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ON THE STABILITY OF TWO-DIMENSIONAL NONISENTROPIC ELASTIC VORTEX SHEETS

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Dedicated to Professor Shuxing Chen on the Occasion of His 80th Birthday

Abstract. We are concerned with the stability of vortex sheet solutions for the two-dimensional nonisentropic compressible flows in elastodynamics. This is a nonlinear free boundary hyperbolic problem with characteristic discontinuities, which has extra difficulties when considering the effect of entropy. The addition of the thermal effect to the system makes the analysis of the Lopatinskiĭ determinant extremely complicated. Our results are twofold. First, through a qualitative analysis of the roots of the Lopatinskiĭ determinant for the linearized problem, we find that the vortex sheets are weakly stable in some supersonic and subsonic regions. Second, under the small perturbation of entropy, the nonlinear stability can be adapted from the previous two-dimensional isentropic elastic vortex sheets [6] by applying the Nash-Moser iteration. The two results confirm the strong elastic stabilization of the vortex sheets. In particular, our conditions for the linear stability (1) ensure that a stable supersonic regime as well as a stable subsonic one always persist for any given nonisentropic configuration, and (2) show how the stability condition changes with the thermal fluctuation. The existence of the stable subsonic bubble, a phenomenon not observed in the Euler flow, is specially due to elasticity.

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1. **Introduction.** In this paper, we investigate the stability of vortex sheets for the two-dimensional (2D) nonisentropic compressible inviscid flows in elastodynamics ([12]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div} \mathbb{T}, \\ (\rho E + \frac{1}{2}\rho|\mathbf{u}|^2)_t + \operatorname{div}((\rho E + \frac{1}{2}\rho|\mathbf{u}|^2)\mathbf{u}) = \operatorname{div}(\mathbb{T}\mathbf{u}), \\ (\rho \mathrm{F}_j)_t + \operatorname{div}(\rho \mathrm{F}_j \otimes \mathbf{u} - \mathbf{u} \otimes \rho \mathrm{F}_j) = 0, \end{cases} \tag{1.1}$$

where ρ denotes the density, $\mathbf{u} = (v, u) \in \mathbb{R}^2$ is the velocity, $\mathbb{T}$ is the Cauchy stress, E is the internal energy, and F_j is the jth column of the deformation gradient $\mathrm{F} = (F_{ij}) \in \mathbf{M}^{2\times 2}$. For the neo-Hookean elastodynamics, the internal energy takes the form

$$E(\mathrm{F}, S) = \sum_{i,j=1}^{2} \frac{1}{2} F_{ij}^2 + e(\rho, S),$$

where $e(\rho, S)$ is the thermodynamic energy and S is the entropy. The Cauchy stress $\mathbb{T}$ and the pressure p are given by

$$\mathbb{T} = \rho \mathrm{F}\mathrm{F}^\top - pI_2, \quad p = p(\rho, S) = \rho^2 \frac{\partial e(\rho, S)}{\partial \rho},$$

where I_2 is the 2×2 identity matrix, and $p_\rho(\rho, S) > 0$ for $\rho > 0$. The sound speed $c = c(\rho, S)$ is defined as

$$c(\rho, S) = \sqrt{p_\rho(\rho, S)}. \tag{1.2}$$

The system (1.1) admits the following involution condition:

$$\operatorname{div}(\rho \mathrm{F}_j) = 0, \quad j = 1, 2.$$

From such an involution, the system (1.1) can be transformed into the following equations:

$$\begin{cases} \partial_t p + \mathbf{u} \cdot \nabla p + \rho c^2 \nabla \cdot \mathbf{u} = 0, \\ \rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p = \rho \sum\limits_{j=1}^{2} (\mathrm{F}_j \cdot \nabla)\mathrm{F}_j, \\ \rho(\partial_t \mathrm{F}_j + \mathbf{u} \cdot \mathrm{F}_j) = \rho(\mathrm{F}_j \cdot \nabla \mathrm{u}), \\ \partial_t S + \mathbf{u} \cdot \nabla S = 0. \end{cases} \tag{1.3}$$

More discussions on the physical background and equations of elastodynamics can be found in the book of Dafermos [12]. We are interested in a special class of solutions to (1.3), namely the vortex sheet solutions. These solutions describe interfaces between two compressible flows passing along each other. They are contact discontinuities, where the tangential velocity field has a jump, while the normal component of velocity remains continuous. Mathematically, vortex sheets are fundamental waves in the study of general entropy solutions to multi-dimensional hyperbolic conservation laws (see, for example, [8, 12]).

Let us give the precise definition of vortex sheet solutions to (1.3). A weak solution $U(t, x_1, x_2) = (p, \mathbf{u}, \mathrm{F}, S)(t, x_1, x_2)$ is said to be a vortex sheet solution to system (1.3) provided that it is a classical solution on both sides of a smooth curve $\Gamma = \{x_2 = \psi(t, x_1)\}$ and satisfies the jump conditions described below. Denote by $\partial_i = \partial_{x_i}, i = 1, 2,$ the partial derivatives, $\nu = (-\partial_1 \psi, 1)$ the normal vector on Γ, and

$$U(t, x_1, x_2) = \begin{cases} U^+(t, x_1, x_2), & \text{when } x_2 > \psi(t, x_1), \\ U^-(t, x_1, x_2), & \text{when } x_2 < \psi(t, x_1), \end{cases}$$

where $U^{\pm} = (p^{\pm}, \mathbf{u}^{\pm}, \mathrm{F}^{\pm}, S^{\pm})(t, x_1, x_2)$. For the vortex sheet solutions, we require that

$$[\mathbf{u} \cdot \nu] = 0, \quad [\mathbf{u}] \neq 0, \quad \text{and} \quad \psi_t = \mathbf{u}^{\pm} \cdot \nu \Big|_{\Gamma},$$

where $[f] := (f^+ - f^-)|_{\Gamma}$ denotes the jump of f across Γ. Therefore, for a vortex sheet in thermoelastic flows, the Rankine-Hugoniot conditions hold at the free surface:

$$p^+ = p^- := p, \quad \psi_t = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu. \tag{1.4}$$

We note that the normal velocity and the pressure are continuous across the interface, the only possible jumps of a contact discontinuity solution are given by the tangential velocity and the entropy, i.e., $[S] \neq 0$.

