$W^{1,\infty}$ INSTABILITY OF H^1-STABLE PEAKONS IN THE NOVIKOV EQUATION

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Abstract. It is known from the previous works that the peakon solutions of the Novikov equation are orbitally and asymptotically stable in H^1. We prove, via the method of characteristics, that these peakon solutions are unstable under $W^{1,\infty}$-perturbations. Moreover, we show that small initial $W^{1,\infty}$-perturbations of the Novikov peakons can lead to the finite time blow-up of the corresponding solutions.

1. Introduction

The integrable Novikov equation

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx} \tag{1.1}$$

is proposed by Novikov [25] from a Lie symmetry analysis of nonlocal partial differential equations. Reformulating (1.1) in terms of the momentum density $m = u - u_{xx}$ yields the following evolution form

$$m_t + u^2 m_x + \frac{3}{2}(u^2)_x m = 0. \tag{1.2}$$

Hence, this Novikov equation can be regarded as a cubic nonlinear generalization of the Camassa–Holm (CH) equation [3] (derived earlier in [14]):

$$m_t + um_x + 2u_x m = 0. \tag{1.3}$$

The Novikov equation shares many common analytical properties with the CH equation. It belongs to the class of completely integrable equations thanks to the existence of the Lax pair [18, 25] and the bi-Hamiltonian structure [18]. The Novikov equation can exhibit the phenomenon of wave-breaking [19] (see also recent work in [5]). Another remarkable feature of the Novikov equation is the existence of peaked traveling wave solutions (called *peakons*):

$$u(t, x) = \varphi_c(x - ct - x_0), \quad c > 0, \ \ x_0 \in \mathbb{R} \tag{1.4}$$

with

$$\varphi_c(x) = \sqrt{c}e^{-|x|}, \quad x \in \mathbb{R}, \tag{1.5}$$

with corner singularities at the peaks [15, 17, 18]. In what follows, we will be dealing with the peakons propagating with the unit speed, for which we denote $\varphi := \varphi_{c=1}$.

Date: November 20, 2019.

1.1. **Previous works.** The (local) well-posedness theory for strong solutions to the Novikov equation (1.1) is a well-studied subject [16, 24, 26, 27, 29]. However, these results are not applicable to the scopes of our work since we have to consider weak solutions due to the wave breaking occurrence and the presence of peakons.

The Novikov equation (1.1) can be rewritten in the convolution form

$$u_t + u^2 u_x + (1-\partial_x^2)^{-1}\partial_x \left(\frac{3}{2} u u_x^2 + u^3\right) + (1-\partial_x^2)^{-1}\left(\frac{1}{2} u_x^3\right) = 0, \tag{1.6}$$

which suggests $H^1 \cap W^{1,3}$ as a natural space for weak solutions. It turns out that, by incorporating one of the conservation laws

$$E(u) := \int_{\mathbb{R}} \left(u^2 + u_x^2\right) \, dx, \tag{1.7}$$

the existence and uniqueness of global weak solutions can be established in $H^1 \cap W^{1,\infty}$ under an additional constraint on the initial datum u_0 that $m_0 := u_0 - u_{0xx}$ is a positive Radon measure [27, 28]. The sign condition $m_0 \geq 0$ was replaced by $u_0 \geq 0$ in [20] and a weak solution in $H^1 \cap W^{1,4}$ with the one-sided L^∞ bound on the gradient of u is obtained through a viscous approximation, at the price of losing the conservation of E and hence the uniqueness of solutions.

If another conservation law

$$F(u) := \int_{\mathbb{R}} \left(u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4\right) \, dx \tag{1.8}$$

is taken into account, the global weak solution theory can be casted in $H^1 \cap W^{1,4}$ without any restrictions on the initial datum [4]. The data-to-solution map is shown to be Lipschitz continuous on bounded sets of $H^1 \cap W^{1,4}$ under an optimal transport metric [2].

The importance of the two conservation laws $E(u)$ and $F(u)$ is also manifested in the stability analysis of the peakons. In [21], a Lyapunov function was constructed from the two conserved quantities, through which an H^1-orbital stability of peakons was established. Among various assumptions on the initial perturbation $u_0 \in H^s$ with $s \geq 3$, a crucial one in [21] was positivity of $m_0 := u_0 - u_{0xx}$. Such a sign property is preserved in the time evolution of the Novikov equation, from which one can control $|u_x(t,x)| \leq |u(t,x)| \leq E(u_0)$, leading to a global solution in H^s, $s \geq 3$. The same sign condition is a key to the construction of the Lyapunov function for peakons in [21].

Applying this orbital stability and utilizing the finite speed propagation property, an H^1-asymptotic stability of the Novikov peakon was obtained in [6] for the initial datum $u_0 \in H^1$ with m_0 being a nonnegative Radon measure.

The sign condition on m_0, and hence the boundedness of $|u_x(t,x)|$, presents a serious obstacle in the analysis of $W^{1,\infty}$-instability of peakons and might even exclude this kind of instability. Therefore, for our work we need an H^1 orbital stability result for the initial datum without the sign condition on m_0. In a recent work [12], such a sign constraint was removed, at the price that the global strong solutions in [21] were replaced by the local strong solutions. The following theorem records the corresponding result from [12].

Theorem A(H^1-orbital stability) *For every $0 < \varepsilon \ll 1$ and for every $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$ satisfying*

$$\|u_0 - \varphi\|_{H^1} < \varepsilon^4,$$

the corresponding solution $u \in C([0,T), H^s)$ *to the Novikov equation* (1.1) *with initial datum* u_0 *and the maximal existence time* $T > 0$ *satisfies*

$$\sup_{t\in[0,T)} \|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < 2\left(4 + \|u_{0x}\|_{L^\infty}^{1/2}\right)\varepsilon$$

where $\xi(t)$ *is a point of maximum of* $u(t,\cdot)$.

Theorem A only considers smooth solutions, whereas for our instability argument we need to control the evolution of solutions that are only Lipschitz. For this purpose, we need to reexamine the H^1 stability in a weaker regularity framework, which we do in Theorem 3.9.

1.2. **Main results and methodology.** The purpose of the current work is to understand the stability of peakons in the Novikov equation under the $W^{1,\infty}$ perturbations which preserve the original smoothness of peakons. In particular, we will consider piecewise C^1 perturbations to a single peakon and study their evolution under both the linearized and nonlinear flows associated to the Novikov equation (1.1). As is formulated in the following two theorems, we will prove that piecewise C^1 perturbations to a single peakon may grow in the $W^{1,\infty}$ norm in spite of being bounded in the H^1 norm both in the linearized and nonlinear flows.

First we derive in Section 2.1 the Cauchy problem for the *linearized* evolution of a perturbation $v(t,x)$ to the peakon $\varphi(x)$ in the form

$$\begin{cases} v_t + (\varphi^2 - 1)v_x + \varphi_x\left[v(t,0) - \varphi v\right] = 0, \\ v|_{t=0} = v_0, \end{cases} \tag{1.9}$$

which, following the idea of [23], motivates us to work in the space $C_0^1 \subset W^{1,\infty}$ defined as

$$C_0^1 := \left\{v \in C(\mathbb{R}) \cap C^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^-) : \quad v, v_x \in L^\infty\right\}. \tag{1.10}$$

Hence $v_0 \in C_0^1$ may have at most one peak at $x = 0$, which is also a location of the peak of φ. The method of characteristics can thus be implemented to provide an explicit solution to (1.9) in $H^1 \cap C_0^1$, allowing one to obtain the following result.

Theorem 1.1 (Linear instability). *For any given initial datum* $v_0 \in H^1 \cap C_0^1$*, there exists a unique global solution* $v \in C(\mathbb{R}, H^1 \cap C_0^1)$ *to the linearized problem* (1.9) *such that*

$$\|v(t,\cdot)\|_{H^1(\mathbb{R}^\pm)}^2 = \|v_0\|_{H^1(\mathbb{R}^\pm)}^2 \qquad \text{(linear } H^1 \text{ stability)} \tag{1.11}$$

and

$$\|v_x(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} \geq |v_0(0) + v_{0x}(0^+)|e^t - |v_0(0)| \quad \text{(linear } W^{1,\infty} \text{ instability)} \tag{1.12}$$

for all $t > 0$.

The nonlinear analysis is more delicate. The Cauchy problem for the Novikov equation can be formulated as

$$\begin{cases} u_t + u^2 u_x + \mathcal{Q}[u] = 0, & t > 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1.13}$$

where

$$\mathcal{Q}[u] := \frac{1}{2}\varphi_x * \left(\frac{3}{2}uu_x^2 + u^3\right) + \frac{1}{4}\varphi * u_x^3. \tag{1.14}$$

Similarly as in the linear analysis, we would like to first establish a well-posedness theory of the evolution of the perturbation v in $H^1 \cap C_0^1$. Compared with the Camassa–Holm case [23], the Cauchy problem for v in H^1 given by (3.16) was not studied before, hence we cannot use the previous well-posedness results. By a careful retooling of the method of characteristics, the Cauchy problem for v can be transformed to a dynamical system (3.26) where the vector field on the right-hand side consists of local terms of polynomial type and nonlocal terms that can be shown to be locally Lipschitz. Hence standard ODE theory applies to imply local well-posedness if solutions for v in $H^1 \cap C^1_0$ established in Theorem 3.13.

The H^1 orbital stability result (Theorem A) suggests that in order for the peakons to be $W^{1,\infty}$-unstable, it is necessary to track the dynamics of the gradient v_x of the perturbation and look to show that $\|v_x\|_{L^\infty}$ exhibit substantial growth. However Theorem A only treats strong solutions, and therefore a similar result in the weak solution framework is needed and is established in Theorem 3.9.

The key ingredient in proving the H^1 orbital stability is to construct a Lyapunov function using the two conservation laws E and F similar to what is done in [12]. For strong solutions, the conservation laws can be easily checked by utilizing the bi-Hamiltonian structure of the equation. However for weak solutions this becomes more delicate. Our strategy is based on regularizing the system and commuting the regularization with nonlinearity. The conservation laws can then be realized by deriving crucial commutator estimates in order to show that the remainder terms converge to zero as the regularization parameter tends to zero as is done in Lemma 3.8.

It turns out that the dynamics of v_x simplifies when restricted at the peak location, see equation (3.28). The corresponding differential equation consists of a Ricatti-like term, the terms that involve interaction with v, and a nonlocal term. The orbital stability ensures that all the interaction terms are small. Another important consequence of the orbital stability is that the nonlocal term is also small. This way a Ricatti-type inequality can be obtained, which in turn leads to a finite time blow-up.

