad - \left(\frac{k_1}{3} + \frac{k_2}{2}\right)\widehat{m}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right](t, q(t,x_0)). \end{aligned} \tag{5.14}$$

Proof. The proof of (5.13) can be immediately obtained from the equation (1.11).

In view of (1.11), it follows that

$$u_t = -k_1 p * \left[(u^2 - u_x^2)m\right]_x - k_2 p * \left(u^2 m_x + 3uu_x m\right). \tag{5.15}$$

The structure of the right-hand side of the above equation suggests that we may recall the results from [6] and [7]. First, from [6, (3.1)] we know that

$$p * \left[(u^2 - u_x^2)m\right]_x = (u^2 - u_x^2)u_x + \frac{2}{3}u_x^3 - \frac{1}{3}\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right].$$

From [7, (3.7)] we have

$$p * \left(u^2 m_x + 3uu_x m\right) = u^2 u_x - \frac{1}{2}\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right].$$

Plugging the above two into (5.15) we obtain (5.11).

The proof of (5.12) can be proceeded the same way. Differentiating (5.15) we obtain

$$u_{xt} = -k_1 p * \left[(u^2 - u_x^2)m\right]_{xx} - k_2 p * \left(u^2 m_x + 3uu_x m\right)_x. \tag{5.16}$$

From [6, (3.2)], it follows that

$$p * \left[(u^2 - u_x^2)m\right]_{xx} = (u^2 - u_x^2)u_{xx} + \left(\frac{1}{3}u^3 - uu_x^2\right) - \frac{1}{3}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right].$$

From [7, (3.8)], we know

$$p * \left(u^2 m_x + 3uu_x m\right)_x = u^2 u_{xx} - \frac{u}{2}(u^2 - u_x^2) - \frac{1}{2}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right].$$

Therefore (5.12) is obtained by combining the above two equations.

Finally (5.14) can be derived from (5.12) and (5.13). □

5.3. **Choice of data and blow-up:** $2k_1 + 3k_2 \neq 0$**.** Note that this parameter regime is consistent with what appears in (1.6), where $k_1 \simeq -15.2$ and $k_2 = 13.8$.

The blow-up criterion (5.3) together with the conservation law $H_1[u]$ indicates two possible scenarios for the formation of singularity, namely the wave-breaking ($|u_x| \to \infty$) or curvature blow-up ($|m| \to \infty$) in finite time. Here in this section we seek data which lead to the latter one.

5.3.1. *General data.* We start by considering a general momentum density m_0 and look for the blow-up data. In this case we make use of the conservation laws $H_1[u]$ and $H_2[u]$, which will be the key to obtain the convolution estimates. Such a control of the nonlocal terms allows us to propagate certain monotonicity property that can lead to a Riccati dynamics.

$$\frac{1}{3}\|u_x\|_{L^4}^4 = \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2\right) dx - H_2[u_0] \leq 2\|u\|_{L^\infty}^2 H_1[u_0] - H_2[u_0] \leq H_1^2[u_0] - H_2[u_0].$$

Therefore

$$\|u_x\|_{L^4}^4 \leq 3\left(H_1^2[u_0] - H_2[u_0]\right). \tag{5.17}$$

Therefore the convolution estimates follow as

$$\begin{aligned} \left|p_\pm * (u \mp u_x)^3\right| &\leq \|p_\pm\|_{L^\infty} \left\|(u \mp u_x)^3\right\|_{L^1} \leq 2\left(\|u\|_{L^3}^3 + \|u_x\|_{L^3}^3\right) \\ &\leq \sqrt{2H_1^{3/2}[u_0]} + 2\sqrt{3H_1[u_0](H_1^2[u_0] - H_2[u_0])} =: K. \end{aligned} \tag{5.18}$$

The blow-up result in this section is the following.

Theorem 5.3. *Suppose $k_1 < 0$ and $-\frac{2}{3}k_1 < k_2 < -2k_1$. Let $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$. Assume that there exists an $x_0 \in \mathbb{R}$ and some $0 < \delta < 1$ such that*

$$\begin{aligned} & m_0(x_0) \geq -\frac{3k_2}{2k_1(1-\delta)}\sqrt{\frac{H_1[u_0]}{2}}, \quad u_0(x_0) > 0, \quad u_{0,x}(x_0) \geq \sqrt[3]{A_1}, \quad \text{and} \\ & u_0(x_0)u_{0,x}^2(x_0) \geq -\frac{2k_1+3k_2}{3(2k_1+k_2)}A_2, \end{aligned} \tag{5.19}$$

where

$$A_1 := -\frac{2k_1+3k_2}{2k_1}K, \qquad A_2 := 2K + \left(\frac{H_1[u_0]}{2}\right)^{3/2},$$

and K is given in (5.18). Then the solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1\delta m_0(x_0)u_{0,x}(x_0)}.$$

Remark 5.1. Note that clearly Theorem 5.3 applies to the case where k_1 and k_2 are obtained in Section 3 (and hence in equation (1.6)).

Proof. Plugging (5.18) in (5.11) and (5.12) we obtain that

$$\begin{aligned} \widehat{u}' &\geq -\frac{2}{3}k_1\widehat{u_x}^3 - \frac{2k_1+3k_2}{3}K, \\ \widehat{u_x}' &\geq -\left(k_1+\frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 - \frac{2k_1+3k_2}{6}\left[2K + \left(\frac{H_1[u_0]}{2}\right)^{3/2}\right]. \end{aligned}$$

Hence we know that $\widehat{u}$ is increasing when $\widehat{u_x}^3 \geq A_1$, and $\widehat{u_x}$ is increasing when

$$-\left(k_1+\frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 \geq \frac{2k_1+3k_2}{6}A_2.$$

From the assumption (5.19) we know that the above two conditions are satisfied initially. Hence a continuity argument yields that over the time of existence of solutions, $\widehat{u}(t)$ and $\widehat{u_x}(t)$ are both increasing. In particular,

$$\widehat{u}(t) \geq u_0(x_0) > 0, \qquad \widehat{u_x}(t) \geq u_{0,x}(x_0) \geq \sqrt[3]{A_1} > 0. \tag{5.20}$$

Recall that $\widehat{m}$ satisfies $\widehat{m}' = -\widehat{u_x}\widehat{m}(2k_1\widehat{m} + 3k_2\widehat{u})$. At the initial time we see from (5.19) that

$$2k_1\widehat{m}(0) \leq -\frac{3k_2}{1-\delta}\sqrt{\frac{H_1[u_0]}{2}} \leq -\frac{3k_2}{1-\delta}\widehat{u}(0) < 0,$$

and hence $2k_1\widehat{m}(0) + 3k_2\widehat{u}(0) < 0$. Together with (5.20) we see that $\widehat{m}(t)$ increases initially. Then a continuity argument ensures that $\widehat{m}$ increases (and hence is positive) over some time interval $[0, t_*]$ for $t_* > 0$. Therefore on $[0, t_*]$

$$2k_1\widehat{m}(t) \leq 2k_1\widehat{m}(0) \leq -\frac{3k_2}{1-\delta}\sqrt{\frac{H_1[u_0]}{2}} \leq -\frac{3k_2}{1-\delta}\widehat{u}(t),$$

leading to

$$2k_1\widehat{m}(t) + 3k_2\widehat{u}(t) \leq 2k_1\delta\widehat{m}(t) < 0 \quad \text{on} \quad [0, t_*].$$

Thus another application of the continuity argument yields that $\widehat{m}$ increases over the entire time of existence, and the dynamics of $\widehat{m}(t)$ gives

$$\widehat{m}' = -\widehat{u_x}\widehat{m}(2k_1\widehat{m} + 3k_2\widehat{u}) \geq -2k_1\delta\widehat{u_x}\widehat{m}^2 \geq -2k_1\delta u_{0,x}(x_0)\widehat{m}^2.$$

Hence $\widehat{m}(t)$ blows up to $+\infty$ in finite time with an estimate on the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1\delta m_0(x_0)u_{0,x}(x_0)}.$$

Since $\widehat{u_x}(t) \geq u_{0,x}(x_0) > 0$ and $\widehat{u}(t)$ is bounded, we see that in fact

$$k_1\widehat{m}(t)\widehat{u_x}(t) + 2k_2\widehat{u}(t)\widehat{u_x}(t) \to -\infty, \quad \text{as} \quad t \to T^*.$$

Hence from Lemma 5.2 we see that the solution blows up in finite time, which completes the proof of the theorem. □

Using similar techniques but with less restrictive assumption on the initial momentum m_0 one can prove the following result when $k_1k_2 > 0$.