Now we give a brief review of the literature on the study of vortex sheets for the Euler equations in the gas dynamics. Using the normal mode analysis, Miles [19,20] first observed that the vortex sheets are linearly stable when the Mach number $M > \sqrt{2}$ and violently unstable when $M < \sqrt{2}$. Through a delicate linear analysis and an application of the Nash–Moser iteration, Coulombel and Secchi [10,11] later provided a rigorous proof for the nonlinear stability and local-in-time existence of 2D compressible vortex sheet solutions when the Mach number is sufficiently large under the assumption that the initial data is the small perturbations of a planar vortex sheet. Further development on the nonlinear stability of 2D compressible nonisentropic vortex sheet solutions can be found in [9,23,24]. In the recent work of [4–6] the authors proved the nonlinear stability of 2D vortex sheet solutions in an elastic medium, confirming the stabilization effect from the elasticity. The stability of current vortex sheet solutions to the three-dimensional MHD was obtained by Chen-Wang [1] and Trakhinin [27] under the magnetic stabilization effect in the anisotropic Sobolev space H^m_* compensating the loss of normal derivative for characteristic free boundary problems. This function space was introduced first in Chen [7] from his key observation that the normal differentiation of order one of the solution results from the tangential differentiation of order two. We would also like to mention the works of [2,3,13,15–18,21,22,25–31] on other types of contact discontinuities and free boundary problems.

This paper focuses on the stability of 2D vortex sheets in compressible nonisentropic elastodynamics. Some notable mathematical challenges can be summarized as follows. First, this is a nonlinear hyperbolic problem with a characteristic free boundary. The Lopatinskiĭ conditions hold only in a weak sense, and we can only control the partial trace (noncharacteristic part) of the solutions. When taking the Fourier-Laplace transformation to the system in the linear analysis, we obtain an ODE system which only governs the evolution of the noncharacteristic part of the solution. Due to the fact that the boundary is characteristic, the symbol of the ODE system is singular in the sense that it has poles. To achieve stability, it is required that the Kreiss Lopatinskiĭ determinant associated with the ODE system has no root in the interior of the frequency space (since otherwise, the linearized problem will be violently unstable and thus ill-posed). It turns out that the addition of the thermal effect to the system makes the analysis of the Lopatinskiĭ determinant extremely complicated. For example, double roots may occur at the transition from stability to instability which causes loss of derivatives in the a *priori* estimates; the collapsing between a pole and a root is also possible, leading to additional loss of regularity.

Following the approach of [10] and the upper-triangularization method of [4], we prove the energy estimate of solutions to the linearized problem. As is mentioned above, the algebraic system for the nonisentropic problem is much more complicated than the one studied in [4], and it is more likely that the poles and roots collapse. It turns out that a detailed rigorous analytical verification of the Lopatinskiĭ condition is beyond our reach at this moment. Instead, we resort to the qualitative analysis to find (non-sharp) regions for stability. These findings confirm the strong stabilization effect from elasticity.

More specifically, from the qualitative analysis we observe that the vortex sheets are stable in both supersonic and subsonic regions. These regions agree with the one found in [4] in the isentropic limit, cf. Theorem 4.1. Though the stability criteria we find are not sharp (see Remark 4.2), an observation can already be made indicating a 'thermal destabilization', cf. Remark 4.3. In a perturbative regime, we prove that the elastic stabilization is robust enough to accommodate a sufficiently small amount of thermal fluctuation. Based on the nonlinear stability of [6] we confirm a nonlinear stability for the elastic vortex sheets in the near-isentropic regime, cf. Theorem 4.2.

The paper is organized as follows. In Section 2, we introduce the formulation of the free boundary problem, transform it to a fixed domain with flat boundary, linearize the resulting system around a piecewise constant background solution, and perform the separation of modes. In Section 3, we present the ODE system for the non-characteristic part of the unknown after the Laplace-Fourier transformation, and provide a qualitative analysis of the roots of Kreiss Lopatinskiĭ determinants. In Section 4, we derive a set of sufficient conditions for the linear stability for the nonisentropic elastic vortex sheets and obtain the nonlinear stability in the near-isentropic regime.

2. **Formulation of the linearized problem.** We are interested in the stability of nonisentropic vortex sheets (1.3) with the initial data being a small perturbation of piecewise constant vortex sheets. Under Galilean transformation and the change of the scale of measurement, we can transform the original piecewise constant solutions to the following form:

$$\begin{aligned}
&\bar{U}^+ = (p^r, v^r, 0, F^r_{11}, 0, F^r_{12}, 0, S^r)^\top,\\
&\bar{U}^- = (p^l, v^l, 0, F^l_{11}, 0, F^l_{12}, 0, S^l)^\top,\\
&\bar{\Phi}^\pm(t, x_1, x_2) = \pm x_2.
\end{aligned}\tag{2.1}$$

Here, the constants $p^{r,l}$, $v^{r,l}$, $F^{r,l}_{11}$, $F^{r,l}_{12}$ satisfy

$$p^r = p^l := p > 0, \quad v^r + v^l = 0, \quad F^r_{11} + F^l_{11} = F^r_{12} + F^l_{12} = 0,$$

and $F^r_{11}, F^r_{12} \neq 0$.