Theorem 1.2 (Nonlinear instability). *For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap C^1_0$ satisfying*

$$\|u_0 - \varphi\|_{H^1} + \|u_{0x} - \varphi_x\|_{L^\infty} < \delta, \tag{1.15}$$

such that the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ to the Cauchy problem (1.13) with the initial datum u_0 and the maximal existence time $T > t_0$ satisfies $u(t, \cdot + a(t)) \in C^1_0$ for $t \in [0,T)$ and

$$\|u_x(t_0, \cdot) - \varphi_x(\cdot - a(t_0))\|_{L^\infty} > 1, \tag{1.16}$$

where $a(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$ such that $a(0) = 0$. Moreover, there exist initial datum u_0 satisfying (1.15) *such that $T < \infty$ for the corresponding solution $u \in C([0,T), H^1 \cap W^{1,\infty})$.*

Remark 1.3. The results of Theorems 1.1 and 1.2 are very similar to the results found in [23] for the CH equation (1.3) except that the H^1 norm of the peaked perturbation grows in the linear evolution of the CH equation, whereas the H^1 norm does not grow for the linearized Novikov equation. The discrepancy between the two results confirm the previous intuition [10] that the linearized evolution in H^1 does not imply anything for the nonlinear evolution of the quasilinear equations with peakons and wave breaking.

Remark 1.4. An interesting outcome of our instability theorem is that it provides a new way to generate wave breaking in the *weak* solution setting. To the best of the authors' knowledge, so far the vast literature on the blow-up analysis for quasilinear integrable equations, like the Camassa–Holm equation [1, 3, 7, 8, 14], the Degasperis-Procesi equation [5, 11, 13, 22], and the Novikov equation [5, 19], is performed in the framework of strong solutions. It is plausible that the idea used here can be extended to other peakon models.

2. Linear analysis

Here we investigate the linear stability of peakons and prove Theorem 1.1. For simplicity, we consider a single peakon (1.4) traveling with the unit speed $c = 1$ and denote it by $\varphi(x) \equiv \varphi_{c=1}(x) = e^{-|x|}$. Note that $\frac{1}{2}\varphi(x)$ is the Green's function of $1 - \partial_x^2$ on $\mathbb{R}$, that is,

$$(1-\partial_x^2)\varphi = 2\delta_0, \qquad (1-\partial_x^2)^{-1} f = \frac{1}{2}\varphi * f. \tag{2.1}$$

Some further properties of φ are given by

$$\varphi_x^2(x) = \varphi^2(x), \quad x \in \mathbb{R}\backslash\{0\} \tag{2.2}$$

and

$$\|\varphi\|_{L^2} = \|\varphi\|_{L^2} = \|\varphi\|_{L^\infty} = \|\varphi_x\|_{L^\infty} = 1. \tag{2.3}$$

In what follows, we derive the linearized problem (1.9), solve it by means of characteristics, and finally obtain relevant estimates for the proof of Theorem 1.1.

2.1. Derivation of the linearized problem.

To study the linearization of (1.6) around φ, we decompose $u(t,x)$ as the sum of a modulated peakon and its perturbation v in the form:

$$u(t,x) = \varphi(x-a(t)) + v(t, x-a(t)). \tag{2.4}$$

The stationary equation for peakon φ is defined for every $x \neq 0$ in the form:

$$(\varphi^2 - 1)\varphi' + \mathcal{Q}[\varphi] = 0, \tag{2.5}$$

where $\mathcal{Q}$ is given by (1.14). When we plug in (2.4) and (2.5) into (1.6) and truncate at the linear terms in v, we obtain the linearized equation for v in the form:

$$(1-\dot{a})\varphi_x + v_t - \dot{a} v_x + (\varphi^2 v)_x + \frac{3}{2}\varphi_x * \left(\varphi^2 v + \frac{1}{2}\varphi_x^2 v + \varphi\varphi_x v_x\right) + \frac{3}{4}\varphi * \left(\varphi_x^2 v_x\right) = 0. \tag{2.6}$$

The following proposition allows us to simplify the nonlocal terms in (2.6) and write it in the local form (1.9).

Proposition 2.1. *For $v \in H^1$ we have*

$$\frac{3}{2}\varphi_x * \left(\varphi^2 v + \frac{1}{2}\varphi_x^2 v + \varphi\varphi_x v_x\right) + \frac{3}{4}\varphi * \left(\varphi_x^2 v_x\right) = 3\varphi_x\left[v(0) - \varphi v\right]. \tag{2.7}$$

Proof. By using (2.2) and integrating by parts, we obtain

$$\frac{3}{4}\varphi * \left(\varphi_x^2 v_x\right) = \frac{3}{4}\varphi * \left(\varphi^2 v_x\right) = \frac{3}{4}\varphi_x * \left(\varphi^2 v\right) - \frac{3}{4}\varphi * \left[(\varphi^2)_x v\right].$$

From (2.1) we see that $\varphi_{xx} = \varphi - 2\delta_0$, and hence further using $\varphi(0) = 1$ and integrating by parts, we obtain

$$\begin{aligned}\frac{3}{2}\varphi_x * (\varphi\varphi_x v_x) &= \frac{3}{4}\varphi_x * \left[(\varphi^2)_x v_x\right] = \frac{3}{4}\varphi_{xx} * \left[(\varphi^2)_x v\right] - \frac{3}{4}\varphi_x * \left[(\varphi^2)_{xx} v\right] \\ &= \frac{3}{4}\varphi * \left[(\varphi^2)_x v\right] - \frac{3}{2}(\varphi^2)_x v - 3\varphi_x * \left[(\varphi^2 - \delta_0\varphi)v\right] \\ &= \frac{3}{4}\varphi * \left[(\varphi^2)_x v\right] - 3\varphi_x * \left(\varphi^2 v\right) - 3\varphi\varphi_x v + 3\varphi_x v(0).\end{aligned}$$

Substituting the two representations into the left-hand side of (2.7) completes the proof of the proposition. □

From Proposition 2.1 we rewrite the nonlocal term in (2.6) in the local form:

$$(1 - \dot{a})\varphi_x + v_t - \dot{a}v_x + (\varphi^2 v)_x + 3\varphi_x\left[v(t, 0) - \varphi v\right] = 0, \tag{2.8}$$

where if $v \in C(\mathbb{R})$, then the last term is continuous everywhere including $x = 0$ thanks to $\varphi(0) = 1$. Since φ_x is continuous everywhere except at the origin, the other terms of the linearized equation (2.8) are continuous at $x = 0$ if

$$\dot{a}(t) = 1 + 2v(t, 0) + \mathcal{O}(v(t, 0)^2) \tag{2.9}$$

where the remainder term in (2.9) is truncated at the linear approximation. Plugging (2.9) into (2.8) and keeping only the linear terms in v, we finally obtain the Cauchy problem (1.9) for the linearized equation at a single peakon.

2.2. **Solution to the linearized problem.** Following the idea of [23], we will solve the linearized problem (1.9) using the method of characteristics. For this, we first define the characteristic curves $q(t, s)$ as

$$\begin{cases}\dfrac{dq}{dt} = \varphi^2(q) - 1, \\ q(0, s) = s.\end{cases} \tag{2.10}$$

For any fixed $s \in \mathbb{R}$, the initial-value problem (2.10) has a unique solution since φ is Lipschitz. Moreover, it follows that

$$q_s(t, s) = \exp\left(\int_0^t 2\varphi\varphi_x(q(\tau, s))\, d\tau\right) > 0 \tag{2.11}$$

hence $q(t, \cdot)$ is a diffeomorphism on $\mathbb{R}$ for any $t \in \mathbb{R}$.

Since $\varphi(0) = 1$, we have $q(t, 0) = 0$ for any $t \in \mathbb{R}$, meaning that the location of the peak of φ is invariant under the flow of system (2.10). Solving (2.10) explicitly, we obtain that

$$q(t, s) = \begin{cases}\dfrac{1}{2}\log\left[1 + \left(e^{2s} - 1\right)e^{-2t}\right], & s > 0, \\ 0, & x = 0, \\ -\dfrac{1}{2}\log\left[1 + \left(e^{-2s} - 1\right)e^{2t}\right], & s < 0.\end{cases} \tag{2.12}$$

From (2.12) it follows that $q(t, s) \to 0$ as $s \to 0^{\pm}$. Define

$$V(t, s) := v(t, q(t, s)). \tag{2.13}$$

From (2.12) we know that when solving (1.9) along the characteristics q, we can consider characteristics with $s > 0$ separately from characteristics with $s < 0$. This corresponds to partitioning of $\mathbb{R}$ into $\mathbb{R}^+$ and $\mathbb{R}^-$ in the physical space and suggests us to consider solutions $v(t,\cdot) \in H^1 \cap C^1_0$ for any $t \in \mathbb{R}$, where $C^1_0 \subset W^{1,\infty}$ is given by (1.10). It follows from (1.9) and (2.10) that $V(t,s)$ satisfy

$$\begin{cases} \dfrac{dV}{dt} = \varphi_x(q) \left[\varphi(q) V - V(t,0)\right], \\ V(0,s) = v_0(s), \end{cases} \tag{2.14}$$

where we have used that $V(t,0) = v(t,q(t,0)) = v(t,0)$. It follows from (2.14) as $s \to 0^+$ that if $V(t,\cdot) \in C(\mathbb{R})$ for $t \in \mathbb{R}$, then $V(t,0) = V(0,0) = v(0,0) = v_0(0)$. Therefore, for $s > 0$ we are solving

$$\begin{cases} \dfrac{dV}{dt} = -e^{2q(t,s)} V + e^{-q(t,s)} v_0(0), \\ V(0,s) = v_0(s). \end{cases} \tag{2.15}$$

Direct computation yields the unique solution to the initial-value problem (2.15) in the form:

$$V(t,s) = \frac{v_0(s) + v_0(0)(e^t - 1)e^{-s}}{\sqrt{1 + (e^{2t} - 1)e^{-2s}}}, \quad s > 0. \tag{2.16}$$

Clearly we see that $\lim\limits_{s \to 0^+} V(t,s) = v_0(0)$. Similarly, for $s < 0$ we obtain the unique solution in the form:

$$V(t,s) = \frac{v_0(s) - v_0(0)(1 - e^{-t})e^{s}}{\sqrt{1 - (1 - e^{-2t})e^{2s}}}, \quad s < 0, \tag{2.17}$$

satisfying $\lim\limits_{s \to 0^-} V(t,s) = v_0(0)$.

One can also compute explicitly the evolution of v_x along the characteristics. Define

$$W(t,s) := v_x(t, q(t,s)). \tag{2.18}$$

Chain rule implies that

$$W(s,t) = \frac{V_s(t,s)}{q_s(t,s)}. \tag{2.19}$$

From (2.12), (2.16), and (2.19) we obtain that

$$\begin{aligned} W(t,s) =& \sqrt{1 + (e^{2t} - 1)e^{-2s}} \left[v_0'(s) - v_0(0)(e^t - 1)e^{-s}\right] \\ &+ \frac{(e^{2t} - 1)e^{-2s} \left[v_0(s) + v_0(0)(e^t - 1)e^{-s}\right]}{\sqrt{1 + (e^{2t} - 1)e^{-2s}}}, \quad s > 0. \end{aligned} \tag{2.20}$$

It follows from (2.20) as $s \to 0^+$ that

$$\lim_{s \to 0^+} W(t,s) = v_0(0)(e^t - 1) + v_0'(0^+)e^t. \tag{2.21}$$

Hence, the gradient $\lim_{x \to 0^+} v(t,x) = \lim_{s \to 0^+} W(t,s)$ grows exponentially in time. Similarly, from (2.12), (2.17), and (2.19) we obtain that

$$\begin{aligned} W(t,s) =& \sqrt{1 + (e^{-2t} - 1)e^{2s}} \left[v_0'(s) - v_0(0)(1 - e^{-t})e^{s}\right] \\ &+ \frac{(1 - e^{-2t})e^{2s} \left[v_0(s) - v_0(0)(1 - e^{-t})e^{s}\right]}{\sqrt{1 - (1 - e^{-2t})e^{2s}}}, \quad s < 0, \end{aligned} \tag{2.22}$$

from which we obtain

$$\lim_{s\to 0^-} W(t,s) = v_0(0)(1-e^{-t}) + v_0'(0^-)e^{-t}. \tag{2.23}$$

Hence, the gradient $\lim_{x\to 0^-} v(t,x) = \lim_{s\to 0^-} W(t,s)$ decays exponentially in time.