Corollary 5.4. *Suppose $k_1k_2 > 0$. Let $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$. Assume that there exists an $x_0 \in \mathbb{R}$ such that*

$$\begin{aligned} &m_0(x_0) > 0, \quad u_0(x_0) > 0, \quad u_0(x_0)u_{0,x}^2(x_0) \geq \frac{2k_1+3k_2}{3(2k_1+k_2)}A_2 \quad \text{and} \\ &u_{0,x}(x_0)\begin{cases} \geq \sqrt[3]{B_1}, & \text{when } k_1,k_2<0, \\ \leq -\sqrt[3]{B_1}, & \text{when } k_1,k_2>0, \end{cases} \end{aligned} \tag{5.21}$$

where

$$B_1 := \frac{2k_1+3k_2}{2k_1}K$$

and A_2 and K are given in Theorem 5.3. Then the solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1m_0(x_0)u_{0,x}(x_0)}.$$

Proof. We will sketch the argument for the proof. For simplicity we only consider the case when $k_1, k_2 < 0$. The other case can be dealt in the same way. The dynamics of $\widehat{u}$ and $\widehat{u_x}$ yield

$$\begin{aligned} \widehat{u}' &\geq -\frac{2}{3}k_1\widehat{u_x}^3 + \frac{2k_1+3k_2}{3}K, \\ \widehat{u_x}' &\geq -\left(k_1+\frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 + \frac{2k_1+3k_2}{6}\left[2K + \left(\frac{H_1[u_0]}{2}\right)^{3/2}\right]. \end{aligned}$$

Hence by a similar argument as in the proof of Theorem 5.3 we conclude that as (5.21) holds, $\widehat{u}$ and $\widehat{u_x}$ are both increasing:

$$\widehat{u}(t) \geq u_0(x_0) > 0, \qquad \widehat{u_x}(t) \geq u_{0,x}(x_0) \geq \sqrt[3]{B_1} > 0. \tag{5.22}$$

Plugging the above into the dynamics of $\widehat{m}$ and using (5.21) again indicates that $\widehat{m}$ increases (and hence is positive) over the time of existence. Therefore

$$\widehat{m}' = \widehat{u_x}(-2k_1\widehat{m}^2 - 3k_2\widehat{m}\widehat{u}) \geq -2k_1\widehat{u_x}\widehat{m}^2 \geq -2k_1u_{0,x}(x_0)\widehat{m}^2.$$

Hence $\widehat{m}(t)$ blows up to $+\infty$ in finite time with an estimate on the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1m_0(x_0)u_{0,x}(x_0)},$$

which completes the proof of the corollary. □

5.3.2. *Non-sign-changing data.* Next we will utilize the sign-persistence property, cf. Corollary 5.2, to consider data with positive momentum $m_0 \geq 0$. From the identities

$$u(t,x) = p * m(t,x), \qquad u_x(t,x) = p_x * m(t,x)$$

we have

$$u(t,x) \geq 0, \qquad u \pm u_x = 2p_\pm * m \geq 0. \tag{5.23}$$

This allows us to control the convolution terms in Lemma 5.3, and we can obtain

Theorem 5.5. *Suppose that $k_1 < 0$, $2k_1/3 < k_2 < -2k_1/9$. Let $u_0 \in H^s(\mathbb{R})$ for $s > 5/2$ and $m_0 \geq 0$. Assume that there exists some point $x_0 \in \mathbb{R}$ such that $m_0(x_0) > 0$ and*

$$u_{0,x}(x_0) \geq u_0(x_0) \cdot \max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}. \tag{5.24}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1 m_0(x_0) u_{0,x}(x_0)}.$$

Proof. From Corollary 5.2 we know that $m(t,x) \geq 0$ and $\widehat{m} > 0$. It then follows from (5.23) and Sobolev embedding that

$$\sqrt{H_1[u]/2} \geq u(t,x) \geq |u_x(t,x)| \geq 0, \qquad \widehat{u}(t) > 0. \tag{5.25}$$

Therefore u_x does not blow up, and then Lemma 5.2 indicates that it suffices to consider the quantity $M(t,x) = (mu_x)(t,x)$.

From the condition of the theorem, (5.25), and (5.14) it holds that

$$\begin{aligned} \widehat{M}' &= -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2\right] \\ &\quad - \left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{m}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right](t, q(t,x_0)) \\ &\geq -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2\right]. \end{aligned} \tag{5.26}$$

Since $\widehat{u}, \widehat{m} > 0$, it is now clear that in order to arrive at a Riccati-type inequality $\widehat{M}' \gtrsim \widehat{M}^2$, one would like to have $(2k_1+3k_2) - (6k_1+21k_2)\widehat{u_x}^2/\widehat{u}^2 \geq 0$. From the assumptions on k_1 and k_2 we see that such a condition can be written as,

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_1+3k_2}{6k_1+21k_2}, \tag{5.27}$$

which involves the competition between u and its derivative u_x along the characteristics. In particular, a finite-time blow-up of $\widehat{M}$ can be realized if the ration $|u_x/u|$ stays reasonably big along the characteristics. A quick computation shows that

$$\begin{aligned} \left(\frac{\widehat{u_x}}{\widehat{u}}\right)' &= \widehat{u}^2\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right) - \left(k_1+\frac{k_2}{2}\right)\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 + \frac{2k_1}{3}\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^4\right] \\ &\quad - \frac{2k_1+3k_2}{6\widehat{u}^2}\left[(\widehat{u}+\widehat{u_x})p_+ * (u-u_x)^3 + (\widehat{u}-\widehat{u_x})p_- * (u+u_x)^3\right] \\ &\geq \widehat{u}^2\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right) - \left(k_1+\frac{k_2}{2}\right)\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 + \frac{2k_1}{3}\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^4\right] \\ &= \frac{2k_1}{3}\widehat{u}^2\left[\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 - 1\right]\left[\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2 - \frac{2k_1+3k_2}{4k_1}\right]. \end{aligned} \tag{5.28}$$

From (5.24), we have chosen the initial data so that

$$\left(\frac{\widehat{u_x}}{\widehat{u}}\right)(0) \geq \max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}.$$

Recall from (5.25) that $\left|\frac{u_x}{u}\right| \leq 1$. The assumptions on k_1 and k_2 ensure that the right-hand side of the above is less than 1. Therefore $\frac{\widehat{u_x}}{\widehat{u}}$ increases initially, and a continuity argument implies that it decreases for later time, and hence

$$\left(\frac{\widehat{u_x}}{\widehat{u}}\right)(t) \geq \left(\frac{\widehat{u_x}}{\widehat{u}}\right)(0) \geq \max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}.$$

In particular we have

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \geq \frac{2k_1+3k_2}{6k_1+21k_2}. \tag{5.29}$$

Plugging this into (5.26) it yields that $\widehat{M}'(t) \geq -2k_1\widehat{M}^2$, and thus $\widehat{M}(t)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1\widehat{M}(0)} = -\frac{1}{2k_1 m_0(x_0)u_{0,x}(x_0)},$$

completing the proof of the theorem. □

Remark 5.2. Using a similar argument one can prove the finite time blow-up for data such that $m_0 \leq 0$, $m_0(x_0) < 0$ and

$$u_{0,x}(x_0) \leq u_0(x_0)\cdot\max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}.$$

Recall from Lemma 5.2 that when m does not change sign, the true blow-up quantity is $k_1 m u_x$. In the setting of Theorem 5.5 and Remark 5.2 where $k_1 > 0$, we seek data which lead to $mu_x \to -\infty$. Thus using a similar argument we can handle the case when $k_1 < 0$, as indicated in the following corollary.