Now, we start discussing our nonlinear stability problem. As a first step, we introduce the transformation $\Phi^\pm(t, x_1, x_2)$ to straighten the unknown free boundary Γ. Consider the class of functions $\Phi(t, x_1, x_2)$ such that $\inf\{\partial_2\Phi\} > 0$, and $\Phi(t, x_1, 0) = \psi(t, x_1)$. Then, we define

$$U^\pm_\natural = (p^\pm_\natural, \mathbf{u}^\pm_\natural, \mathrm{F}^\pm_\natural, S^\pm_\natural)(t, x_1, x_2) = (p, \mathbf{u}, \mathrm{F}, S)(t, x_1, \Phi(t, x_1, \pm x_2)),$$

for $x_2 \geq 0$. We shall drop the subscript $\natural$ for simplicity. Define $\Phi^\pm(t, x_1, x_2) = \Phi(t, x_1, \pm x_2)$ for convenience. To obtain a convenient expression of our problem, we can choose Φ satisfying the following eikonal equations:

$$\partial_t\Phi^\pm + v^\pm\partial_1\Phi^\pm - u^\pm = 0,$$

for $x_2 \geq 0$, which is inspired by [10]. With this variable transform, we rewrite the system (1.1) as

$$\partial_t U^\pm + A_1(U^\pm)\partial_1 U^\pm + \frac{1}{\partial_2 \Phi^\pm}\Big(A_2(U^\pm) - \partial_t \Phi^\pm I - \partial_1 \Phi^\pm A_1(U^\pm)\Big)\partial_2 U^\pm = 0, \quad (2.2)$$

for $x_2 > 0$ with the fixed boundary $x_2 = 0$, where

$$A_1(U) = \begin{bmatrix} v & \rho c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\rho} & v & 0 & -F_{11} & 0 & -F_{12} & 0 & 0 \\ 0 & 0 & v & 0 & -F_{11} & 0 & -F_{12} & 0 \\ 0 & -F_{11} & 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & -F_{11} & 0 & v & 0 & 0 & 0 \\ 0 & -F_{12} & 0 & 0 & 0 & v & 0 & 0 \\ 0 & 0 & -F_{12} & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix},$$
$$A_2(U) = \begin{bmatrix} u & 0 & \rho c^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & -F_{21} & 0 & -F_{22} & 0 & 0 \\ \frac{1}{\rho} & 0 & u & 0 & -F_{21} & 0 & -F_{22} & 0 \\ 0 & -F_{21} & 0 & u & 0 & 0 & 0 & 0 \\ 0 & 0 & -F_{21} & 0 & u & 0 & 0 & 0 \\ 0 & -F_{22} & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & -F_{22} & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u \end{bmatrix}. \quad (2.3)$$

We can write

$$\begin{cases} \mathcal{L}(U^\pm, \Phi^\pm) = 0, & \text{if } x_2 > 0, \\ \mathcal{B}(U^\pm, \psi) = 0, & \text{if } x_2 = 0, \\ (U^\pm, \psi)|_{t=0} = (U_0^\pm, \psi_0), \end{cases}$$

where

$$\mathcal{L}(U, \Phi) = L(U, \Phi)U, \quad (2.4)$$

$$L(U, \Phi) := \partial_t + A_1(U)\partial_1 + \tilde{A}_2(U, \Phi)\partial_2, \quad (2.5)$$

$$\tilde{A}_2(U, \Phi) = \frac{1}{\partial_2 \Phi}\big(A_2(U) - \partial_t \Phi I - \partial_1 \Phi A_1(U)\big), \quad (2.6)$$

$$\mathcal{B}(U^\pm, \psi) = \begin{bmatrix} (v^+ - v^-)\partial_1 \psi - (u^+ - u^-) \\ \partial_t \psi + v^+ \partial_1 \psi - u^+ \\ p^+ - p^- \end{bmatrix}. \quad (2.7)$$

After straightened variables, the piecewise constant vortex sheet (2.1) corresponds to

$$\bar{U}^\pm = (\bar{p}, \pm\bar{v}, 0, \pm\bar{F}_{11}, 0, \pm\bar{F}_{12}, 0, \bar{S}^\pm)^\top, \quad \bar{\Phi}^\pm(t, x_1, x_2) = \pm x_2, \quad (2.8)$$

with $\bar{p} > 0$, and $\bar{v} > 0$. Set $\bar{c}_\pm = c(\bar{p}, \bar{S}^\pm)$ for the sound speeds corresponding to the constant states $\bar{U}^\pm$. The standard procedure in studying the nonlinear stability and existence of compressible vortex sheets starts with proving the energy estimates for the constant coefficient linearized problem corresponding to the linear stability of the piecewise constant background states, and then boosting the linear stability to background states which are perturbative to the piecewise constant ones. Finally through constructing smooth approximate solutions by imposing suitable compatible initial data, and using the modification of the Nash-Moser iteration scheme, one can conclude the existence of solutions in the usual Sobolev spaces.