The following lemma justifies the solution constructed in (2.12), (2.16), (2.17), (2.20), and (2.22) and provides useful estimates.

Lemma 2.2. *For any $v_0 \in H^1 \cap C^1_0$, the Cauchy problem* (1.9) *admits a unique global solution $v \in C(\mathbb{R}; H^1 \cap C^1_0)$ satisfying the estimates:*

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} \leq |v_0(0)| + \|v_0\|_{L^\infty(\mathbb{R}^+)}, \tag{2.24}$$

$$\|v_x(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} \geq |v_0(0) + v_0'(0^+)|e^t - |v_0(0)|, \tag{2.25}$$

for any $t > 0$.

Proof. If $v_0 \in H^1 \cap C^1_0$, then the solution in (2.16) and (2.17) satisfies $V(t,\cdot) \in H^1 \cap C^1_0$ for any $t \in \mathbb{R}$ so that $V(t,\cdot)$ is locally Lipschitz continuous everywhere on $\mathbb{R}$. By the existence and uniqueness theory for differential equations, $V(t,s)$ is the unique solution of the initial-value problem (2.14) in this class of functions. Moveover, thanks to the property (2.11) and the property $q(t,s) \sim s$ as $|s| \to \infty$, we have $v(t,\cdot) \in H^1 \cap C^1_0$ for any $t \in \mathbb{R}$.

Since q is a diffeomorphism on $\mathbb{R}^+ \to \mathbb{R}^+$ and $\mathbb{R}^- \to \mathbb{R}^-$, we have

$$\|v\|_{L^\infty(\mathbb{R}^\pm)} = \|V\|_{L^\infty(\mathbb{R}^\pm)}, \qquad \|v_x\|_{L^\infty(\mathbb{R}^\pm)} = \|W\|_{L^\infty(\mathbb{R}^\pm)}.$$

From (2.16) we infer that

$$|V(t,s)| \leq |v_0(s)| + |v_0(0)|, \quad s > 0,$$

which yields (2.24). It follows from (2.21) that

$$\|W(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} \geq \lim_{s\to 0^+} |W(t,s)| \geq \left|v_0(0) + v_0'(0^+)\right| e^t - |v_0(0)|,$$

which yields (2.25). □

Remark 2.3. Even if $v_0 \in H^1 \cap C^1$, the solution of the linearized problem (1.9) only exists in $v(t,\cdot) \in H^1 \cap C^1_0$ because the jump of the derivative v_x across $x = 0$ appears instantaneously in time:

$$[v_x(t,x)]^+_- := \lim_{x\to 0^+} v_x(t,x) - \lim_{x\to 0^-} v_x(t,x) = 2v_0(0)(\cosh t - 1) + 2v_0'(0)\sinh t,$$

where $v_0'(0) = \lim_{x\to 0^+} v_{0x}(x) = \lim_{x\to 0^-} v_{0x}(x)$.

2.3. H^1 **conservation of** v**.** Estimate (2.25) in Lemma 2.2 indicates the linear $W^{1,\infty}$ instability of the Novikov peakons. For the Camassa–Holm peakons it is showed [23] that the perturbation are also H^1 linearly unstable. However for Novikov peakons, we will prove that the H^1 norm of the linearized perturbation v satisfying (1.9) is conserved for all time.

Lemma 2.4. *The unique global solution $v \in C(\mathbb{R}; H^1 \cap C^1_0)$ in Lemma 2.2 satisfies*

$$\|v(t,\cdot)\|^2_{H^1(\mathbb{R}^\pm)} = \|v_0\|^2_{H^1(\mathbb{R}^\pm)} \tag{2.26}$$

for every $t \in \mathbb{R}$.

Proof. Multiplying the linearized equation (1.9) by v and integrating on $\mathbb{R}^+$ using integration by parts we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^+)} - 2\int_0^\infty \varphi\varphi_x v^2\,dx + v(t,0)\int_0^\infty \varphi_x v\,dx = 0. \tag{2.27}$$

Differentiating (1.9) with respect to x yields

$$v_{xt} + (\varphi^2 v_x)_x - v_{xx} + \varphi_{xx}v(0) - (\varphi\varphi_x v)_x = 0 \tag{2.28}$$

Multiplying (2.28 by v_x and integrating over $\mathbb{R}^+$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_x\|^2_{L^2(\mathbb{R}^+)} + \int_0^\infty \left((\varphi^2)_x v_x^2 + \varphi^2 v_x v_{xx}\right)\,dx - \int_0^\infty v_x v_{xx}\,dx + v(t,0)\int_0^\infty \varphi v_x\,dx \\ - \int_0^\infty \left(2\varphi^2 v v_x + \varphi\varphi_x v_x^2\right)\,dx = 0.\end{aligned} \tag{2.29}$$

where we have used that $\varphi_{xx} = \varphi$ and $\varphi_x^2 = \varphi^2$ on $\mathbb{R}^+$. Using the fact that $\varphi(0) = 1$, $\varphi_x(0^+) = -1$, we integrate by parts and simplify (2.29) to the form:

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2_{L^2(\mathbb{R}^+)} + 2\int_0^\infty \varphi\varphi_x v^2\,dx - v(t,0)\int_0^\infty \varphi_x v\,dx = 0. \tag{2.30}$$

Adding (2.27) and (2.30) yields

$$\frac{d}{dt}\|v\|^2_{H^1(\mathbb{R}^+)} = 0, \quad \Rightarrow \quad \|v(t,\cdot)\|^2_{H^1(\mathbb{R}^+)} = \|v_0\|^2_{H^1(\mathbb{R}^+)}, \quad \text{for all} \quad t > 0.$$

Similarly we can prove the same result on $\mathbb{R}^-$, and hence we conclude the proof. □

Remark 2.5. Lemma 2.4 can be proven by integrating the explicit solutions (2.16) and (2.20) on $\mathbb{R}^+$ along the characteristics (2.12) with the chain rule:

$$\|v(t,\cdot)\|^2_{H^1(\mathbb{R}^+)} = \int_0^\infty \left[V(t,s)^2 + W(t,s)^2\right] q_s(t,s)ds = \|v_0\|^2_{H^1(\mathbb{R}^+)},$$

and similarly with the explicit solutions (2.17) and (2.22) on $\mathbb{R}^-$.

Proof of Theorem 1.1. Lemma 2.2 gives the existence of the unique solution $v \in C(\mathbb{R}, H^1 \cap C_0^1)$ to the linearized problem (1.9) for any initial datum $v_0 \in H^1 \cap C_0^1$ satisfying the estimate (1.12). Lemma 2.4 gives the H^1 conservation (2.26). □

3. Nonlinear analysis

Here we investigate the nonlinear dynamics of perturbations near a single peakon and prove Theorem 1.2. In what follows, we review weak solutions for the Cauchy problem (1.13), obtain an improved version of the H^1-orbital stability of a single peakon compared to Theorem A, derive the nonlinear system for peaked perturbations to a single peakon, solve this system with the method of characteristics, and obtain relevant estimates for the proof of Theorem 1.2.

3.1. **Weak solution theory.** Let's first recall two known results for global weak solutions to the Cauchy problem of the Novikov equation (1.13). The first result holds for initial datum $u_0 \in H^1$ and assumes the sign condition on $m_0 := u_0 - u_{0,xx}$.

Theorem 3.1 ([28]). *For any $u_0 \in H^1$ with $m_0 \in \mathcal{M}_+(\mathbb{R})$, where $\mathcal{M}_+$ is the space of non-negative finite Radon measures on $\mathbb{R}$, the Cauchy problem (1.13) admits a unique global weak solution $u \in W^{1,\infty}(\mathbb{R}^+ \times \mathbb{R}) \cap C(\mathbb{R}^+; H^1(\mathbb{R}))$ such that $m(t,\cdot) \in \mathcal{M}^+(\mathbb{R})$ for all $t > 0$, where $m := u - u_{xx}$. Moreover, $E(u)$ and $F(u)$ are conservation laws.*

Remark 3.2. The statement we give in Theorem 3.1 is stronger than the original statement of [28, Theorem 3.1]. Firstly, the solution constructed in [28] has weaker regularity $u \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$. However one can improve it to the strong topology $u \in C(\mathbb{R}^+; H^1(\mathbb{R}))$ by further using the conservation of $E(u)$. Secondly, [28] only asserts the conservation of $E(u)$. In fact a direct computation, see the proof of Lemma 3.8, allows one to further prove the conservation of $F(u)$.

The next result holds for the initial datum u_0 in the natural energy space $H^1 \cap W^{1,4}$ without the sign condition on m_0.

Theorem 3.3 ([4]). *Given $u_0 \in H^1 \cap W^{1,4}$. Then the Cauchy problem (1.13) admits a unique global weak solution $u(t,\cdot) \in H^1 \cap W^{1,4}$ for all $t \geq 0$. Moreover, $E(u)$ is a conservation law.*

Remark 3.4. For the instability analysis, we need to work with the initial datum u_0 in the restrictive function space $H^1 \cap W^{1,\infty}$ without the sign condition on m_0, for which neither Theorem 3.1 nor Theorem 3.3 is applicable. One of the reasons is that weak solutions in $H^1 \cap W^{1,\infty}$ enjoy (spatial) Lipschitz regularity which suits well for the standard theory for solvability of differential equations along the characteristics. The other reason is due to the fact that while $E(u)$ conserves for the weak solutions in $H^1 \cap W^{1,4}$, $F(u)$ is only conserved for almost every $t > 0$ [4]. Although no previous local well-posedness theory has been developed for the Cauchy problem (1.13) in $H^1 \cap W^{1,\infty}$, we will obtain the local well-posedness from the method of characteristics under the assumption that our solution in $H^1 \cap W^{1,\infty}$ consists of a single peakon perturbed by a single-peaked piecewise C^1 function, see Theorem 3.13.

Next we state the regularity of the nonlocal terms in (1.13). A similar argument as in [23, Lemma 5] combined with the estimates in [4, Section 2] leads to

Lemma 3.5. *If $u \in H^1 \cap W^{1,\infty}$, then $\mathcal{Q}[u] \in C(\mathbb{R})$. If $u \in H^1 \cap C^1_0$, then $\mathcal{Q}[u] \in C^1_0$.*

Following [23], the function class we use here is C^1_0 which is suited for capturing the single peak in the peaked solution u. Similarly to [23, Lemma 6], the location of the peak moves with its local characteristic speed.