Corollary 5.6. *Suppose that $k_1 > 0$, $-2k_1/9 < k_2 < 2k_1/3$. Let $u_0 \in H^s(\mathbb{R})$ for $s > 5/2$ and $m_0 \geq 0$. Assume that there exists some point $x_0 \in \mathbb{R}$ such that $m_0(x_0) > 0$ and*

$$u_{0,x}(x_0) \leq -u_0(x_0)\cdot\max\left\{\sqrt{\frac{2k_1+3k_2}{4k_1}}, \sqrt{\frac{2k_1+3k_2}{6k_1+21k_2}}\right\}. \tag{5.30}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1 m_0(x_0)u_{0,x}(x_0)}.$$

Proof. We still consider the dynamics of $\widehat{M}$ and look to have $\widehat{M} \to -\infty$ in finite time.

$$\begin{aligned}\widehat{M}' &= -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}}{6}\left[(2k_1+3k_2)\widehat{u}^2 - (6k_1+21k_2)\widehat{u_x}^2\right] \\ &\quad - \left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{m}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right](t, q(t,x_0)) \\ &\leq -2k_1\widehat{M}^2 + \frac{\widehat{m}\widehat{u}^3}{6}\left[(2k_1+3k_2) - (6k_1+21k_2)\frac{\widehat{u_x}^2}{\widehat{u}^2}\right].\end{aligned} \tag{5.31}$$

Now the goal is to have $(2k_1+3k_2)\widehat{u}^2-(6k_1+21k_2)\widehat{u_x}^2 \le 0$, that is,

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \ge \frac{2k_1+3k_2}{6k_1+21k_2}, \tag{5.32}$$

and this again leads to considering $\widehat{u_x}/\widehat{u}$. From (5.28) we have

$$\begin{aligned}\left(\frac{\widehat{u_x}}{\widehat{u}}\right)' &= \frac{\widehat{u}^2-\widehat{u_x}^2}{\widehat{u}^2}\left[\left(\frac{k_1}{3}+\frac{k_2}{2}\right)\widehat{u}^2-\frac{2k_1}{3}\widehat{u_x}^2\right] \\ &\quad -\frac{2k_1+3k_2}{6\widehat{u}^2}\left[(\widehat{u}+\widehat{u_x})p_+ * (u-u_x)^3+(\widehat{u}-\widehat{u_x})p_- * (u+u_x)^3\right] \\ &\le \frac{2k_1}{3}\widehat{u}^2\left[\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2-1\right]\left[\left(\frac{\widehat{u_x}}{\widehat{u}}\right)^2-\frac{2k_1+3k_2}{4k_1}\right].\end{aligned} \tag{5.33}$$

Therefore we know that when (5.36) is satisfied, $\widehat{u_x}/\widehat{u}$ decreases, and thus

$$\frac{\widehat{u_x}^2}{\widehat{u}^2} \ge \max\left\{\frac{2k_1+3k_2}{4k_1}, \frac{2k_1+3k_2}{6k_1+21k_2}\right\}.$$

This way we obtain the desired Riccati inequality for $\widehat{M}$

$$\widehat{M}'(t) \le -2k_1\widehat{M}^2,$$

which implies that $\widehat{M}(t) \to -\infty$ as $t \to T^*$ where $T^* \le -\dfrac{1}{2k_1 m_0(x_0)u_{0,x}(x_0)}$. □

Remark 5.3. Note that when $k_2=0$, equation (1.11) becomes the mCH equation. Condition (5.24) becomes $u_{0,x}(x_0) \le -u_0(x_0)/\sqrt{2}$, which agrees with the one obtained in [6, Theorem 1.1].

5.4. **Choice of data and blow-up:** $2k_1+3k_2=0$**.** In the previous section, we require that $2k_1+3k_2 \neq 0$. In fact when $2k_1+3k_2=0$, the dynamics in Lemma 5.3 can be simplified as

$$\begin{aligned}\widehat{u}' &= -\frac{2}{3}k_1\widehat{u_x}^3, \\ \widehat{u_x}' &= -\left(k_1+\frac{k_2}{2}\right)\widehat{u}\widehat{u_x}^2 = -\frac{2}{3}k_1\widehat{u}\widehat{u_x}^2, \\ \widehat{m}' &= -(2k_1\widehat{m}\widehat{u_x}+3k_2\widehat{u}\widehat{u_x})\widehat{m} = -2k_1\widehat{m}\widehat{u_x}(\widehat{m}-\widehat{u}), \\ \widehat{M}' &= -2k_1\widehat{M}^2+\frac{4}{3}k_1\widehat{u}\widehat{u_x}\widehat{M} = -2k_1\widehat{u_x}\widehat{M}\left(\widehat{m}-\frac{2}{3}\widehat{u}\right).\end{aligned} \tag{5.34}$$

In particular, the convolution terms all vanish and the dynamics is completely local. However, the dynamics of $\widehat{M}$ does not immediately lead to a Riccati type inequality. Instead, it involves the competition between $\widehat{u}$ and $\widehat{m}$.

5.4.1. *The case when* $k_1<0$. Note from (5.34) that when $k_1<0$,

$$\operatorname{sign}(\widehat{u}') = \operatorname{sign}(\widehat{u_x}), \quad \operatorname{sign}(\widehat{u_x}') = \operatorname{sign}(\widehat{u}). \tag{5.35}$$

Using this we first derive the following theorem which requires m to be non-sign-changing.

Theorem 5.7. *Suppose that* $k_1<0$, $2k_1+3k_2=0$. *Let* $u_0 \in H^s(\mathbb{R})$ *for* $s>5/2$. *Assume that*

(a) $m_0 \ge 0$ *and there exists some point* $x_0 \in \mathbb{R}$ *such that*

$$m_0(x_0)>0, \quad u_{0,x}(x_0)>0, \quad m_0(x_0) \ge \frac{4}{3}u_0(x_0), \quad \textit{or} \tag{5.36}$$

(b) $m_0 \le 0$ *and there exists some point* $x_0 \in \mathbb{R}$ *such that*

$$m_0(x_0)<0, \quad u_{0,x}(x_0)<0, \quad m_0(x_0) \ge \frac{4}{3}u_0(x_0). \tag{5.37}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \le -\frac{1}{k_1 m_0(x_0) u_{0,x}(x_0)}.$$

Proof. Because $k_1 < 0$, the goal is to show that $\widehat{M} \to +\infty$ in finite time.