Following a similar manner in [4], we linearize the system (2.2)–(2.7) around the above constant states (2.8). Let

$$\dot{U}^{\pm} = (\dot{p}^{\pm}, \dot{\mathbf{u}}^{\pm}, \dot{\mathrm{F}}^{\pm}, \dot{S}^{\pm}) = U^{\pm} - \bar{U}^{\pm}, \quad \dot{\Phi}^{\pm} = \Phi^{\pm} - \bar{\Phi}^{\pm}$$

be the small perturbation of the constant solution and consider the following linearized problem:

$$\partial_t \dot{U}^{\pm} + A_1(\bar{U}^{\pm})\partial_1 \dot{U}^{\pm} \pm A_2(\bar{U}^{\pm})\partial_2 \dot{U}^{\pm} = 0$$

in $x_2 > 0$, with the boundary condition at $x_2 = 0$:

$$\begin{cases} (v^r - v^l)\partial_1\varphi - (\dot{u}^+ - \dot{u}^-) = 0; \\ \partial_t\varphi + v^r\partial_1\varphi - \dot{u}^+ = 0; \\ \dot{p}^+ = \dot{p}^-, \end{cases} \tag{2.9}$$

where $\varphi = (\Phi^{\pm} - \dot{\Phi}^{\pm})|_{x_2=0} = \psi$, at $x_2 = 0$. Therefore, we have

$$\begin{cases} \tilde{\mathcal{L}}\dot{U} = 0, & \text{if } x_2 > 0, \\ \mathcal{B}(\dot{U}, \varphi) = 0, & \text{if } x_2 = 0, \end{cases} \tag{2.10}$$

where

$$\begin{aligned} \tilde{\mathcal{L}}\dot{U} = \partial_t \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix} + \begin{bmatrix} A_1(\bar{U}^+) & 0 \\ 0 & A_1(\bar{U}^-) \end{bmatrix} \partial_1 \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix} \\ + \begin{bmatrix} A_2(\bar{U}^+) & 0 \\ 0 & -A_2(\bar{U}^-) \end{bmatrix} \partial_2 \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix}, \\ \mathcal{B}(\dot{U}, \varphi) = \begin{bmatrix} (v^r - v^l)\partial_1\varphi - (\dot{u}^+ - \dot{u}^-) \\ \partial_t\varphi + \dot{v}^+\partial_1\varphi - \dot{u}^+ \\ \dot{p}^+ - \dot{p}^- \end{bmatrix}. \end{aligned}$$

The next step is to symmetrize the system (2.10). Consider the following change of variables,

$$W = \begin{bmatrix} T_r & 0 \\ 0 & T_l \end{bmatrix} \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix}, \tag{2.11}$$

where $T_{r,l}$ is a matrix with following form:

$$T_{r,l} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\rho c_{r,l}^2} & 0 & \frac{1}{2c_{r,l}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2\rho c_{r,l}^2} & 0 & \frac{1}{2c_{r,l}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.12}$$

with $c_{r,l} = c(p, S^{r,l})$ and c the sound speed given by (1.2). For simplicity, we denote the components of the new variable by

$$W = (W_1, W_2, \cdots, W_{16})^{\top},$$

and

$$W^c = (W_1, W_4, \cdots, W_8, W_9, W_{12}, \cdots, W_{16})^{\top},$$

$$W^{nc} = (W_2, W_3, W_{10}, W_{11})^\top. \tag{2.13}$$

Thus (2.10) can be transformed into a linear system for W, and the linear stability of the background solution (2.8) follows from an energy estimate on W.

3. Qualitative analysis of roots of Lopatinskiĭ Determinant.

Following the argument as in [4], we can decompose the linear system W and eliminate the wave front from the boundary conditions. Performing a Laplace-Fourier transform and analyzing the normal modes lead to the following differential equations for $\widehat{W}^{nc}$:

$$\frac{d}{dx_2}\widehat{W}^{nc} = A\widehat{W}^{nc} \tag{3.1}$$

on the whole frequency space

$$\Pi = \{(\tau,\eta) \in \mathbb{C}\times\mathbb{R} :\ |\tau|^2 + \eta^2 \neq 0,\ \Re\tau \geq 0\},$$

where

$$A = \begin{bmatrix} n^r & -m^r & 0 & 0 \\ m^r & -n^r & 0 & 0 \\ 0 & 0 & -n^l & m^l \\ 0 & 0 & -m^l & n^l \end{bmatrix} \tag{3.2}$$

and

$$n^{r,l} = \frac{2(k_1^{r,l})^2 + k_2^{r,l}}{2c_{r,l}k_1^{r,l}} + \frac{c_{r,l}k_1^{r,l}\eta^2}{2((k_1^{r,l})^2 + k_2^{r,l})}, \quad m^{r,l} = \frac{c_{r,l}k_1^{r,l}\eta^2}{2((k_1^{r,l})^2 + k_2^{r,l})} - \frac{k_2^{r,l}}{2c_{r,l}k_1^{r,l}},$$

with

$$k_1^{r,l} = \tau + i\eta v^{r,l}, \quad k_2^{r,l} = ((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2)\eta^2.$$

The homogeneity structure of the problem suggests that it suffices to work on the hemisphere

$$\Sigma = \{(\tau,\eta) \in \mathbb{C}\times\mathbb{R} :\ |\tau|^2 + (v^r)^2\eta^2 = 1,\ \Re\tau \geq 0\}.$$

From the classical hyperbolic theory we know that the boundary estimate of $\widehat{W}^{nc}$ relies on the bound of its components on the stable subspace of A. Hence we derive the following lemma of Hersh-type [14] on the explicit description of the stable subspace of A on Σ.