Lemma 3.6. *Assume that there exists the unique weak solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ to the Cauchy problem (1.13) for some $T > 0$ with a jump of u_x across $x = a(t)$ such that $u(t,\cdot + a(t)) \in C^1_0$, $t \in [0,T)$. Then, we have $a \in C^1(0,T)$ and $a'(t) = u^2(t,a(t))$, for $t \in [0,T)$.*

Assuming local well-posedness of the Cauchy problem (1.13) for $u_0 \in H^1 \cap W^{1,\infty}$, we shall extend the result of Theorem A to prove the orbital stability of the single peakon φ in H^1, see Theorem 3.9.

3.2. **H^1-orbital stability of peakons for single-peaked perturbations.** Let us first recall the following characterization of $W^{1,p}$ functions in terms of the integrability of their spatial shifts.

Theorem 3.7 ([30] Theorem 2.1.6). *Let $1 \le p < \infty$. Then $u \in W^{1,p}(\mathbb{R}^d)$ if and only if $u \in L^p(\mathbb{R}^d)$ and the quantity*

$$\int_{\mathbb{R}^d} \left| \frac{u(x+h)-u(x)}{h} \right|^p dx$$

remains bounded for all $h \in \mathbb{R}^d$.

We show now that the two functionals $E(u)$ and $F(u)$ are still conserved for the same weak solutions as those assumed in Lemma 3.6.

Lemma 3.8. *Assume that there exists the unique weak solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ to the Cauchy problem* (1.13) *for some $T > 0$. Then, the values of $E(u)$ and $F(u)$ are conserved.*

Proof. Rewrite the convolution form (1.6) of the Novikov equation as follows:

$$u_t + u^2 u_x + \partial_x P_1(u, u_x) + P_2(u_x) = 0, \tag{3.1}$$

where

$$P_1(u,u_x) := \frac{1}{2}\varphi * \left(\frac{3}{2} u u_x^2 + u^3\right), \quad P_2(u_x) := \frac{1}{4}\varphi * (u_x^3).$$

Differentiating (3.1) in x and using that $(1-\partial_x^2)\varphi = 2\delta$ we obtain

$$u_{xt} + (u^2 u_x)_x - \left(\frac{3}{2} u u_x^2 + u^3\right) + P_1(u,u_x) + \partial_x P_2(u_x) = 0. \tag{3.2}$$

For analysis of conservation laws, we will regularize the evolution equations (3.1) and (3.2). Let $\varepsilon > 0$ and define

$$\overline{u}(x) := \eta_\varepsilon * u(x),$$

where $\eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right)$ and $\eta \ge 0$ is a smooth even function compactly supported in a ball of radius 1, and with integral equal to 1.

Applying the mollifier η_ε to (3.1) and using the cummutative and associative properties of the convolution, we obtain

$$\overline{u}_t + \overline{u}^2 \overline{u_x} + \partial_x P_1(\overline{u}, \overline{u_x}) + P_2(\overline{u_x}) = \mathcal{R}_1, \tag{3.3}$$

where

$$\mathcal{R}_1 := \overline{u}^2 \overline{u_x} - \overline{u^2 u_x} + \partial_x P_1(\overline{u}, \overline{u_x}) - \partial_x \overline{P_1(u,u_x)} + P_2(\overline{u_x}) - \overline{P_2(u_x)}.$$

Similarly, from (3.2) we have

$$\overline{u_{xt}} + (\overline{u}^2 \overline{u_x})_x - \left(\frac{3}{2}\overline{u}\ \overline{u_x}^2 + \overline{u}^3\right) + P_1(\overline{u}, \overline{u_x}) + \partial_x P_2(\overline{u_x}) = \mathcal{R}_2, \tag{3.4}$$

where

$$\mathcal{R}_2 := (\overline{u}^2 \overline{u_x})_x - (\overline{u^2 u_x})_x - \left(\frac{3}{2}\overline{u}\ \overline{u_x}^2 + \overline{u}^3\right) + \overline{\frac{3}{2} u u_x^2 + u^3}$$
$$+ P_1(\overline{u}, \overline{u_x}) - \overline{P_1(u,u_x)} + \partial_x P_2(\overline{u_x}) - \partial_x \overline{P_2(u_x)}.$$

We are now able to verify conservation of $E(u)$ and $F(u)$.

Conservation of $E(u)$: Following [4, Section 2], multiplying (3.3) by $\overline{u}$ and (3.4) by $\overline{u_x}$ we obtain a regularized local conservation law:

$$\partial_t \left(\frac{\overline{u}^2 + \overline{u_x}^2}{2} \right) + \partial_x \left(\frac{\overline{u}^2 \overline{u_x}^2}{2} + \overline{u} P_1(\overline{u}, \overline{u_x}) + \overline{u} \partial_x P_2(\overline{u_x}) \right) = \overline{u} \mathcal{R}_1 + \overline{u_x} \mathcal{R}_2. \tag{3.5}$$

Integration over $\mathbb{R}$ then gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E(\overline{u}) = \int_{\mathbb{R}} \bigg\{ & \frac{1}{3} \overline{u} \left(\overline{u}^3 - \overline{u^3} \right)_x + \frac{1}{3} \overline{u_x} \left(\overline{u}^3 - \overline{u^3} \right)_{xx} + \frac{3}{2} \overline{u_x} \left(\overline{u} \; \overline{u_x}^2 - \overline{u u_x^2} \right) \\ & + \overline{u_x} \left(\overline{u}^3 - \overline{u^3} \right) + \frac{1}{2} \overline{u} \left(\overline{u_x}^3 - \overline{u_x^3} \right) \bigg\} \, dx \end{aligned} \tag{3.6}$$

Note that $u \in H^1 \cap W^{1,\infty}$, and hence $u, u_x \in L^p$ for any $2 \le p \le \infty$. The properties of smooth approximation imply that

$$\|\overline{u} - u\|_{L^p(\mathbb{R})} \to 0, \quad \|\overline{u_x} - u_x\|_{L^p(\mathbb{R})} \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{3.7}$$

This way we know that the first, fourth, and fifth terms in the right-hand side of (3.6) all converge to zero as $\varepsilon \to 0$.

For the third term, note that we can write

$$\begin{aligned} \overline{u} \; \overline{u_x}^2 - \overline{u u_x^2} &= \overline{u} \left(\overline{u_x}^2 - \overline{u_x^2} \right) + \left(\overline{u} \; \overline{u_x^2} - \overline{u u_x^2} \right) \\ &= \overline{u} \left(\overline{u_x}^2 - u_x^2 \right) + \overline{u} \left(u_x^2 - \overline{u_x^2} \right) + \left(\overline{u} \; \overline{u_x^2} - \overline{u u_x^2} \right). \end{aligned}$$

The first two terms of the above can be treated using (3.7). For the last term, we can recall [9, Lemma 3], which states that if f is uniformly continuous and bounded, and $\mu \in \mathcal{M}(\mathbb{R})$, then $\overline{f\mu} - \overline{f}\overline{\mu} \to 0$ in L^1. Since u is Lipschitz and $u_x^2 \in L^1$, we have that $\overline{u} \; \overline{u_x^2} - \overline{u u_x^2} \to 0$ in L^1. Therefore, the third term in the right-hand side of (3.6) converges to zero as $\varepsilon \to 0$.

Finally we look to show that

$$\int_{\mathbb{R}} \overline{u_x} \left(\overline{u}^3 - \overline{u^3} \right)_{xx} dx = \int_{\mathbb{R}} \overline{u}_{xxx} \left(\overline{u}^3 - \overline{u^3} \right) dx \;\to\; 0 \quad \text{as} \;\; \varepsilon \to 0.$$

Since the above obviously holds for smooth functions, one can use the Banach–Steinhaus theorem to observe that it is enough to show that $\left\| \overline{u}_{xxx} (\overline{u}^3 - \overline{u^3}) \right\|_{L^1}$ is uniformly bounded.

To this end, note that

$$\begin{aligned} |\overline{u}_{xxx}(x)| &= \frac{1}{\varepsilon^4} \left| \int_{-\varepsilon}^{\varepsilon} u(x-y) \eta''' \left(\frac{y}{\varepsilon} \right) dy \right| = \frac{1}{\varepsilon^4} \left| \int_{-\varepsilon}^{\varepsilon} \left(u(x-y) - u(x) \right) \eta''' \left(\frac{y}{\varepsilon} \right) dy \right| \\ &\le \frac{1}{\varepsilon^4} \int_{-\varepsilon}^{\varepsilon} \frac{|u(x-y) - u(x)|}{|y|} \left| \eta''' \left(\frac{y}{\varepsilon} \right) \right| |y| \, dy \lesssim \frac{1}{\varepsilon^2} \|u_x\|_{L^\infty}. \end{aligned} \tag{3.8}$$

It is also straightforward to check that

$$\overline{u^3} - \overline{u}^3 = \overline{r}_3(u) + (u - \overline{u})^3 + 3u \overline{r}_2(u) - 3u(u - \overline{u})^2,$$

where

$$\overline{r}_n(u) := \int_{-\varepsilon}^{\varepsilon} \left(u(x-y) - u(x) \right)^n \eta_\varepsilon(y) \, dy$$

This way

$$
\begin{aligned}
|\overline{r}_n(u)(x)| &\le \int_{-\varepsilon}^{\varepsilon} \left(\frac{|u(x-y)-u(x)|}{|y|}\right)^n \eta_\varepsilon(y)|y|^n\,dy \\
&\le \varepsilon^n \int_{-\varepsilon}^{\varepsilon} \left(\frac{|u(x-y)-u(x)|}{|y|}\right)^n \eta_\varepsilon(y)\,dy.
\end{aligned}
$$

Moreover it follows from Hölder's inequality that

$$
\begin{aligned}
|\overline{u}-u|^n &= \left|\int_{-\varepsilon}^{\varepsilon} (u(x-y)-u(x))\,\eta_\varepsilon(y)\,dy\right|^n \\
&\le \left(\int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y)\,dy\right)^{n-1} \left(\int_{-\varepsilon}^{\varepsilon} |u(x-y)-u(x)|^n\,\eta_\varepsilon(y)\,dy\right) \\
&= |\overline{r}_n(u)(x)|\,.
\end{aligned}
$$

An application of Fubini Theorem together with Theorem 3.7 implies that

$$
\begin{aligned}
\|\overline{r}_n(u)\|_{L^1} &\le \varepsilon^n \int_{\mathbb{R}}\int_{-\varepsilon}^{\varepsilon} \left(\frac{|u(x-y)-u(x)|}{|y|}\right)^n \eta_\varepsilon(y)\,dydx \\
&\le \varepsilon^n \int_{-\varepsilon}^{\varepsilon} \left[\int_{\mathbb{R}} \left(\frac{|u(x-y)-u(x)|}{|y|}\right)^n dx\right] \eta_\varepsilon(y)\,dy \\
&\le \varepsilon^n \|u\|_{W^{1,n}}^n.
\end{aligned} \tag{3.9}
$$

From (3.8) and (3.9) it follows that

$$
\left\|\overline{u}_{xxx}(\overline{u}^3-\overline{u^3})\right\|_{L^1} \lesssim \|u_x\|_{L^\infty}\left(\varepsilon\|u\|_{W^{1,3}}^3 + \|u\|_{H^1}^2\right).
$$

Putting together the above estimates we obtain that

$$
\frac{d}{dt}E(\overline{u}) \to 0, \quad \text{as} \quad \varepsilon \to 0,
$$

which proves the conservation of $E(u)$.