(a) Since now $m \ge 0$, $\widehat{m} > 0$ and $k_1 < 0$, we know from (5.35) that $\widehat{u} > 0$ and hence $\widehat{u_x}' > 0$. So $\widehat{u_x}(t) > 0$ if $\widehat{u_x}(0) > 0$. Then the last equation in (5.34) suggests that in order to derive a Riccati type inequality for $\widehat{M}$, one would like to have $\widehat{m} - \frac{2}{3}\widehat{u} \ge \varepsilon \widehat{m}$, for some $\varepsilon > 0$, that is,

$$\frac{\widehat{m}}{\widehat{u}} \ge \frac{2}{3(1-\varepsilon)}. \tag{5.38}$$

Now we can check the dynamics of $\widehat{m}/\widehat{u}$.

$$\left(\frac{\widehat{m}}{\widehat{u}}\right)' = -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \widehat{u}^2 - \frac{1}{3}\widehat{u_x}^2\right) \ge -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \frac{4}{3}\widehat{u}^2\right), \tag{5.39}$$

where we have used $|u_x| \le u$ to obtain the last inequality.

Therefore $\widehat{m}/\widehat{u}$ increases when $\widehat{m} \ge \frac{4}{3}\widehat{u}$. So when $\widehat{m}(0) \ge \frac{4}{3}\widehat{u}(0)$ we have

$$\frac{\widehat{m}}{\widehat{u}}(t) \ge \frac{\widehat{m}}{\widehat{u}}(0) \ge \frac{4}{3},$$

indicating that we may take $\varepsilon = \frac{1}{2}$ in (5.38). Thus from the last equation in (5.34) we have

$$\widehat{M}' \ge -k_1 \widehat{M}^2,$$

leading to $\widehat{M}(t) \to +\infty$ as $t \to T^*$ where T^* satisfies

$$T^* \le -\frac{1}{k_1 m_0(x_0) u_{0,x}(x_0)},$$

proving part (a).

(b) Similarly as in (a), we can deduce from (5.37) that

$$\widehat{m}(t) < 0, \quad \widehat{u}(t) \le \widehat{u}(0) < 0, \quad \widehat{u_x}(t) \le \widehat{u_x}(0) < 0. \tag{5.40}$$

To obtain a Riccati type inequality for $\widehat{M}$, it suffices to ask that $\widehat{m} - \frac{2}{3}\widehat{u} \le \varepsilon \widehat{m}$, for some $\varepsilon > 0$, which leads to (5.38) again.

Following the dynamics of $\widehat{m}/\widehat{u}$ and keeping track of the signs as in (5.40) it follows that (5.39) still holds. Hence the rest of the argument goes the same way as in (a). $\square$

5.4.2. *The case when $k_1 > 0$.* In this case it follows from (5.34) that

$$\operatorname{sign}(\widehat{u}') = -\operatorname{sign}(\widehat{u_x}), \quad \operatorname{sign}(\widehat{u_x}') = -\operatorname{sign}(\widehat{u}). \tag{5.41}$$

The corresponding blow-up results are as follows.

Theorem 5.8. *Suppose that $k_1 > 0$, $2k_1 + 3k_2 = 0$. Let $u_0 \in H^s(\mathbb{R})$ for $s > 5/2$. Assume that*

(a) *$m_0 \ge 0$ and there exists some point $x_0 \in \mathbb{R}$ such that*

$$m_0(x_0) > 0, \quad u_{0,x}(x_0) < 0, \quad m_0(x_0) \ge \frac{4}{3}u_0(x_0), \tag{5.42}$$

or

(b) *$m_0 \le 0$ and there exists some point $x_0 \in \mathbb{R}$ such that*

$$m_0(x_0) < 0, \quad u_{0,x}(x_0) > 0, \quad m_0(x_0) \le \frac{4}{3}u_0(x_0), \tag{5.43}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \le -\frac{1}{k_1 m_0(x_0) u_{0,x}(x_0)}. \tag{5.44}$$

Proof. Tracking the dynamics of $\widehat{M}$ and using (5.41) we see that to obtain a Riccati type inequality for $\widehat{M}$ it suffices to have (5.38) for some $\varepsilon > 0$, for both cases (a) and (b). Thus computing $(\widehat{m}/\widehat{u})'$ and using that $|u_x| \le u$ we get

$$\left(\frac{\widehat{m}}{\widehat{u}}\right)' = -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \widehat{u}^2 - \frac{1}{3}\widehat{u_x}^2\right) \ge -\frac{2k_1 \widehat{m}\widehat{u_x}}{\widehat{u}^2}\left(\widehat{m}\widehat{u} - \frac{4}{3}\widehat{u}^2\right),$$

which implies that

$$\frac{\widehat{m}}{\widehat{u}} \text{ increases if } \frac{\widehat{m}}{\widehat{u}} \ge \frac{4}{3}. \tag{5.45}$$

This in turn leads to $\widehat{M}' \le -k_1 \widehat{M}^2$ and hence the blow-up of $\widehat{M}$, with an estimate of the blow-up time as (5.44).

Finally the theorem is proved by realizing that (5.45) is satisfied if (5.42) or (5.43) holds. □

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Appendix A. Computation for the derivation of the η equation

In this section we provide the details in deriving (2.9). Recall the asymptotic expansion of (2.3) at orders $O(\varepsilon^i\mu^j)$ in Section 2 for $i+2j \le 4$. For the $O(\varepsilon^3\mu^1)$-order approximation when $0 < z < 1$, the following system is obtained

$$\begin{cases} -p_{31,\xi} = -u_{31,\xi} + u_{21,\tau} + (u_{00}u_{21} + u_{10}u_{11} + u_{20}u_{01})_\xi \\ \qquad\qquad + w_{00}u_{21,z} + w_{10}u_{11,z}, \\ -p_{31,z} = -w_{20,\xi} + w_{10,\tau} + u_{00}w_{10,\xi} + u_{10}w_{00,\xi} + (w_{00}w_{10})_z, \\ u_{31,\xi} + w_{31,z} = 0, \\ u_{31,z} - w_{20,\xi} = 0. \end{cases} \tag{A.1}$$

The boundary condition on $z = 0$ is $w_{31} = 0$, and on $z = 1$, the conditions read

$$\begin{cases} \eta_{31} = p_{31} + \eta_{21}p_{00,z} + \eta_{00}p_{21,z} + \eta_{11}p_{10,z} + \eta_{10}p_{11,z} + \eta_{20}p_{01,z} + \eta_{01}p_{20,z} + \frac{1}{2}\eta_{00}^2 p_{11,zz}, \\ w_{31} + \eta_{21}w_{00,z} + \eta_{00}w_{21,z} + \eta_{11}w_{10,z} + \eta_{10}w_{11,z} + \eta_{20}w_{01,z} + \eta_{01}w_{20,z} - \eta_{21,\tau} + \frac{\eta_{00}^2}{2}w_{11,zz} \\ \qquad = -\eta_{31,\xi} + u_{21}\eta_{00,\xi} + u_{00}\eta_{21,\xi} + u_{20}\eta_{01,\xi} + u_{01}\eta_{20,\xi} + u_{10}\eta_{11,\xi} + u_{11}\eta_{10,\xi} + \eta_{00}\eta_{00,\xi}u_{11,z}. \end{cases}$$

Next, we plug $w_{i0} = -z\eta_{i0,\xi} (i = 0, 1, 2)$ which can easily be obtained from [30] into the second equation in (A.1). It takes the form of

$$p_{31,z} = -\, zu_{20,\xi\xi} + zu_{10,\xi\tau} + zu_{00}u_{10,\xi\xi} + zu_{10}\eta_{00,\xi\xi} - (w_{00}w_{10})_z.$$

Taking the ξ derivative of the above and integrating in z on $[1, z]$, we know

$$\begin{aligned} p_{31,\xi} =& \int_1^z p_{31,z'\xi} dz' + p_{31,\xi}|_{z=1} \\ =& \frac{z^2-1}{2}\big(- u_{20,\xi\xi\xi} + u_{10,\xi\xi\tau} + (u_{00}u_{10,\xi\xi} + u_{10}\eta_{00,\xi\xi})_\xi\big) + (w_{00}w_{10})_\xi|_{z=1} \\ & - (w_{00}w_{10})_\xi + \eta_{31,\xi} + \Big(\eta_{10}\eta_{00,\xi\xi} + \eta_{00}\eta_{10,\xi\xi} + 2\eta_{00}\eta_{00,\xi}^2 + \frac{1}{2}\eta_{00}^2\eta_{00,\xi\xi}\Big)_\xi. \end{aligned} \tag{A.2}$$