Lemma 3.1. *For $(\tau,\eta) \in \Sigma$ and $\Re\tau > 0$, the matrix A defined in (3.2) admits four eigenvalues $\pm\omega^r$ and $\pm\omega^l$, where $\Re\omega^r$ and $\Re\omega^l$ are negative. Moreover, the following dispersion relations hold:*

$$\begin{aligned}
(\omega^r)^2 &= (n^r)^2 - (m^r)^2 = \frac{1}{c_r^2}\big((\tau + iv^r\eta)^2 + ((F_{11}^r)^2 + (F_{12}^r)^2)\eta^2\big) + \eta^2, \\
(\omega^l)^2 &= (n^l)^2 - (m^l)^2 = \frac{1}{c_l^2}\big((\tau + iv^l\eta)^2 + ((F_{11}^l)^2 + (F_{12}^l)^2)\eta^2\big) + \eta^2.
\end{aligned} \tag{3.3}$$

Furthermore, the eigenvectors of $\omega^r, -\omega^r, \omega^l, -\omega^l$ take the following forms:

$$\begin{aligned}
E_-^r &= (a^r, b^r, 0, 0)^\top, \quad E_+^r = (a^r, c^r, 0, 0)^\top, \\
E_-^l &= (0, 0, b^l, a^l)^\top, \quad E_+^l = (0, 0, c^l, a^l)^\top,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
a^{r,l} &= m^{r,l}\alpha^{r,l}, \quad b^{r,l} = (n^{r,l} - \omega^{r,l})\alpha^{r,l}, \quad c^{r,l} = (n^{r,l} + \omega^{r,l})\alpha^{r,l}, \\
\alpha^{r,l} &= (\tau + iv^{r,l}\eta)\big((\tau + iv^{r,l}\eta)^2 + ((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2)\eta^2\big).
\end{aligned}$$

Both ω^r and ω^l can be extended continuously to all the points $(\tau,\eta)\in\Sigma$ such that $\Re\tau=0$, so can $E^r_\pm$ and $E^l_\pm$. Moreover, two vectors E^r_- and E^l_- are linearly independent for all frequency $(\tau,\eta)\in\Sigma$.

Following [4], we can obtain the following proposition ensuring the separation of modes everywhere on Σ. The proof is very similar to that of [4, Proposition 4.1], and hence we omit it.

Proposition 3.1. *For $\omega^{r,l}$ defined in Lemma 3.1, we have*

$$(\tau+iv^{r,l}\eta)\omega^{r,l}-c_{r,l}((\omega^{r,l})^2-\eta^2)\neq 0,$$

for all $(\tau,\eta)\in\Sigma$.

The above proposition allows us to perform an upper-triangularization on the matrix A given in (3.2). Therefore the boundary estimate of $\widehat{W}^{nc}$ reduces to an estimate on its incoming mode $\widehat{W}^{in}|_{x_2=0}$, which relies on the invertibility of the corresponding 2×2 boundary matrix $\beta[E^r_-\ E^l_-]$ where

$$\beta=\begin{bmatrix} -1 & 1 & 1 & -1\\ -c_r(\tau-iv^r\eta) & -c_r(\tau-iv^r\eta) & c_l(\tau+iv^r\eta) & c_l(\tau+iv^r\eta)\end{bmatrix}. \tag{3.5}$$

The determinant of the matrix $\beta[E^r_-\ E^l_-]$, namely the Lopatinskiĭ determinant, is

$$\begin{aligned}\Delta=\det(\beta[E^r_-\ E^l_-])=&c_r^2c_l^2(\tau+iv^r\eta)(\tau+iv^l\eta)\Big((\tau+iv^r\eta)\omega^r-c_r((\omega^r)^2-\eta^2)\Big)\\ &\times\left(c_l((\omega^l)^2-\eta^2)-(\tau+iv^l\eta)\omega^l\right)(\omega^l\omega^r-\eta^2)(\omega^r+\omega^l).\end{aligned}\tag{3.6}$$

We see that the Lopatinskiĭ determinant Δ can vanish at multiple points in Σ. Therefore, the uniform Lopatinskiĭ condition fails.

Let $\tau=\gamma+i\delta,\gamma=\Re\tau$. In the following we provide a detailed analysis on each factor of the Lopatinskiĭ determinant (3.6).

Step 1: The third and fourth factors $(\tau+iv^{r,l}\eta)\omega^{r,l}-c_{r,l}((\omega^{r,l})^2-\eta^2)$**.** Note that $(\tau+iv^{r,l}\eta)\omega^{r,l}-c_{r,l}((\omega^{r,l})^2-\eta^2)$ has the same expression in Proposition 3.1. Thus, we can see that they are not zero.

Step 2: The first and second factors $\tau+iv^{r,l}\eta$.
Obviously, $\tau=-iv^{r,l}\eta$ are the simple roots to $\tau+iv^{r,l}\eta=0$, respectively.

Step 3: The fifth factor $\omega^r\omega^l-\eta^2$.
We assume that

$$\omega^r\omega^l-\eta^2=0. \tag{3.7}$$

If $\eta=0$, we have $\omega^r\omega^l-\eta^2=\frac{\tau^2}{c_rc_l}$, for $\tau\neq 0$. Hence, $\omega^r\omega^l-\eta^2\neq 0$. Therefore, $\eta\neq 0$. If $\delta+v^{r,l}\eta=0$, for example, $\delta+v^r\eta=0$, then from (3.3), this implies $\Im\omega^l\neq 0$, thus $\omega^r\omega^l$ cannot be a real number. This violates (3.7). We obtain that ω^r is real and negative. However, since $\eta\neq 0$, we know that $\delta+v^l\eta\neq 0$, therefore (3.7) cannot happen for $\eta=0$ and $\delta+v^{r,l}\eta=0$. We introduce the following two variables

$$V=\frac{\tau}{i\eta},\quad \Omega^{r,l}=\frac{\omega^{r,l}}{i\eta}. \tag{3.8}$$