Conservation of $F(u)$**:** Similarly as before, to get the conservation law for $F(\overline{u})$ we multiply (3.1) by $4\overline{u}^3 + 2\overline{u}\ \overline{u_x}^2$, multiply (3.2) by $-\frac{4}{3}\overline{u_x}^3 + 2\overline{u}^2\overline{u_x}$ and integrate over $\mathbb{R}$ we have

$$
\frac{d}{dt}F(\overline{u}) = \int_{\mathbb{R}} \left[(4\overline{u}^3 + 2\overline{u}\ \overline{u_x}^2)\mathcal{R}_1 + \left(2\overline{u}^2\overline{u_x} - \frac{4}{3}\overline{u_x}^3\right)\mathcal{R}_2\right] dx.
$$

The rest of the proof follows in a similar way. □

Since the proof of Theorem A in [12] only makes use of the continuity of the solution and the conservation of E and F, we can recast the same idea in our current regularity setting to obtain the following result.

Theorem 3.9. *For every $0<\varepsilon\ll 1$, let $u_0 \in H^1 \cap C_0^1$ satisfy*

$$
\|u_0-\varphi\|_{H^1} < \varepsilon^4.
$$

Assume existence of the unique weak solution $u \in C([0,T), H^1\cap W^{1,\infty})$ to the Cauchy problem (1.13) with the initial datum u_0 and the maximal existence time $T>0$ such that $u(t,\cdot+a(t)) \in$

C^1_0, $t \in [0,T)$ *for some* $a \in C^1([0,T)$ *with* $a(0) = 0$. *The corresponding solution* u *satisfies*

$$\sup_{t\in[0,T)} \|u(t,\cdot\,) - \varphi(\,\cdot - a(t))\|_{H^1} < 2\left(4 + \|u_{0x}\|^{1/2}_{L^\infty}\right)\varepsilon.$$

Remark 3.10. Because $u(t,\cdot + a(t)) \in C^1_0$ is H^1 close to φ in Theorem 3.9, it follows from continuous embedding of H^1 to C^0 and monotonicity of φ with $\lim_{x\to 0^\pm}\varphi_x(x) = \mp 1$ that the location of the peak at $a(t)$ in Theorem 3.9 coincides with the location of the maximum of u at $\xi(t)$ in Theorem A.

3.3. **Derivation of the evolution problem for perturbations to a single peakon.** We shall construct a unique weak solution $u \in C([0,T), H^1\cap W^{1,\infty})$ to the Cauchy problem (1.13) for some $T > 0$ with a single jump of u_x across $x = a(t)$. We use the same decomposition (2.4) and look for the modulation $a \in C^1(0,T)$ and the perturbation $v \in C([0,T), H^1 \cap W^{1,\infty})$ to the peakon φ. If $v(t,\cdot) \in C^1_0$ for all $t \in [0,T)$, then the solution u satisfies $u(t,\cdot + a(t)) \in C^1_0$, $t \in [0,T)$ so that Lemma 3.6 implies that $a \in C^1(0,T)$ satisfies the following modulation equation:

$$\dot{a}(t) = u^2(t,a(t)) = (\varphi(0) + v(t,0))^2 = (1 + v(t,0))^2. \tag{3.10}$$

Note that the linear part of this modulation equation has already been used in the linearized equation (2.9). Thus, the problem of constructing the local solution $u \in C([0,T), H^1\cap W^{1,\infty})$ is now replaced by the problem of constructing the local solution $v \in C([0,T), H^1 \cap W^{1,\infty})$ such that $v(t,\cdot) \in C^1_0$ for all $t \in [0,T)$.

Substituting (2.4) and (3.10) into (1.13) yields the following equation:

$$v_t - (1 + v(t,0))^2(\varphi_x + v_x) + \varphi^2\varphi_x + (\varphi^2 v + \varphi v^2)_x + v^2 v_x + \mathcal{Q}[\varphi + v] = 0. \tag{3.11}$$

Canceling the stationary equation (2.5) for φ and grouping the linear, quadratic, and cubic terms together, we obtain the evolution equation for v in the form:

$$v_t + \mathcal{N}_1(v) + \mathcal{N}_2(v) + \mathcal{N}_3(v) = 0, \tag{3.12}$$

where

$$\mathcal{N}_1(v) = \left[\varphi^2 - 1\right] v_x + 2\varphi_x\left[\varphi v - v(t,0)\right] + \frac{3}{2}\varphi_x * \left(\varphi^2 v + \frac{1}{2}\varphi_x^2 v + \varphi\varphi_x v_x\right) + \frac{3}{4}\varphi * \left(\varphi_x^2 v_x\right),$$

$$\mathcal{N}_2(v) = 2\left[\varphi v - v(t,0)\right] v_x + \varphi_x\left[v^2 - v^2(t,0)\right] + \frac{1}{2}\varphi_x * \left(\frac{3}{2}\varphi v_x^2 + 3\varphi_x v v_x + 3\varphi v^2\right) + \frac{3}{4}\varphi * \left(\varphi_x v_x^2\right)$$

$$\mathcal{N}_3(v) = \left[v^2 - v^2(t,0)\right] v_x + \mathcal{Q}[v].$$

By Proposition 2.1, the linear part is reduced to the local form:

$$\mathcal{N}_1(v) = \left[\varphi^2 - 1\right] v_x - \varphi_x\left[\varphi v - v(t,0)\right]. \tag{3.13}$$

In order to simplify the quadratic part, we use the following proposition

Proposition 3.11. *Let* $f \in L^1(\mathbb{R})$. *Then*

$$\varphi_x * (\varphi f) + \varphi * (\varphi_x f) = -2\varphi\int_0^x \varphi(y)f(y)\,dy. \tag{3.14}$$

Proof. Since $\varphi_x = -\mathrm{sgn}(x)\varphi$, direct computation shows that

$$\begin{aligned}\varphi_x * (\varphi f) + \varphi * (\varphi_x f) &= e^{-x}\int_{-\infty}^{x} (\varphi_y - \varphi)\, f(y)\, dy + e^{x}\int_{x}^{\infty} (\varphi_y + \varphi)\, f(y)\, dy \\ &= e^{-x}\int_{0}^{x} (\varphi_y - \varphi)\, f(y)\, dy + e^{x}\int_{x}^{0} (\varphi_y + \varphi)\, f(y)\, dy \\ &= -2\varphi \int_0^x \varphi(y) f(y)\, dy,\end{aligned}$$

which is (3.14). ∎

Using Proposition 3.11, we prove the following proposition:

Proposition 3.12. *For $v \in H^1$ we have*

$$\begin{aligned}&\frac{1}{2}\varphi_x * \left(\frac{3}{2}\varphi v_x^2 + 3\varphi_x v v_x + 3\varphi v^2\right) + \frac{3}{4}\varphi * \left(\varphi_x v_x^2\right) \\ &= -\frac{3}{2}\varphi_x \left[v^2 - v^2(t,0)\right] - \frac{3}{2}\varphi \int_0^x \varphi(v^2 + v_y^2)\, dy.\end{aligned}$$

Proof. Integrating by parts and using (2.1), we obtain

$$\frac{3}{2}\varphi_x * (\varphi_x v v_x) = \frac{3}{4}\varphi * \left(\varphi_x v^2\right) - \frac{3}{2}\varphi_x v^2 - \frac{3}{4}\varphi_x * \left(\varphi v^2\right) + \frac{3}{2}\varphi_x v^2(t,0).$$

Combining with other convolution terms, we obtain

$$\frac{3}{4}\varphi_x * \left(\varphi(v^2 + v_x^2)\right) + \frac{3}{4}\varphi * \left(\varphi_x(v^2 + v_x^2)\right) = -\frac{3}{2}\varphi \int_0^x \varphi(v^2 + v_y^2)\, dy,$$

where the result of Proposition 3.11 has been used. ∎

By Proposition 3.12, the quadratic part is reduced to the simple form:

$$\mathcal{N}_2(v) = 2\left[\varphi v - v(t,0)\right] v_x - \frac{1}{2}\varphi_x \left[v^2 - v^2(t,0)\right] - \frac{3}{2}\varphi \int_0^x \varphi(v^2 + v_y^2)\, dy. \tag{3.15}$$

Putting (3.13) and (3.15) into (3.12), we obtain the Cauchy problem for the perturbation v to the peakon φ in the following form:

$$\begin{cases} v_t + \left[(\varphi + v)^2 - (1 + v(t,0))^2\right] v_x - \varphi_x\left(\varphi v - v(t,0)\right) - \dfrac{1}{2}\varphi_x\left(v^2 - v^2(t,0)\right) \\ \qquad - \dfrac{3}{2}\varphi \displaystyle\int_0^x \varphi(v^2 + v_y^2)\, dy + \mathcal{Q}[v] = 0, \\ v(0,x) = v_0(x). \end{cases} \tag{3.16}$$

As discussed above, the small initial datum v_0 belongs to the space $H^1 \cap C_0^1$ and we are looking for the unique local weak solution $v \in C([0,T), H^1 \cap W^{1,\infty})$ to the evolution problem (3.16) for some $T > 0$ such that $v(t,\cdot) \in C_0^1$ for all $t \in [0,T)$. The following result states local well-posedness of the Cauchy problem (3.16).

Theorem 3.13. *For every initial datum $v_0 \in H^1 \cap C_0^1$, there exist the maximal existence time $T > 0$ and the unique solution $v \in C([0,T), H^1 \cap C_0^1)$ to the Cauchy problem (3.16) that depends continuously on the initial datum $v_0 \in H^1 \cap C_0^1$.*

Theorem 3.13 is proven next by using the method of characteristics.

3.4. **Solution to the evolution problem.** The evolution problem (3.16) suggests us to work with the characteristics $q(t,s)$ which satisfy the following evolution problem:

$$\begin{cases} \dfrac{dq}{dt} = [\varphi(q) + v(t,q)]^2 - [1 + v(t,0)]^2, \\ q(0,s) = s. \end{cases} \tag{3.17}$$

Compared to the linearized evolution problem (2.10), we cannot solve the nonlinear evolution problem (3.17) explicitly. However, we can analyze if the slope function

$$f(t,q) := [\varphi(q) + v(t,q)]^2 - [1 + v(t,0)]^2 \tag{3.18}$$

defines a well-posed initial-value problem in the correct solution space for v, as is done in the following lemma.

Lemma 3.14. *Assume that $v \in C([0,T), H^1 \cap C^1_0)$ with some maximal existence time $T > 0$. There exists the unique solution $q \in C^1([0,T), C^1_0)$ to system (3.17) such that the mapping $\mathbb{R} \ni s \mapsto q(t,\cdot) \in C^1_0$ is invertible for every $t \in [0,T)$ and satisfies $q(t,0) = 0$ and* $\lim_{|s|\to\infty} q_s(t,s) = 1$.