On the other hand, we have $u_{21,z} = w_{10,\xi}$ and $u_{11,z} = w_{00,\xi}$ from [30]. Then the first equation in (A.1) becomes

$$-p_{31,\xi} = - u_{31,\xi} + u_{21,\tau} + (u_{00}u_{21} + u_{10}u_{11} + u_{20}u_{01})_\xi + (w_{00}w_{10})_\xi. \tag{A.3}$$

Combining (A.2) with (A.3), it leads to

$$\begin{aligned} 0 =& - u_{31,\xi} + u_{21,\tau} + (u_{00}u_{21} + u_{10}u_{11} + u_{20}u_{01})_\xi + \eta_{31,\xi} + (w_{00}w_{10})_\xi|_{z=1} \\ & + \Big(\eta_{10}\eta_{00,\xi\xi} + \eta_{00}\eta_{10,\xi\xi} + 2\eta_{00}\eta_{00,\xi}^2 + \frac{1}{2}\eta_{00}^2\eta_{00,\xi\xi}\Big)_\xi \\ & + \frac{z^2-1}{2}\big(- u_{20,\xi\xi\xi} + u_{10,\xi\xi\tau} + (u_{00}u_{10,\xi\xi} + u_{10}\eta_{00,\xi\xi})_\xi\big). \end{aligned} \tag{A.4}$$

Now we will simplify equation (A.4). Because the fourth equation in (A.1) gives that

$$u_{31,\xi} = -\frac{z^2}{2}u_{20,\xi\xi\xi} + \partial_\xi\Phi_{31}(\tau, \xi).$$

for some $\Phi_{31}(\tau, \xi)$ independent of z, the third equation in (A.1) and the boundary condition on $\{z = 0\}$ for w_{31} yield that

$$w_{31} = w_{31}|_{z=0} + \int_0^z w_{31,z'}\, dz' = - \int_0^z u_{31,\xi}\, dz' = \frac{z^3}{6}u_{20,\xi\xi\xi} - z\partial_\xi\Phi_{31}(\tau, \xi).$$

Hence, combining with the boundary condition for w_{31} on $\{z = 1\}$, we have

$$\frac{1}{6}u_{20,\xi\xi\xi} - \partial_\xi\Phi_{31}(\tau, \xi) = -\eta_{31,\xi} + \eta_{21,\tau} + H_{4,\xi}|_{z=1} - \frac{1}{2}(\eta_{00}^2\eta_{00,\xi\xi})_\xi,$$

where $H_4 := u_{00}\eta_{21} + u_{21}\eta_{00} + u_{20}\eta_{01} + u_{01}\eta_{20} + u_{11}\eta_{10} + u_{10}\eta_{11}$. Therefore $\Phi_{31}(\tau, \xi)$ satisfies

$$\partial_\xi\Phi_{31}(\tau, \xi) = \eta_{31,\xi} - \eta_{21,\tau} + \frac{1}{6}u_{20,\xi\xi\xi} - H_{4,\xi}|_{z=1} + \frac{1}{2}(\eta_{00}^2\eta_{00,\xi\xi})_\xi.$$

This in turn implies that

$$u_{31,\xi} = \eta_{31,\xi} - \eta_{21,\tau} - \left(\frac{z^2}{2} - \frac{1}{6}\right) u_{20,\xi\xi\xi} - H_{4,\xi}|_{z=1} + \frac{1}{2}(\eta_{00}^2\eta_{00,\xi\xi})_\xi.$$

It then follows from (A.4) that

$$\begin{aligned} 0 =& 2\eta_{21,\tau} + (u_{20}\eta_{01} + \eta_{00}\eta_{21} + u_{10}\eta_{11})_\xi - \frac{1}{2}(u_{10}\eta_{00}\eta_{01})_\xi + (\eta_{00}\eta_{10,\xi\xi})_\xi - \frac{1}{3}(\eta_{00}u_{10,\xi\xi})_\xi \\ & + \frac{5}{6}(\eta_{10}\eta_{00,\xi\xi}) + H_{4,\xi}|_{z=1} - \left(u_{00}\int \eta_{11,\tau}\, d\xi\right)_\xi - (u_{00}H_2|_{z=1})_\xi - \int \eta_{11,\tau\tau}\, d\xi - H_{2,\tau}|_{z=1} \\ & - \frac{1}{3}u_{10,\xi\xi\tau} + \frac{1}{3}u_{20,\xi\xi\xi} + (\eta_{00,\xi}\eta_{10,\xi})_\xi + \frac{1}{24}(\eta_{00}^2\eta_{00,\xi\xi})_\xi + \frac{3}{2}(\eta_{00}\eta_{00,\xi}^2)_\xi, \end{aligned} \tag{A.5}$$

where $H_2 := u_{00}\eta_{11} + u_{11}\eta_{00} + u_{10}\eta_{01} + u_{01}\eta_{10}$.

From [30], it is easy to see that

$$\begin{aligned}&-2\left(\eta_{00}\int \eta_{11,\tau}\,d\xi\right)_\xi\\&\quad=3(\eta_{00}^2\eta_{11}+\eta_{00}\eta_{10}\eta_{01})_\xi-\frac{3}{4}(\eta_{00}^3\eta_{01})_\xi+\frac{1}{3}(\eta_{00}\eta_{10,\xi\xi})_\xi+\frac{13}{24}(\eta_{00}\eta_{00,\xi}^2)_\xi+\frac{5}{6}(\eta_{00}^2\eta_{00,\xi\xi})_\xi.\end{aligned}\tag{A.6}$$

and

$$\begin{aligned}&\int \eta_{11,\tau\tau}\,d\xi+H_{2,\tau}|_{z=1}\\&=\frac{3}{4}(\eta_{00}^3\eta_{01})_\xi-\frac{3}{4}\left(\eta_{00}^2\eta_{11}\right)_\xi-\frac{1}{12}\eta_{00}\eta_{10,\xi\xi\xi}-\frac{1}{12}\eta_{10}\eta_{00,\xi\xi\xi}-\frac{1}{6}\eta_{10,\xi\xi\tau}-\frac{3}{2}(\eta_{00}\eta_{10}\eta_{01})_\xi\\&\quad+\frac{13}{16}(\eta_{00,\xi}^2\eta_{00})_\xi+\frac{53}{48}(\eta_{00}^2\eta_{00,\xi\xi})_\xi-\frac{3}{8}\eta_{00}^2\eta_{00,\xi\xi\xi}.\end{aligned}\tag{A.7}$$

To obtain an equation for η only, we substitute $u_{10},u_{01},u_{20},u_{21}$ and (A.6), (A.7) into (A.5) to get that

$$\begin{aligned}0=\;&2\eta_{21,\tau}+3(\eta_{00}\eta_{21}+\eta_{01}\eta_{20}+\eta_{10}\eta_{11})_\xi+\frac{1}{3}\eta_{20,\xi\xi\xi}-\frac{3}{4}(\eta_{00}^2\eta_{11}+2\eta_{01}\eta_{10}\eta_{00})_\xi\\&+\frac{3}{8}(\eta_{00}^3\eta_{01})_\xi+\frac{5}{6}(\eta_{10}\eta_{00,\xi\xi\xi}+\eta_{00}\eta_{10,\xi\xi\xi})+\frac{23}{12}(\eta_{00,\xi}\eta_{10,\xi\xi}+\eta_{00,\xi\xi}\eta_{10,\xi})\\&+\frac{21}{16}(\eta_{00,\xi}^3)-\frac{5}{16}(\eta_{00}\eta_{00,\xi}\eta_{00,\xi\xi})-\frac{3}{4}(\eta_{00}^2\eta_{00,\xi\xi\xi}).\end{aligned}\tag{A.8}$$