From (3.7), we have that $\Omega^r\Omega^l=-1$, and hence $(\Omega^r)^2(\Omega^l)^2=1$. By (3.3), and let

$$g=(F^r_{11})^2+(F^r_{12})^2,$$

we obtain that

$$\begin{aligned}&V^4-(2(v^r)^2+2g+c_r^2+c_l^2)V^2+2v^r(c_r^2-c_l^2)V+(v^r)^4\\ &\quad-(2g+c_r^2+c_l^2)(v^r)^2+g^2+g(c_l^2+c_r^2)=0.\end{aligned}\tag{3.9}$$

Let

$$X := \frac{V}{c_r}, \quad M := \frac{v^r}{c_r}, \quad \Theta := \frac{c_l^2}{c_r^2} - 1, \quad \tilde{g} = \frac{g}{c_r^2}. \tag{3.10}$$

Here we denote $\Theta > -1$ as the thermal fluctuation. Then, we can write (3.9) into the following form

$$\begin{aligned}\mathcal{P}(X) = X^4 &- (2M^2 + 2\tilde{g} + \Theta + 2)X^2 - 2M\Theta X \\ &+ M^2(M^2 - 2\tilde{g} - \Theta - 2) + \tilde{g}^2 + \tilde{g}(\Theta + 2) = 0. \end{aligned} \tag{3.11}$$

This is a fourth-order equation in X. The following lemma provides some sufficient conditions on the parameters which guarantee that equation (3.11) has four distinct real roots.

Lemma 3.2. (i) *Given any $\tilde{g} > 0$ and $\Theta > -1$, if the Mach number M satisfies*

$$M^2 > \tilde{g} + \Theta + 2 \qquad \text{or} \qquad M^2 < \tilde{g}, \tag{3.12}$$

then the equation (3.11) *has four distinct real roots.*

(ii) *Under the assumptions of* (i), *denote the four roots of* (3.11) *by $X_1 < X_2 < X_3 < X_4$. The corresponding roots to* (3.7) *(and hence to the Lopatinskĭi determinant Δ) are $(\tau, \eta) \in \Sigma$ where each τ is purely imaginary. Moreover τ takes the form of $\tau = iV_k\eta$ where $V_k = c_r X_k$ (k may not take all values of $\{1, 2, 3, 4\}$).*

Proof. Direct calculation yields that

$$\begin{aligned} \mathcal{P}(0) &= (M^2 - \tilde{g})\left[M^2 - (\tilde{g} + \Theta + 2)\right], \\ \mathcal{P}(M + \sqrt{\tilde{g}}) &= -4M(M + \sqrt{\tilde{g}})(\Theta + 1) < 0, \\ \mathcal{P}(-M - \sqrt{\tilde{g}}) &= -4M(M + \sqrt{\tilde{g}}) < 0, \\ \mathcal{P}(M + \sqrt{\tilde{g} + \Theta + 2}) &= 4M(M + \sqrt{\tilde{g} + \Theta + 2}) > 0, \\ \mathcal{P}(-M - \sqrt{\tilde{g} + \Theta + 2}) &= 4M(M + \sqrt{\tilde{g} + \Theta + 2})(\Theta + 1) > 0. \end{aligned}$$

Therefore when (3.12) holds we have $\mathcal{P}(0) > 0$, ensuring that $\mathcal{P}(X) = 0$ has four distinct real roots.

Part (ii) follows directly from the definition of τ in (3.8). ☐

Step 4: The sixth factor $\omega^r + \omega^l$. Finally we consider the roots to $\omega^r + \omega^l = 0$. Setting $\Omega^{r,l} = \frac{\omega^{r,l}}{i\eta}$, we have $(\Omega^r)^2 = (\Omega^l)^2$. Then, we have $p^r = p^l$.

From Lemma 3.1, if $\Re\tau > 0$, then $\Re\omega^{r,l} < 0$. Then, we have $\omega^r + \omega^l \neq 0$. Therefore we only consider the case $\Re\tau = \gamma = 0$. From (3.3) it follows that

$$(\omega^r)^2 = \frac{1}{c_r^2}\big(- (\delta + v^r\eta)^2 + ((F_{11}^r)^2 + (F_{12}^r)^2)\eta^2\big) + \eta^2,$$

$$(\omega^l)^2 = \frac{1}{c_l^2}\big(- (\delta + v^l\eta)^2 + ((F_{11}^l)^2 + (F_{12}^l)^2)\eta^2\big) + \eta^2.$$

When $\eta = 0, \delta \neq 0$, we have

$$(\omega^r)^2 - (\omega^l)^2 = -\left(\frac{1}{c_r^2} - \frac{1}{c_l^2}\right)\delta^2 \neq 0.$$

This contradicts with $\omega^r + \omega^l = 0$. Simple calculation yields that

$$(\omega^r)^2 - (\omega^l)^2 = -\left(\frac{1}{c_r^2} - \frac{1}{c_l^2}\right)\delta^2 + 2\left(\frac{1}{c_r^2} + \frac{1}{c_l^2}\right)\delta v^l\eta - \left(\frac{1}{c_r^2} - \frac{1}{c_l^2}\right)(v^l)^2\eta^2$$

$$+\left(\frac{1}{c_r^2}-\frac{1}{c_l^2}\right)\left((F_{11}^r)^2+(F_{12}^r)^2\right)\eta^2.$$

Assume that $\eta \neq 0$. Let $Y = \frac{\delta}{v^l\eta}$. Then, we consider

$$\frac{(\omega^r)^2-(\omega^l)^2}{(v^l\eta)^2} = -\left(\frac{1}{c_r^2}-\frac{1}{c_l^2}\right)Y^2+2\left(\frac{1}{c_r^2}+\frac{1}{c_l^2}\right)Y-\left(\frac{1}{c_r^2}-\frac{1}{c_l^2}\right)+\left(\frac{1}{c_r^2}-\frac{1}{c_l^2}\right)\frac{g}{(v^r)^2}.$$