Proof. Since $\varphi \in C^1_0 \subset W^{1,\infty}$ and $v(t,\cdot) \in C^1_0 \subset W^{1,\infty}$ for $t \in [0,T)$, then f is Lipschitz in q and continuous in t for every $t \in [0,T)$. By existence, uniqueness, and continuous dependence theory for differential equations, the initial-value problem (3.17) admits the unique solution $q(t,s)$ satisfying $q(\cdot,s) \in C^1(0,T)$ for any $s \in \mathbb{R}$ and $q(t,\cdot) \in C^1_0$ for any $t \in [0,T)$. Moreover, $f(t,0) = 0$, hence $q(t,0) = 0$ holds for all $t \in [0,T)$.

Differentiating the initial-value problem (3.17) with respect to s piecewise for $s > 0$ and $s < 0$ yields

$$\begin{cases} \dfrac{dq_s}{dt} = 2\,[\varphi(q) + v(t,q)]\,[\varphi_x(q) + v_x(t,q)]\,q_s, \\ q_s(0,s) = 1, \end{cases} \qquad s \in \mathbb{R}\backslash\{0\}, \tag{3.19}$$

with the unique solution for every $s \in \mathbb{R}\backslash\{0\}$:

$$q_s(t,s) = \exp\left(2\int_0^t [\varphi(q) + v(\tau,q)]\,[\varphi_x(q) + v_x(\tau,q)]\,d\tau\right) > 0, \tag{3.20}$$

hence $q(t,\cdot)$ is invertible on $\mathbb{R}$ for $t \in [0,T)$. Moreover we have $\lim_{|s|\to\infty} q_s(t,s) = 1$ for $t \in [0,T)$ because $v_x(t,\cdot) \in L^\infty$ and $v(t,q) \to 0$ as $|q| \to \infty$ for $t \in [0,T)$ thanks to the Sobolev embedding of $H^1(\mathbb{R})$ to the space of continuous and decaying functions. ☐

Setting $V(t,s) := v(t,q(t,s))$ as in (2.13), then it follows from (3.16) that evolution of V along the characteristics q is given by

$$\begin{cases} \dfrac{dV}{dt} = \varphi_x(q)\,[\varphi(q)V - V(t,0)] + \dfrac{1}{2}\varphi_x(q)\left[V^2 - V^2(t,0)\right] \\ \qquad\quad + \dfrac{3}{2}\varphi(q)\displaystyle\int_0^q \varphi(v^2 + v_y^2)\,dy - \mathcal{Q}[v](q) \\ V(0,s) = v_0(s). \end{cases} \tag{3.21}$$

Denote $V^0(t) := V(t,0) = v(t,0)$, where the last equality follows from $q(t,0) = 0$. It follows from the initial-value problem (3.21) as $s \to 0$ from either side that V^0 satisfies the limiting initial-value problem

$$\begin{cases} \dfrac{dV^0}{dt} = -\mathcal{Q}[v](0), \\ V^0(0) = v_0(0). \end{cases} \tag{3.22}$$

In order to control solvability of the solution in (3.20), we need to control v_x, and hence V_s along the characteristics. Therefore we need to differentiate (3.16) in order to derive the evolution equation for v_x. The appearance of φ' in (3.16) presents severe trouble when differentiating. The way to overcome that is to "cut out" the origin and consider solving the evolution equation for $w := v_x$ separately on $\mathbb{R}^+$ and $\mathbb{R}^-$. This agrees with Lemma 3.14, which suggests that for the solution $v \in C([0,T), H^1 \cap C^1_0)$ the spatial domain $\mathbb{R}$ can be partitioned into $\mathbb{R}^+$ and $\mathbb{R}^-$ on two sides from the peaked wave φ invariantly in time t.

Computing derivative of (3.16) separately on $\mathbb{R}^+$ and $\mathbb{R}^-$ and using the fact that $\varphi'' = \varphi$ on $\mathbb{R}\backslash\{0\}$, we derive the evolution equation for $x \neq 0$:

$$\begin{cases} w_t + \left[(\varphi+v)^2 - (1+v(t,0))^2\right] w_x - \varphi\left(\varphi v - v(t,0)\right) - \dfrac{1}{2}\varphi\left(v^2 - v^2(t,0)\right) \\ \qquad + \varphi\varphi_x w - \varphi^2 v + \varphi_x v w + 2\varphi w^2 + \dfrac{1}{2} v w^2 - v^3 \\ \qquad - \dfrac{3}{2}\varphi^2(v^2+w^2) - \dfrac{3}{2}\varphi_x \displaystyle\int_0^x \varphi(v^2+w^2)\,dy + \mathcal{P}[v] = 0, \\ w(0,x) = v_{0x}(x), \end{cases} \tag{3.23}$$

where

$$\mathcal{P}[v](x) := \frac{1}{2}\varphi * \left(\frac{3}{2} v v_x^2 + v^3\right) + \frac{1}{4}\varphi_x * v_x^3. \tag{3.24}$$

Setting $W(t,s) := v_x(t,q(t,s))$ as in (2.18), then it follows that W satisfies (2.19). If the mapping $\mathbb{R} \ni s \mapsto q \in C^1_0$ is invertible as in Lemma 3.14, we have $\|V\|_{L^\infty} = \|v\|_{L^\infty}$ and $\|W\|_{L^\infty} = \|v_x\|_{L^\infty}$. Writing the evolution problem (3.23) at the characteristics yields for $s \neq 0$:

$$\begin{cases} \dfrac{dW}{dt} = \varphi(q)\left[\varphi(q)V - V(t,0)\right] + \dfrac{1}{2}\varphi_x(q)\left[V^2 - V^2(t,0)\right] \\ \qquad -\varphi(q)\varphi_x(q)W + \varphi(q)^2 V - \varphi_x(q) V W - 2\varphi(q) W^2 - \dfrac{1}{2} V W^2 + V^3 \\ \qquad + \dfrac{3}{2}\varphi(q)^2(V^2+W^2) + \dfrac{3}{2}\varphi_x(q)\displaystyle\int_0^q \varphi(v^2+w^2)\,dy - \mathcal{P}[v](q), \\ W(0,s) = v_{0x}(s). \end{cases} \tag{3.25}$$

Compared to the linearized evolution problem (2.14), we cannot solve the nonlinear evolution problems (3.21) and (3.25) explicitly. Nevertheless, we can analyze the vector field for the evolution system

$$\frac{d}{dt}\begin{bmatrix} q \\ V \\ W \end{bmatrix} = \begin{bmatrix} f^{(q)}(q,V) \\ f^{(V)}(q,V,W) \\ f^{(W)}(q,V,W) \end{bmatrix} =: F(q,V,W), \tag{3.26}$$

where components of $F(q,V,W)$ are given by

$$\begin{aligned} f^{(q)}(q,V,) &:= [\varphi(q)+V]^2 - \left[1+V^0\right]^2, \\ f^{(V)}(q,V,W) &:= \varphi_x(q)\left[\varphi(q)V - V^0\right] + \frac{1}{2}\varphi_x(q)\left[V^2-(V^0)^2\right] \\ &\quad + \frac{3}{2}\varphi(q)\int_0^q \varphi(v^2+w^2)\,dy - \mathcal{Q}[v](q), \\ f^{(W)}(q,V,W) &:= \varphi(q)\left[\varphi(q)V - V^0\right] + \frac{1}{2}\varphi_x(q)\left[V^2-(V^0)^2\right] \\ &\quad -\varphi(q)\varphi_x(q)W + \varphi(q)^2 V - \varphi_x(q)VW - 2\varphi(q)W^2 - \frac{1}{2}VW^2 + V^3 \\ &\quad + \frac{3}{2}\varphi(q)^2(V^2+W^2) + \frac{3}{2}\varphi_x(q)\int_0^q \varphi(v^2+w^2)\,dy - \mathcal{P}[v](q). \end{aligned}$$

The dynamical system (3.26) is equipped with the initial datum:

$$\left.\begin{bmatrix} q \\ V \\ W \end{bmatrix}\right|_{t=0} = \begin{bmatrix} s \\ v_0(s) \\ v_{0x}(s) \end{bmatrix}, \quad s \in \mathbb{R}. \tag{3.27}$$

Because of the nonlocal terms in $f^{(V)}$ and $f^{(W)}$, the vector field $F(q,V,W)$ computed for solutions to the dynamical system (3.26) with the initial datum (3.27) with one value of $s \in \mathbb{R}$ requires global information about solutions (q,V,W) computed for all other values of s on $\mathbb{R}$.

The nonlocal terms are treated with the chain rule $v(q(s)) = V(s)$ and $v_x(q(s)) = W(s)$ provided that the mapping $\mathbb{R} \ni s \mapsto q \in C^1_0$ is invertible. In addition, we use $V^0 = V(0)$. The following lemma show that the vector field $F(q,V,W)$ is locally Lipschitz with respect to (q,V,W) and preserves properties of the mapping $\mathbb{R} \ni s \mapsto q \in C^1_0$ and properties of the solution (v,w).

Lemma 3.15. *For every $q \in C^1_0$ satisfying $q(0)=0$, $\inf_{s\in\mathbb{R}} q_s(s) > 0$, and $\lim_{|s|\to\infty} q_s(s) = 1$ and every $v \in H^1 \cap C^1_0$, the vector field $F(q,V,W)$ is locally Lipschitz in (q,V,W) separately for $q \in \mathbb{R}^+$ and $q \in \mathbb{R}^-$. Moreover, we have*

(i) $f^{(q)}(0,V^0) = 0$, $f^{(V)}(0,V^0,W) = -\mathcal{Q}[v](0)$,
(ii) $f^{(V)}(q(\cdot),V(\cdot),W(\cdot)) \in L^2$, $f^{(W)}(q(\cdot),V(\cdot),W(\cdot)) \in L^2$,
(iii) $\partial_s f^{(q)}(q(s),V(s)) = G(s)q_s(s)$ *with* $G \in L^\infty$ *satisfying* $\lim_{|s|\to\infty} G(s) = 0$.

Proof. Thanks to the assumption $q_s(s) > 0$ for every $s \in \mathbb{R}$, the mapping $\mathbb{R} \ni s \mapsto q \in C^1_0$ is invertible, hence $V(s) = v(q(s))$ belongs to C^1_0 and $W(s) = v_x(q(s))$ is bounded and continuous for $s \in \mathbb{R}^+$ and $s \in \mathbb{R}^-$. Thanks to the assumption $\lim_{|s|\to\infty} q_s(s) = 1$ and the chain rule, it follows from $v \in H^1$ that $V \in L^2$ and $W \in L^2$. Thanks to the assumption $q(0) = 0$, the vector field $F(q,V,W)$ in system (3.26) can be considered separately for $q \in \mathbb{R}^+$ and $q \in \mathbb{R}^-$.