The asymptotic expansion introduced before shows

$$\eta:=\eta_{00}+\varepsilon\eta_{10}+\varepsilon^2\eta_{20}+\varepsilon^3\eta_{30}+\mu\eta_{01}+\varepsilon\mu\eta_{11}+\varepsilon^2\mu\eta_{21}+O(\varepsilon^4,\mu^2).$$

In view of [30], the η_{ij} equations are given by

$$\begin{aligned}&2\eta_{00,\tau}+3\eta_{00}\eta_{00,\xi}=0,\\&2\eta_{01,\tau}+3(\eta_{00}\eta_{01})_\xi+\frac{1}{3}\eta_{00,\xi\xi\xi}=0,\\&2\eta_{10,\tau}+3(\eta_{00}\eta_{10})_\xi-\frac{1}{4}(\eta_{00}^3)_\xi=0,\\&2\eta_{20,\tau}+3(\eta_{00}\eta_{20}+\frac{3}{2}\eta_{10}^2)_\xi-\frac{3}{4}(\eta_{00}^2\eta_{10})_\xi+\frac{3}{32}(\eta_{00}^4)_\xi=0\\&2\eta_{30,\tau}+3(\eta_{00}\eta_{30}+\eta_{10}\eta_{20})_\xi-\frac{3}{4}(\eta_{00}^2\eta_{20}+\eta_{00}\eta_{10}^2)_\xi+\frac{3}{8}(\eta_{00}^3\eta_{10})_\xi+\frac{23}{192}(\eta_{00}^5)_\xi=0,\\&2\eta_{11,\tau}+3(\eta_{00}\eta_{11}+\eta_{10}\eta_{01})_\xi+\frac{1}{3}\eta_{10,\xi\xi\xi}-\frac{3}{4}(\eta_{00}^2\eta_{01})_\xi+\frac{23}{24}(\eta_{00,\xi}^2)_\xi+\frac{5}{6}(\eta_{00}\eta_{00,\xi\xi\xi})=0,\end{aligned}$$

and hence (2.9) is obtained.

Appendix B. Computation of the higher order terms in the u equation

In this section we provide the detailed computation for the asymptotic expansion of the surface equation (2.11) when substituting the Kodama transformation (3.2). In Section 3 we already computed the coefficients in lower order terms. In the following we continue to proceed to the higher order terms.

Step 4. We now consider $O_3(\varepsilon\mu)$ term. Choose $C=\lambda_4u_x^2+\lambda_5uu_{xx}$. From (3.6), the expression for u_t is given by

$$u_t=-u_x-\frac{3}{2}\varepsilon uu_x-\frac{1}{2}(O_0(\mu)+O_0(\varepsilon\mu))+\lambda_1u\varepsilon O_0(\mu)+O(\varepsilon^2\mu,\mu^2,\varepsilon^2).$$

This operation produces $O_3(\varepsilon\mu)$ of the form

$$\begin{aligned}
&\varepsilon\mu(C_x+C_t)-\lambda_1\varepsilon\mu u(2\lambda_2(u_{xxx}+u_{xxt}))\\
=&-3\lambda_4\varepsilon^2\mu u_x(uu_x)_x-\frac{3}{2}\lambda_5\varepsilon^2\mu uu_xu_{xx}-\left(\frac{3}{2}\lambda_5-3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx}\\
&-\lambda_4\varepsilon\mu u_x(O_0(\mu))_x-\lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu)-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu uO_0(\mu)_{xx}\\
&-\lambda_4\varepsilon\mu u_x\big(O_0(\varepsilon\mu)+2\lambda_1u\varepsilon O_0(\mu)\big)_x-\frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu)+\lambda_5\lambda_1uu_{xx}\varepsilon^2\mu O_0(\mu)\\
&-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu uO_0(\varepsilon\mu)_{xx}+(\lambda_5-2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}.
\end{aligned}$$

The $\varepsilon\mu$-order term turns out to be

$$\varepsilon\mu\left[\left(\frac{3}{2}\lambda_2+\frac{1}{2}A_4-\frac{3}{2}\nu\lambda_2\right)uu_{xxx}+\left(\lambda_1+\frac{3}{2}\lambda_2+\frac{1}{2}A_3-\frac{9}{2}\nu\lambda_2\right)u_xu_{xx}\right].$$

Denote the coefficients of uu_{xxx} and u_xu_{xx} by

$$\begin{cases}I_{uu_{xxx}}:=\frac{3}{2}\lambda_2+\frac{1}{2}A_4-\frac{3}{2}\nu\lambda_2,\\ I_{u_xu_{xx}}:=\lambda_1+\frac{3}{2}\lambda_2+\frac{1}{2}A_3-\frac{9}{2}\nu\lambda_2.\end{cases}\tag{B.1}$$

The terms in asymptotic order are

$$\begin{aligned}
O_4(\mu^2):=&O_3(\mu^2),\qquad O_4(\varepsilon^3):=O_3(\varepsilon^3),\qquad O_4(\varepsilon^4):=O_3(\varepsilon^4),\\
O_4(\varepsilon^2\mu):=&O_3(\varepsilon^2\mu)-3\lambda_4\varepsilon^2\mu u_x(uu_x)_x-\frac{3}{2}\lambda_5\varepsilon^2\mu uu_xu_{xx}-\left(\frac{3}{2}\lambda_5-3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx},\\
O_4(\varepsilon\mu^2):=&O_3(\varepsilon\mu^2)-\lambda_4\varepsilon\mu u_x(O_0(\mu))_x-\lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu)-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu u(O_0(\mu))_{xx},\\
O_4(\varepsilon^2\mu^2):=&O_3(\varepsilon^2\mu^2)-\lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu)+2\lambda_1\varepsilon uO_0(\mu))_x-\frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu)+\lambda_5\lambda_1\varepsilon^2\mu uu_{xx}O_0(\mu)\\
&-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx}+(\lambda_5-2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}.
\end{aligned}$$

Step 5. Next we consider ε^3-order which has the form

$$\begin{aligned}
O_4(\varepsilon^3)=&\frac{1}{2}O_0(\varepsilon^3)-\lambda_1\varepsilon uO_0(\varepsilon^2)-(6\lambda_3-8\lambda_1^2)\lambda_1\varepsilon^3u^2(uu_x+uu_t)-\left(\frac{9}{2}\lambda_3-6\lambda_1^2\right)\varepsilon^3u^3u_x\\
=&\left(\frac{3}{8}\lambda_3+\frac{1}{8}A_2+\frac{1}{4}\lambda_1A_1\right)\varepsilon^3(u^4)_x+2\lambda_0\varepsilon^3\big((u^4)_x+(u^4)_t\big),
\end{aligned}$$

where we have replaced u_t by $-u_x-\frac{3}{2}\varepsilon uu_x$. The coefficient is denoted by

$$I_{(u^4)_x}:=\frac{3}{8}\lambda_3+\frac{1}{8}A_2+\frac{1}{4}\lambda_1A_1.\tag{B.2}$$

Also, at ε^4-order we have

$$O_5(\varepsilon^4):=\frac{1}{2}O_0(\varepsilon^4)-\lambda_1\varepsilon uO_0(\varepsilon^3)-\left(\frac{3}{2}\lambda_3-2\lambda_1^2\right)\varepsilon^2u^2O_0(\varepsilon^2)-12\lambda_0\varepsilon^4u^4u_x.$$