The right hand side of the above polynomial in Y has two distinct real roots $\bar{Y}_1$, $\bar{Y}_2$:

$$\bar{Y}_1 = \frac{c_r^2+c_l^2+\sqrt{4c_r^2c_l^2+(c_l^2-c_r^2)^2\frac{g}{(v^r)^2}}}{c_l^2-c_r^2} = \frac{\Theta+2+\sqrt{4(\Theta+1)+\Theta^2\frac{\tilde{g}}{M^2}}}{\Theta},$$

$$\bar{Y}_2 = \frac{c_r^2+c_l^2-\sqrt{4c_r^2c_l^2+(c_l^2-c_r^2)^2\frac{g}{(v^r)^2}}}{c_l^2-c_r^2} = \frac{\Theta+2-\sqrt{4(\Theta+1)+\Theta^2\frac{\tilde{g}}{M^2}}}{\Theta}.$$

This gives that $(\omega^r)^2-(\omega^l)^2$ vanishes at the points $(i\delta,\eta)\in\Sigma$ with

$$\delta = -\bar{Y}_1v^r\eta, \quad \text{or } \delta = -\bar{Y}_2v^r\eta, \quad \text{for } \eta\neq 0. \tag{3.13}$$

In conclusion, the factor $\omega^r+\omega^l$ only admits distinct purely imaginary roots, and therefore the correspondingly roots to the Lopatinskiĭ determinant are purely imaginary as well.

4. **Sufficient conditions for stability.** This section is devoted to providing some sufficient conditions for linear and nonlinear stability of the vortex sheets. As is discussed in the Introduction, the complexity of the algebraic system prevents us from obtaining sharp conditions on the root distribution of the Lopatinskiĭ determinant. Therefore we will only work on deriving sufficient linear stability conditions as well as a nonlinear stability in a perturbative regime.

4.1. **Sufficient conditions for linear stability.** With a good understanding of the roots distribution of the Lopatinskiĭ determinant, in particular, Lemma 3.2, we may proceed to the linear stability of the vortex sheets.

For a background solution defined by (2.1), recall the definition (3.10) that

$$M := \frac{v^r}{c_r}, \qquad \Theta := \frac{c_l^2}{c_r^2}-1, \qquad \tilde{g} := \frac{(F_{11}^r)^2+(F_{12}^r)^2}{c_r^2}.$$

Before stating the theorem, let us introduce the function spaces, which is the same as in [4]. Set

$$\mathbb{R}^3_+ := \{(t,x_1,x_2)\in\mathbb{R}^3 : x_2>0\}.$$

Denote $\mathcal{D}'$ to be the set of distributions and define

$$H^s_\gamma(\mathbb{R}^2) := \{u(t,x_1)\in\mathcal{D}'(\mathbb{R}^2) : e^{-\gamma t}u(t,x_1)\in H^s(\mathbb{R}^2)\},$$
$$H^s_\gamma(\mathbb{R}^3_+) := \{v(t,x_1,x_2)\in\mathcal{D}'(\mathbb{R}^3_+) : e^{-\gamma t}v(t,x_1,x_2)\in H^s(\mathbb{R}^3_+)\},$$

for $s\in\mathbb{R},\gamma\geq 1$, with equivalent norms

$$\|u\|_{H^s_\gamma(\mathbb{R}^2)} := \|e^{-\gamma t}u\|_{H^s(\mathbb{R}^2)}, \quad \|v\|_{H^s_\gamma(\mathbb{R}^3_+)} := \|e^{-\gamma t}v\|_{H^s(\mathbb{R}^3_+)}.$$

Define the norm

$$\|u\|^2_{s,\gamma} := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}(\gamma^2+|\xi|^2)^s|\widehat{u}(\xi)|^2d\xi, \ \forall u\in H^s(\mathbb{R}^2),$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. Setting $\tilde{u} = e^{-\gamma t}u$, we see that $\|\tilde{u}\|_{s,\gamma}$ and $\|u\|_{H^s_\gamma(\mathbb{R}^2)}$ are equivalent, denoted by $\|u\|_{H^s_\gamma(\mathbb{R}^2)} \simeq \|\tilde{u}\|_{s,\gamma}$. Finally we define the space $L^2(\mathbb{R}_+; H^s_\gamma(\mathbb{R}^2))$ endowed with the norm

$$|||v|||^2_{L^2(H^s_\gamma)} := \int_0^{+\infty} \|v(\cdot, x_2)\|^2_{H^s_\gamma(\mathbb{R}^2)} dx_2.$$

It follows that

$$|||v|||^2_{L^2(H^s_\gamma)} \simeq |||\tilde{v}|||^2_{s,\gamma} := \int_0^{+\infty} \|\tilde{v}(\cdot, x_2)\|^2_{s,\gamma} dx_2.$$

Theorem 4.1 (Linear stability)**.** *Supposes that the condition* (3.12) *holds for the background solution* (2.8)*, namely, in the physical variables,*

$$\bar{v}^2 > \bar{F}^2_{11} + \bar{F}^2_{12} + \bar{c}^2_+ + \bar{c}^2_- \qquad or \qquad \bar{v}^2 < \bar{F}^2_{11} + \bar{F}^2_{12}. \tag{4.1}$$

Then there is a positive constant C and an integer $m \in \{1, 2\}$ such that for all $\gamma > 1, W \in H^2_\gamma(\mathbb{R}^3_+)$ and $\varphi \in H^2_\gamma(\mathbb{R}^2)$ solving the linear problem (2.10) *with* (2.11)*, the following estimate holds:*

$$\begin{aligned} &\gamma|||W|||^2_{L^2(H^0_\gamma)} + \|W^{nc}|_{x_2=0}\|^2_{0,\gamma} + \|\varphi\|^2_{H^1_\gamma(\mathbb{R}^2)} \\ \le &C\left(\frac{1}{\gamma^{2m+1}}|||\mathcal{L}W|||^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^{2m}}\|\mathcal{B}(W^{nc}|_{x_2=0}, \varphi)\|^2_{H^m_\gamma(\mathbb{R}^2)}\right). \end{aligned} \tag{4.2}$$

Proof. We will only provide a sketch of the ideas since the argument follows along a quite similar way as in [4].