All local terms in $F(q,V,W)$ are locally Lipschitz in (q,V,W) separately for $q \in \mathbb{R}^+$ and $q \in \mathbb{R}^-$. The nonlocal terms in $f^{(V)}(q,V,W)$ are also locally Lipschitz in (q,V,W) for every $q \in \mathbb{R}$, $V \in L^2$, and $W \in L^2$, thanks to integrability of v^2+w^2, invertibility of the mapping $\mathbb{R} \ni s \mapsto q \in C^1_0$, and the chain rule, e.g.

$$\varphi(q)\int_0^q \varphi(v^2+w^2)\,dy = \varphi(q)\int_0^q \varphi(q(s'))(V^2+W^2)(s')q_s(s')ds'$$

and

$$\mathcal{Q}[v](q) = \frac{1}{2}\int_{\mathbb{R}} \varphi_x\big(q-q(s')\big)\left(\frac{3}{2}VW^2+V^3\right)(s')q_s(s')\,ds' + \frac{1}{4}\int_{\mathbb{R}} \varphi\big(q-q(s')\big)W^3(s')q_s(s')\,ds',$$

Similarly, it follows that the nonlocal terms in $f^{(W)}(q,V,W)$ are locally Lipschitz in (q,V,W) for every $q\in\mathbb{R}$, $V\in L^2$, and $W\in L^2$.

It remains to verify items (i), (ii), and (iii). It follows from the factorization formula:

$$f^{(q)}(q,V) = (\varphi(q)+1+V+V^0)(\varphi(q)-1+V-V^0),$$

that $f^{(q)}(q,V)$ is locally Lipschitz at $q=0$ and $V=V^0$ with $f^{(q)}(0,V^0)=0$. Similarly, $f^{(V)}(q,V,W)$ is locally Lipschitz at $q=0$, $V=V^0$, and every $W\in\mathbb{R}$ with $f^{(V)}(0,V^0,W) = -\mathcal{Q}[v](0)$. This verifies item (i). Note that $f^{(W)}(q,V,W)$ is not locally Lipschitz at $q=0$, $V=V^0$, and $W\neq 0$ because of the local terms $-\varphi(q)\varphi_x(q)W$ and $-\varphi_x(q)VW$ in $f^{(W)}(q,V,W)$.

For item (ii), all local terms in $f^{(V)}(q(\cdot),V(\cdot),W(\cdot))$ and $f^{(W)}(q(\cdot),V(\cdot),W(\cdot))$ are in L^2 because $\varphi,\varphi_x,V,W\in L^2\cap L^\infty$. Similarly, nonlocal terms are in L^2 because of invertibility of the mapping $\mathbb{R}\ni s\mapsto q\in C^1_0$ and the chain rule. For instance, we have for $f^{(V)}$,

$$\|\varphi(q(\cdot))\int_0^{q(\cdot)}\varphi(v^2+w^2)\,dy\|_{L^2} \leq \frac{1}{[\inf_{s\in\mathbb{R}}|q_s(s)|]^{1/2}}\|\varphi\|_{L^2}\|\varphi\|_{L^\infty}\|v\|^2_{H^1}$$

and

$$\|\mathcal{Q}[v](q(\cdot))\|_{L^2} \leq \frac{1}{[\inf_{s\in\mathbb{R}}|q_s(s)|]^{1/2}}\left(\frac{3}{4}\|\varphi_x\|_{L^2}\|v\|_{L^\infty}\|v\|^2_{H^1}+\frac{1}{4}\|\varphi\|_{L^2}\|w\|_{L^\infty}\|w\|^2_{L^2}\right),$$

and similar estimates for $f^{(W)}$.

Finally, for item (iii), we have explicitly

$$\partial_s f^{(q)}(q(s),V(s)) = 2(\varphi(q(s))+V(s))(\varphi_x(q(s))+W(s))q_s(s) =: G(s)q_s(s),$$

so that $G\in L^\infty$ and $\lim_{|s|\to\infty}G(s)=0$. $\square$

Theorem 3.13 is proven by using Lemma 3.15.

Proof of Theorem 3.13. We consider the initial datum $v_0\in H^1\cap C^1_0$ for which $v_{0x}\in L^2$ is continuous separately for $x\in\mathbb{R}^+$ and $x\in\mathbb{R}^-$. The dynamical system (3.26) is considered with the initial datum (3.27) which satisfies the assumptions of Lemma 3.15.

By Lemma 3.15, the vector field preserves the assumptions in the sense that if we define

$$\left\{\begin{array}{l} \hat{q}(t,s) = s+\int_0^t f^{(q)}(q(t',s),V(t',s))dt', \\ \hat{V}(t,s) = v_0(s)+\int_0^t f^{(V)}(q(t',s),V(t',s),W(t',s))dt', \\ \hat{W}(t,s) = v_{0x}(s)+\int_0^t f^{(W)}(q(t',s),V(t',s),W(t',s))dt', \end{array}\right.$$

and

$$\hat{q}_s(t,s) = 1+\int_0^t G(s)q_s(t',s)dt',$$

then for every t on a compact interval $[-\tau,\tau]$ with small $\tau>0$, we have $\hat{q}\in C^1_0$ satisfying $\hat{q}(0)=0$, $\inf_{s\in\mathbb{R}}\hat{q}_s(s)>0$, and $\lim_{|s|\to\infty}\hat{q}_s(s)=1$ and $\hat{v}\in H^1\cap C^1_0$. By the existence and uniqueness theory for differential equations, there exists the unique solution

$q \in C^1([0,T), C^1_0)$, $V \in C^1([0,T), H^1 \cap C^1_0)$, and $W \in C^1([0,T), C^0(\mathbb{R}^+) \cap C^0(\mathbb{R}^-))$ to system (3.26) for some maximal existence time $T > 0$. The solution depends continuously on the initial data and preserves invertibility of the mapping $\mathbb{R} \ni s \mapsto q \in C^1_0$ with $q(t,0) = 0$, $\inf_{s\in\mathbb{R}} q_s(t,s) > 0$, and $\lim_{|s|\to\infty} q_s(t,s) = 1$. Therefore, the transformation formulas $V(t,s) = v(t,q(t,s))$ and $W(t,s) = w(t,q(t,s))$ are invertible and the solutions (q,V,W) yields the unique solution $v \in C^1([0,T), H^1 \cap C^1_0)$ to the evolution problem (3.16).

Continuous dependence of the solution $v \in C^1([0,T), H^1 \cap C^1_0)$ on the initial datum $v_0 \in H^1 \cap C^1_0$ is obtained from the continuous dependence theory for differential equations thanks to the Lipschitz continuity of the vector field $F(q,V,W)$ in Lemma 3.15. □

3.5. **Proof of instability.** The characteristics $q = 0$ at $s = 0$ is the breaking point for the initial-value problem (3.25) since W may have a jump discontinuity across $s = 0$. This point corresponds to the peak's location for a perturbed single peakon, according to the decomposition (2.4). As follows from the proof of Theorem 3.13, the dynamical system (3.26) admits the unique solution in the form $W \in C^1([0,T], C^0(\mathbb{R}^+) \cap C^0(\mathbb{R}^-))$. Therefore, we can define the one-sided limits $W^0_\pm \in C^1(0,T)$ by

$$W^0_\pm(t) := \lim_{s\to 0^\pm} W(t,s) = \lim_{s\to 0^\pm} v_x(t,q(t,s)),$$

which satisfies the initial value problems

$$\begin{cases} \dfrac{dW^0_\pm}{dt} = \pm\left(1+V^0\right) W^0_\pm + V^0 - \dfrac{1}{2}\left(1+V^0\right)\left(W^0_\pm\right)^2 + \dfrac{3}{2}\left(V^0\right)^2 + \left(V^0\right)^3 - \mathcal{P}[v](0), \\ W^0_\pm(0) = v_{0x}(0^\pm). \end{cases} \tag{3.28}$$

This initial-value problem is combined with (3.22) which determines evolution of V^0. The following lemma gives estimates for the two nonlocal terms in (3.22) and (3.28).

Lemma 3.16. *Let the assumptions of Theorem 3.9 hold and define $v(t,\cdot) := u(t,\cdot + a(t)) - \varphi$ with $v \in C([0,T), H^1 \cap C^1_0)$. There exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ we have for every $x \in \mathbb{R}$ and every $t \in [0,T)$,*

$$\left|\mathcal{P}[v](t,x) + \mathcal{Q}[v](t,x)\right| < C_0\varepsilon^2(1 + \|u_{0x}\|^{3/2}_{L^\infty} + \varepsilon\|u_{0x}\|^2_{L^\infty}). \tag{3.29}$$

Proof. By Theorem 3.9, it follows for ε small enough that

$$\|v\|_{H^1} < 2\left(4 + \|u_{0x}\|^{1/2}_{L^\infty}\right)\varepsilon. \tag{3.30}$$

Since $\|\varphi\|_{L^\infty} = \|\varphi_x\|_{L^\infty} = 1$, similar to [4, (2.7)–(2.8)] we obtain

$$\begin{aligned} \|v_x\|^4_{L^4} &= 3\int_{\mathbb{R}} (v^4 + 2v^2v_x^2)\,dx - 3F(v) \\ &\le 3\|v\|_{L^\infty}\int_{\mathbb{R}} (v^2 + 2v_x^2)\,dx - 3F(v) \\ &\le 3\Big(2\|v\|^4_{H^1} - F(v)\Big), \end{aligned}$$

indicating that $F(v) \le 2\|v\|^4_{H^1}$, where $F(v)$ is defined in (1.8). Interpolation implies that

$$\|v_x\|^3_{L^3} \le \sqrt{3}\|v\|_{H^1}\sqrt{2\|v\|^4_{H^1} - F(v)},$$

and hence

$$
\begin{aligned}
|\mathcal{P}[v]+\mathcal{Q}[v]| &= \left|\frac{1}{2}(\varphi+\varphi_x)*\left(\frac{3}{2}vv_x^2+v^3\right)+\frac{1}{4}(\varphi+\varphi_x)*v_x^3\right| \\
&\leq \left\|\frac{3}{2}vv_x^2+v^3\right\|_{L^1}+\frac{1}{2}\|v_x^3\|_{L^1} \\
&\leq \frac{3}{2}\|v\|_{L^\infty}\|v\|_{H^1}^2+\frac{1}{2}\|v_x\|_{L^3}^3 \\
&\leq \frac{3}{2}\|v\|_{H^1}^3+\frac{\sqrt{3}}{2}\|v\|_{H^1}\sqrt{2\|v\|_{H^1}^4-F(v)}. \qquad (3.31)
\end{aligned}
$$

Plugging $u=\varphi+v$ into $F(u)$ and using $\|\varphi\|_{L^2}=\|\varphi_x\|_{L^2}=1$, we obtain

$$
\begin{aligned}
|F(v)| &\leq |F(u)-F(\varphi)| \\
&\quad + 2\left|\int_{\mathbb{R}}\left(2v^2v_x\varphi_x+v^2\varphi_x^2+2vv_x^2\varphi+4vv_x\varphi\varphi_x+2v\varphi\varphi_x^2+v_x^2\varphi^2+2v_x\varphi^2\varphi_x\right)dx\right| \\
&\quad + \left|\int_{\mathbb{R}}\left(4v^3\varphi+6v^2\varphi^2+4v\varphi^3\right)dx\right|+\frac{1}{3}\left|\int_{\mathbb{R}}\left(4v_x^3\varphi_x+6v_x^2\varphi_x^2+4v_x\varphi_x^3\right)dx\right| \\
&\leq |F(u)-F(\varphi)|+\frac{4}{3}\left|\int_{\mathbb{R}}v_x^3\varphi_x\,dx\right|+\left(12+\frac{4}{3}\right)\|v\|_{H^1}+20\|v\|_{H^1}^2+10\|v\|_{H^1}^3.
\end{aligned}
$$