Since

$$O_0(\varepsilon^4)=\varepsilon^4\left(\lambda_0+\lambda_1+A_1\lambda_1^2+A_2\lambda_1+\frac{A_8}{5}+A_1\lambda_3\right)(u^5)_x,$$

we can simplify O_5 as

$$\begin{aligned}O_5(\varepsilon^4) &= \frac{1}{2}O_0(\varepsilon^4) - \lambda_1\varepsilon u O_0(\varepsilon^3) - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 u^2 O_0(\varepsilon^2) - 12\lambda_0\varepsilon^4 u^4 u_x \\ &= \frac{1}{2}\big(- 19\lambda_0 + 5\lambda_1 + 5A_1\lambda_1^2 + 5A_2\lambda_1 + A_8 + 5A_1\lambda_3\big)\varepsilon^4 u^4 u_x \\ &\quad - \lambda_1\varepsilon^4\big(12\lambda_3 + 6\lambda_1^2 + 4A_1\lambda_1 + A_2\big)u^4 u_x - \left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\big(9\lambda_1 + A_1\big)u^4 u_x \\ &= \varepsilon^4\frac{1}{10}\Big(- 19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3\Big)(u^5)_x.\end{aligned}$$

Then

$$I_{(u^5)_x} = \frac{1}{10}\left(-19\lambda_0 - \frac{3}{4}\lambda_1^2 + \frac{49}{8}\lambda_1 + \frac{A_8}{5} - \frac{3}{2}\lambda_3 + 24\lambda_1^3 - 51\lambda_1\lambda_3\right), \tag{B.3}$$

and the terms which involve μ remain the same.

Step 6. Finally, we consider the $\varepsilon^2\mu$-order which has the form

$$\begin{aligned}O_4(\varepsilon^2\mu) &= \frac{1}{2}O_0(\varepsilon^2\mu) - \lambda_1\varepsilon u O_0(\varepsilon\mu) - u^2\left(\frac{3}{2}\lambda_3 - 2\lambda_1^2\right)\varepsilon^2 O_0(\mu) + \lambda_2\nu\mu(F_{\varepsilon^2})_{xx} \\ &\quad - 3\lambda_4\varepsilon^2\mu u_x(uu_x)_x - \frac{3}{2}\lambda_5\varepsilon^2\mu u u_x u_{xx} - \left(\frac{3}{2}\lambda_5 - 3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx}.\end{aligned}$$

We choose $G = \lambda_7 u u_x^2 + \lambda_8 u^2 u_{xx}$ to keep the scaling in the equation. From (3.6), the expression for u_t is given by $u_t = -u_x - \lambda_2\mu(u_{xxx} + u_{xxt}) - \frac{1}{6}\mu u_{xxx}$. We eliminate u_t by (3.6) itself, namely

$$u_t = -u_x - \frac{1}{6}\mu u_{xxx} + O(\varepsilon\mu). \tag{B.4}$$

Thereby, there appears the relation

$$\varepsilon^2\mu(G_x + G_t) = -\frac{1}{6}\lambda_7\varepsilon^2\mu^2 u_x^2 u_{xxx} - \frac{1}{3}\lambda_7\varepsilon^2\mu^2 u u_x u_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2 u^2 u_{xxxxx} - \frac{1}{3}\lambda_8\varepsilon^2\mu^2 u u_{xx} u_{xxx}.$$

Hence, $\frac{1}{2}O_0(\varepsilon^2\mu)$ takes the form

$$\begin{aligned}\frac{1}{2}O_0(\varepsilon^2\mu) &= \frac{1}{6}\varepsilon^2\mu\lambda_3(u^3)_{xxx} + \frac{3}{2}\varepsilon^2\mu\lambda_4(uu_x^2)_x + \frac{3}{2}\varepsilon^2\mu\lambda_5(u^2u_{xx})_x + \frac{3}{2}\lambda_2\lambda_1\varepsilon^2\mu(u^2u_{xx})_x \\ &\quad + \frac{1}{2}\varepsilon^2\mu\lambda_2 A_1(u^2u_{xx})_x + \varepsilon^2\mu\lambda_1 A_3(uu_x^2)_x + \frac{1}{2}A_4\lambda_1\varepsilon^2\mu u^2 u_{xxx} + \frac{1}{2}\varepsilon^2\mu\lambda_1 A_4 u(u^2)_{xxx} \\ &\quad + \frac{1}{2}A_5\varepsilon^2\mu u u_x u_{xx} + \frac{1}{2}A_6\varepsilon^2\mu u^2 u_{xxx} + \frac{1}{2}A_7\varepsilon^2\mu u_x^3 - \frac{1}{6}\lambda_7\varepsilon^2\mu^2 u_x^2 u_{xxx} \\ &\quad - \frac{1}{3}\lambda_7\varepsilon^2\mu^2 u u_x u_{xxxx} - \frac{1}{6}\lambda_8\varepsilon^2\mu^2 u^2 u_{xxxxx} - \frac{1}{3}\lambda_8\varepsilon^2\mu^2 u u_{xx} u_{xxx}.\end{aligned}$$

We now deal with $-\lambda_1 u\varepsilon O_0(\varepsilon\mu)$. By definition $C = \lambda_4 u_x^2 + \lambda_5 u u_{xx}$ and (B.4), it follows that

$$-2\lambda_1\varepsilon^2\mu u(C_x + C_t) = -\varepsilon^2\mu^2\left(-\frac{1}{3}\lambda_1\lambda_4 u u_x u_{xxxx} - \frac{1}{6}\lambda_1\lambda_5 u u_{xx} u_{xxx} - \frac{1}{6}\lambda_1\lambda_5 u^2 u_{xxxxx}\right).$$

Then we know

$$\begin{aligned}-\lambda_1 u\varepsilon O_0(\varepsilon\mu) = &- \lambda_1\varepsilon^2\mu\left[\left(\frac{6}{3}\lambda_1 + 3\lambda_2 + A_3\right)uu_x u_{xx} + \left(\frac{2}{3}\lambda_1 + 3\lambda_2 + A_4\right)u^2 u_{xxx}\right] \\ &+ \varepsilon^2\mu^2\left(\frac{1}{3}\lambda_1\lambda_4 u u_x u_{xxxx} + \frac{1}{6}\lambda_1\lambda_5 u u_{xx} u_{xxx} + \frac{1}{6}\lambda_1\lambda_5 u^2 u_{xxxxx}\right).\end{aligned}$$

Similarly, we have

$$-\left(\frac{3}{2}\lambda_3-2\lambda_1^2\right)\varepsilon^2u^2O_0(\mu)=-\left(\frac{1}{2}\lambda_3-\frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu u^2u_{xxx}-\lambda_2\left(\frac{1}{2}\lambda_3-\frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu^2u^2u_{xxxxx},$$

and $\lambda_2\nu\mu(F_{\varepsilon^2})_{xx}=-\lambda_2\nu\varepsilon^2\mu(\frac{1}{2}\lambda_1+\frac{A_1}{6})(u^3)_{xxx}$.