Note that we have

$$
\left|\frac{4}{3}\int_{\mathbb{R}}v_x^3\varphi_x\,dy\right| \leq \frac{4}{3}\|v_x\|_{L^3}^3 \leq \frac{4}{\sqrt{3}}\|v\|_{H^1}\sqrt{2\|v\|_{H^1}^4-F(v)}.
$$

Thus, for $\|v\|_{H^1}\ll 1$ sufficiently small it follows that

$$
|F(v)| \leq |F(u)-F(\varphi)|+\frac{4}{\sqrt{3}}\|v\|_{H^1}\sqrt{2\|v\|_{H^1}^4-F(v)}+15\|v\|_{H^1}. \qquad (3.32)
$$

Thanks to the conservation $F(u)=F(u_0)$, a direct calculation yields that

$$
\begin{aligned}
|F(u)-F(\varphi)| &= |F(u_0)-F(\varphi)| \\
&\leq \left|\int_{\mathbb{R}}\left(u_0^4-\varphi^4\right)dx\right|+2\left|\int_{\mathbb{R}}\left(u_0^2u_{0x}^2-\varphi^2\varphi_x^2\right)dx\right|+\frac{1}{3}\left|\int_{\mathbb{R}}\left(u_{0x}^4-\varphi_x^4\right)dx\right|.
\end{aligned}
$$

Following [12, Lemma 2.4], we estimate the above as follows:

$$
\begin{aligned}
\left|\int_{\mathbb{R}}\left(u_0^4-\varphi^4\right)dx\right| &\leq \|v_0\|_{L^\infty}\|u_0+\varphi\|_{L^\infty}(\|u_0\|_{L^2}^2+\|\varphi\|_{L^2}^2) \\
&\leq \|v_0\|_{H^1}\big(\|v_0\|_{H^1}+2\big)\big(\|v_0\|_{H^1}^2+2\|v_0\|_{H^1}+2\big), \\
\left|\int_{\mathbb{R}}\left(u_0^2u_{0x}^2-\varphi^2\varphi_x^2\right)dx\right| &\leq \|v_0\|_{L^\infty}\|u_0+\varphi\|_{L^\infty}\|u_{0x}\|_{L^2}^2+\|\varphi\|_{L^\infty}^2\|v_{0x}\|_{L^2}\|u_{0x}+\varphi_x\|_{L^2} \\
&\leq \|v_0\|_{H^1}\big(\|v_0\|_{H^1}+2\big)\big(\|v_0\|_{H^1}+1\big)^2+\|v_0\|_{H^1}\big(\|v_0\|_{H^1}+2\big),
\end{aligned}
$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}} \left(u_{0x}^4 - \varphi_x^4\right)\, dx\right| &\leq \left(\int_{\mathbb{R}} (u_{0x}^2 + \varphi_x^2)^2 (u_{0x} + \varphi_x)^2\, dx\right)^{1/2} \|v_{0x}\|_{L^2} \\ &\leq 3\left(\int_{\mathbb{R}} (u_{0x}^6 + \varphi_x^6)\, dx\right)^{1/2} \|v_0\|_{H^1} \leq 3\|v_0\|_{H^1}\sqrt{\|u_{0x}\|_{L^\infty}^4 \|u_{0x}\|_{L^2}^2 + \frac{1}{3}} \\ &\leq \left(3\|u_{0x}\|_{L^\infty}^2 (\|v_0\|_{H^1} + 1) + \sqrt{3}\right) \|v_0\|_{H^1}.\end{aligned}$$

where we have used that $\|\varphi_x\|_{L^6}^6 = \frac{1}{3}$. Putting the above together yields

$$|F(u) - F(\varphi)| \leq \left(2\|u_{0x}\|_{L^\infty}^2 + 15\right) \|v_0\|_{H^1}.$$

Plugging this and (3.34) into (3.32) we have

$$|F(v)| \leq \frac{4}{\sqrt{3}} \|v\|_{H^1} \sqrt{2\|v\|_{H^1}^4 - F(v)} + K,$$

where $K := (30 + 2\|u_{0x}\|_{L^\infty}^2)\, \|v_0\|_{H^1}$. Solving the above we get

$$|F(v)| \leq 6\|v\|_{H^1}^2 + 4\|v\|_{H^1}^3 + \sqrt{6K}\|v\|_{H^1} + K.$$

For ε sufficiently small, we can find some large $C > 0$ such that

$$|F(v)| \leq C\varepsilon^2 (1 + \|u_{0x}\|_{L^\infty} + \varepsilon^2 \|u_{0x}\|_{L^\infty}^2).$$

Plugging this into (3.31) and by further shrinking ε if needed, we obtain (3.29). □

Theorem 1.2 is proven by using Theorem 3.9, Theorem 3.13, and Lemma 3.15.

Proof. of Theorem 1.2. By Theorem 3.13, we consider the unique solution $v \in C([0,T), H^1 \cap C_0^1)$ to the Cauchy problem (3.16). It follows from the bound (1.15) and the decomposition (2.4) with $a(0) = 0$ that the initial datum $v_0 \in H^1 \cap C_0^1$ satisfies the bound

$$\|v_0\|_{H^1} + \|v_{0x}\|_{L^\infty} < \delta. \tag{3.33}$$

Let $\varepsilon > 0$ be a small parameter to be determined below. By Theorem 3.9, we have

$$\text{if} \quad \|v_0\|_{H^1} < \varepsilon^4, \quad \text{then} \quad \|v(t,\cdot)\|_{H^1} < 2\left(4 + \|u_{0x}\|_{L^\infty}^{1/2}\right)\varepsilon, \tag{3.34}$$

From (3.33) we know that for δ sufficiently small,

$$\|u_{0x}\|_{L^\infty}^{1/2} < (1+\delta)^{1/2} < \sqrt{2}. \tag{3.35}$$

Therefore Sobolev embedding implies that

$$|V^0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \|v(t,\cdot)\|_{H^1} < (8 + 2\sqrt{2})\varepsilon < 12\varepsilon. \tag{3.36}$$

Instability. The instability argument relies on the behavior of $v_x(t,x)$ near the peak at $x = 0$ from the right side, where the linear instability result of Theorem 1.1 suggests at least exponential growth. Therefore, picking W_+^0 in (3.28), and using an integrating factor we obtain

$$\begin{aligned}\frac{d}{dt}\left[e^{-t}(V^0 + W_+^0)\right] &= e^{-t}\left[\frac{3}{2}\left(V^0\right)^2 + V^0 W_+^0 - \frac{1}{2}\left(1 + V^0\right)\left(W_+^0\right)^2 + \left(V^0\right)^3 - \mathcal{P}[v](0) - \mathcal{Q}[v](0)\right] \\ &\leq e^{-t}\left[\frac{5}{2}\left(V^0\right)^2 - \frac{1}{4}\left(1 + 2V^0\right)\left(W_+^0\right)^2 + \left(V^0\right)^3 - \mathcal{P}[v](0) - \mathcal{Q}[v](0)\right].\end{aligned}$$

Therefore for ε sufficiently small, it follows from (3.36) that

$$\frac{d}{dt}\left[e^{-t}(V^0+W_+^0)\right] \le e^{-t}\left[3\left(V^0\right)^2 - \mathcal{P}[v](0) - \mathcal{Q}[v](0)\right]. \tag{3.37}$$

Lemma 3.16 yields the control of $\mathcal{P}[v](0)$ and $\mathcal{Q}[v](0)$ in (3.29). By integrating (3.37) and using (3.29), (3.35) and (3.36), we obtain

$$V^0(t) + W_+^0(t) \le e^t\left[V^0(0) + W_+^0(0) + C\varepsilon^2\right], \tag{3.38}$$

for some $C > 0$. Let us pick the initial datum $v_0 \in H^1 \cap C_0^1$ satisfying $v_0(0) = 0$ and

$$\lim_{x\to 0^+} v_{0x}(x) = -\|v_{0x}\|_{L^\infty} = -2C\varepsilon^2. \tag{3.39}$$

This is possible provided that for any given $\delta > 0$ in the initial bound (1.15) (and hence (3.33)), the small parameter $\varepsilon > 0$ is chosen to satisfy the bound:

$$\varepsilon^4 + 2C\varepsilon^2 < \delta.$$

Since $V^0(0) = 0$ and $W_+^0(0) = -2C\varepsilon^2$, we obtain from (3.38) that

$$V^0(t) + W_+^0(t) \le -C\varepsilon^2 e^t,$$

which implies that

$$|V^0(t) + W_0^+(t)| > 2 \quad \text{for} \quad t > t_0 := \log\left(\frac{2}{C\varepsilon^2}\right) > 0.$$

Thanks to the bound (3.36) on $V^0(t)$, this implies that $|W_0^+(t)| > 1$ for $t > t_0$.

If $t_0 < T$, then we have the instability (1.16). If $t_0 > T$, then T is finite and we have $\|v_x(t,\cdot)\|_{L^\infty} \to \infty$ as $t \to T$ due to the fact that $\|v(t,\cdot)\|_{H^1}$ is bounded from the H^1 conservation of solutions. In this case, the existence of another $t_0' \in (0,T)$ such that $\|v_x(t_0',\cdot)\|_{L^\infty} > 1$ follows from the continuity arguments.

Blow-up. Now we want to show that by choosing suitable initial datum satisfying (1.15), the corresponding solution can indeed blow up in finite time.

Recall from (3.28) that we have

$$\frac{dW_+^0}{dt} = -\frac{1}{2}(1+V^0)(W_+^0-1)^2 + \frac{1}{2} + \frac{3}{2}V^0 + \frac{3}{2}(V^0)^2 + (V^0)^3 - \mathcal{P}[v](0).$$

Note from (3.29) and (3.34)–(3.36) that for ε sufficiently small, W_+^0 satisfies the following Ricatti inequality

$$\frac{dW_+^0}{dt} \le -\frac{1}{2}(1-12\varepsilon)(W_+^0-1)^2 + \frac{1}{2} + 20\varepsilon.$$

Therefore it follows from the routine analysis of the differential inequality (see, for example [5, Lemma 3.3]) that if we choose initial datum satisfying

$$W_+^0(0) < 1 - \sqrt{\frac{1+40\varepsilon}{1-12\varepsilon}}, \tag{3.40}$$

then $W_+^0(t)$ tends to $-\infty$ in finite time. To be more precise, let us pick the initial datum $v_0 \in H^1 \cap C_0^1$ satisfying

$$\|v_0\|_{H^1} < \varepsilon^4, \qquad \lim_{x\to 0^+} v_{0x}(x) = -30\varepsilon,$$

with

$$\varepsilon^4 + 30\varepsilon < \delta.$$

Then (3.40) is satisfied, and hence $v_x(t, 0) \to -\infty$ as $t \to T^*$ for some $T^* < \infty$. Hence the maximal existence time T satisfies $T \leq T^* < \infty$. $\square$

Acknowledgments: The work of RMC is partially supported by National Science Foundation under grant DMS-1613375 and DMS-1907584. The work of DEP is partially supported by the NSERC Discovery grant.

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