Putting the above together, we have

$$\begin{aligned}O_5(\varepsilon^2\mu):=&\frac{1}{6}\varepsilon^2\mu\lambda_3(u^3)_{xxx}+\frac{3}{2}\varepsilon^2\mu\lambda_4(uu_x^2)_x+\frac{3}{2}\varepsilon^2\mu\lambda_5(u^2u_{xx})_x+\frac{3}{2}\lambda_2\lambda_1\varepsilon^2\mu(u^2u_{xx})_x\\&+\frac{1}{2}\varepsilon^2\mu\lambda_2A_1(u^2u_{xx})_x+\varepsilon^2\mu\lambda_1A_3(uu_x^2)_x+\frac{1}{2}A_4\lambda_1\varepsilon^2\mu u^2u_{xxx}+\frac{1}{2}\varepsilon^2\mu\lambda_1A_4u(u^2)_{xxx}\\&+\frac{1}{2}A_5\varepsilon^2\mu uu_xu_{xx}+\frac{1}{2}A_6\varepsilon^2\mu u^2u_{xxx}+\frac{1}{2}A_7\varepsilon^2\mu u_x^3-\lambda_1\varepsilon^2\mu(\frac{6}{3}\lambda_1+3\lambda_2+A_3)uu_xu_{xx}\\&-\lambda_1\varepsilon^2\mu\left(\frac{2}{3}\lambda_1+3\lambda_2+A_4\right)u^2u_{xxx}-\left(\frac{1}{2}\lambda_3-\frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu u^2u_{xxx}-\lambda_2\nu\varepsilon^2\mu\left(\frac{1}{2}\lambda_1+\frac{A_1}{6}\right)(u^3)_{xxx}\\&-3\lambda_4\varepsilon^2\mu u_x(uu_x)_x-\frac{3}{2}\lambda_5\varepsilon^2\mu uu_xu_{xx}-\left(\frac{3}{2}\lambda_5-3\lambda_1\lambda_2\right)\varepsilon^2\mu u(uu_x)_{xx}.\end{aligned}$$

More precisely, the coefficient of these terms are

$$\begin{aligned}I_{u^2u_{xxx}}&:=\frac{3}{2}(1-\nu)\lambda_1\lambda_2+\frac{A_1}{2}(1-\nu)\lambda_2+\frac{1}{2}A_4\lambda_1+\frac{1}{2}A_6,\\I_{uu_xu_{xx}}&:=3\lambda_3-3\lambda_5-9(1-\nu)\lambda_2\lambda_1+A_1(1-3\nu)\lambda_2+(A_3+3A_4)\lambda_1+\frac{1}{2}A_5-2\lambda_1^2,\\I_{u_x^3}&:=\frac{1}{2}A_7+\lambda_3-\lambda_2\nu(3\lambda_1+A_1)-\frac{3}{2}\lambda_4+A_3\lambda_1.\end{aligned}\tag{B.5}$$

In the asymptotic order, we have

$$\begin{aligned}O_5(\mu^2)&:=\frac{1}{2}O_0(\mu^2)+\lambda_2\nu\mu(F_\mu)_{xx},\\O_5(\varepsilon\mu^2)&:=\frac{1}{2}O_0(\varepsilon\mu^2)-\lambda_1\varepsilon uO_0(\mu^2)+\lambda_2\nu\mu(F_{\varepsilon\mu})_{xx}\\&\quad-\lambda_4\varepsilon\mu u_x(O_0(\mu))_x-\lambda_5\varepsilon\mu u_{xx}\frac{1}{2}O_0(\mu)-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu uO_0(\mu)_{xx},\\O_5(\varepsilon^2\mu^2)&:=\frac{1}{2}O_0(\varepsilon^2\mu^2)-\lambda_1\varepsilon uO_0(\varepsilon\mu^2)-\left(\frac{3}{2}\lambda_3-2\lambda_1^2\right)\varepsilon^2u^2(O_0(\mu^2)+\lambda_2\nu\mu(F_{\varepsilon^2\mu})_{xx}\\&\quad-\lambda_4\varepsilon\mu u_x(O_0(\varepsilon\mu)+2\lambda_1u\varepsilon O_0(\mu))_x-\frac{\lambda_5}{2}\varepsilon\mu u_{xx}O_0(\varepsilon\mu)+\lambda_5\lambda_1uu_{xx}\varepsilon^2\mu O_0(\mu)\\&\quad-\left(\frac{1}{2}\lambda_5-\lambda_1\lambda_2\right)\varepsilon\mu u(O_0(\varepsilon\mu))_{xx}+(\lambda_5-2\lambda_1\lambda_2)\lambda_1\varepsilon^2\mu u(uO_0(\mu))_{xx}\\&\quad+\varepsilon^2\mu^2\lambda_1\left(\frac{1}{3}\lambda_4uu_xu_{xxxx}+\frac{1}{6}\lambda_5uu_{xx}u_{xxx}+\frac{1}{6}\lambda_5u^2u_{xxxxx}\right)-\lambda_2\left(\frac{1}{2}\lambda_3-\frac{2}{3}\lambda_1^2\right)\varepsilon^2\mu^2u^2u_{xxxxx}\\&\quad-\frac{1}{6}\lambda_7\varepsilon^2\mu^2u_x^2u_{xxx}-\frac{1}{3}\lambda_7\varepsilon^2\mu^2uu_xu_{xxxx}-\frac{1}{6}\lambda_8\varepsilon^2\mu^2u^2u_{xxxxx}-\frac{1}{3}\varepsilon^2\mu^2\lambda_8uu_{xx}u_{xxx}.\end{aligned}$$

This procedure can be continued successively, and finally the coefficients of the terms at the order of $\varepsilon^2\mu^2$-order are obtained as

$$
\begin{aligned}
&I_{u^2u_{xxxxxx}} := C_1, \quad I_{uu_xu_{xxxxx}} := \lambda_8 - 6\lambda_{10} - 4A_1\lambda_6 + C_2,\\
&I_{uu_{xx}u_{xxxx}} := \lambda_7 + \lambda_8 - 10A_1\lambda_6 - 15\lambda_{10} + C_3,\\
&I_{u_x^2u_{xxxx}} := \lambda_7 + \lambda_8 - 10A_1\lambda_6 - 30\lambda_1\lambda_6 - 6\lambda_9 + C_4,\\
&I_{u_xu_{xx}^2} := 2\lambda_7 + \lambda_8 - 15A_1\lambda_6 - 45\lambda_1\lambda_6 - \frac{9}{2}\lambda_9 - \frac{15}{2}\lambda_{11} + C_5,
\end{aligned} \tag{B.6}
$$

where $C_i (i = 1 \dots 5)$ are constants depending on $\lambda_1, \dots, \lambda_5$ and ν, and satisfy the following:

$$
\begin{pmatrix} 0 & 1 & 0 & -6 \\ 1 & 1 & 0 & -15 \\ 1 & 1 & -6 & 0 \\ 2 & 1 & -\frac{9}{2} & 0 \end{pmatrix}
\begin{pmatrix} \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \end{pmatrix}
=
\begin{pmatrix} I_{uu_xu_{xxxxx}} - C_2 + 4A_1\lambda_6 \\ I_{uu_{xx}u_{xxxx}} - C_3 + 10A_1\lambda_6 \\ I_{u_x^2u_{xxxx}} - C_4 + (30\lambda_1 + 10A_1)\lambda_6 \\ I_{u_xu_{xx}^2} - C_5 + \frac{15}{2}\lambda_{11} + (45\lambda_1 + 15A_1)\lambda_6 \end{pmatrix}. \tag{B.7}
$$

Note that the 4×4 matrix $\begin{pmatrix} 0 & 1 & 0 & -6 \\ 1 & 1 & 0 & -15 \\ 1 & 1 & -6 & 0 \\ 2 & 1 & -\frac{9}{2} & 0 \end{pmatrix}$ is invertible. Thus, for any choice of parameters and any choice of λ_6, λ_{11}, there exists unique tuple $\lambda_7, \lambda_8, \lambda_9, \lambda_{10}$ solve the above equation.

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(Robin Ming Chen) Department of Mathematics, University of Pittsburgh, PA 15260, USA
Email address: mingchen@pitt.edu

(Tianqiao Hu) Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019
Email address: tianqiao.hu@uta.edu

(Yue Liu) Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019
Email address: yliu@uta.edu