By the classical hyperbolic theory, one may remove the source term from the linear system for W, and eliminate the wave front φ as well. Thus the linear stability reduces to an energy type estimate on the non-characteristic part W^{nc} of W. In the Laplace-Fourier variables, W^{nc} satisfies

$$\begin{cases} \dfrac{d}{dx_2}\widehat{W}^{nc} = A\widehat{W}^{nc}, \\ \beta\widehat{W}^{nc}|_{x_2=0} = h, \end{cases}$$

where we have recalled (3.1) and (3.5).

The upper-triangularization method in [4] suggests a further decomposition of $\widehat{W}^{nc}$ into the outgoing and incoming modes, which further reduces the problem to estimating the incoming mode $\widehat{W}^{in}$ from

$$\begin{cases} \dfrac{d}{dx_2}\widehat{W}^{in} = G\widehat{W}^{in}, \\ P\widehat{W}^{in}|_{x_2=0} = h, \end{cases}$$

where G is a 2×2 matrix obtained from the upper-triangularization of A, and P is also a 2×2 matrix whose determinant is exactly the Lopatinskiĭ determinant. Clearly the estimate of $\widehat{W}^{in}$ will follow from a control on $\widehat{W}^{in}|_{x_2=0}$. The latter can be achieved as long as P is invertible. This is where we implement the analysis of the Lopatinskiĭ determinant.

The analysis performed in Section 3 (Step 1 through Step 4) provides sufficient conditions (4.1) (or equivalently, (3.12)) under which the roots of the Lopatinskiĭ determinant are purely imaginary and lie on the boundary of Σ. Hence similar estimates like in [4, Lemma 5.2], and in turn the energy estimates as in [4, Section 6] can be carried out, leading to the linear stability. The introduction of the integer index m in the theorem is due to the possibility of having repeated roots, which, if exist, will be double roots. □

Remark 4.1. Theorem 4.1 confirms the strong elastic stabilization of the vortex sheets. In particular, condition (4.1) ensures that a stable supersonic regime as well as a stable subsonic one always persist for any given nonisentropic configuration. The existence of the stable subsonic bubble, a phenomenon not observed in the Euler flow, is particularly due to elasticity. This is consistent with the result of [4] in the isentropic limit $\bar{c}_+^2 - \bar{c}_-^2 \to 0$, where the supersonic region in (4.1) becomes

$$\bar{v}^2 > \bar{F}_{11}^2 + \bar{F}_{12}^2 + 2c^2,$$

Remark 4.2. We want to point out that although in the isentropic limit, (4.1) agrees with the one in [4] which is a necessary and sufficient condition for linear stability, it is not sharp. This can be seen from the comparison with the result for the 2D nonisentropic Euler vortex sheets, where a stability threshold is proved in [9,23,24] stating that the vortex sheets are stable when

$$\bar{v}^2 > \frac{1}{4}\left(\bar{c}_+^{2/3} + \bar{c}_-^{2/3}\right)^3, \tag{4.3}$$

and violently unstable when

$$\bar{v}^2 < \frac{1}{4}\left(\bar{c}_+^{2/3} + \bar{c}_-^{2/3}\right)^3.$$

Note that in absence of elasticity, condition (4.1) becomes

$$\bar{v}^2 > \bar{c}_+^2 + \bar{c}_-^2,$$

which is certainly not as sharp as (4.3).

Remark 4.3. Consistent with the result of [9,23,24], we find from (4.1) or (3.12) how the stability condition changes with the thermal fluctuation. Taking the right (or the '+') state as the reference state, a large thermal fluctuation corresponds to $\bar{c}_-^2 \gg \bar{c}_+^2$ (or $\Theta \gg 1$). In this case we see that the supersonic region shrinks, indicating an attenuation of stability.

Remark 4.4. The above theorem only provides a sufficient condition for the stability, and does not specify the value of m. To obtain a sharper quantitative information about the stable regime requires a more thorough examination of the Lopatinskiĭ determinant, which is beyond the scope of the current paper. We will try to address this issue in the forthcoming works.

4.2. **Nonlinear stability for near-isentropic vortex sheets.** Recall that in the isentropic regime ($\bar{S}^+ = \bar{S}^-$) the two dimensional elastic vortex sheets is nonlinearly stable [6], see [6, Theorem 1.1]. The stability criteria in [6] are open-set properties. Therefore using a perturbative argument it follows that the same result holds for $|\bar{S}^+ - \bar{S}^-| \ll 1$. More precisely, we have

Theorem 4.2. *Let $T_0 > 0$ and $s_0 \geq 14$ be an integer. There exists some $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$ one can find a constant $C_\varepsilon > 0$ sufficiently small, if $|\bar{S}^+ - \bar{S}^-| < \varepsilon$ and the background state (2.8) satisfies the following*

$$\bar{v}^2 > 2\bar{c}_+^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2 + C_\varepsilon, \tag{4.4}$$

or

$$0 < \bar{v}^2 < \min \left\{ \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2}{4}, \frac{\left(\sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2 + \bar{c}_+^2} - \sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2}\right)^2}{4} \right\} - C_\varepsilon, \tag{4.5}$$

then for any initial data $U_0^\pm$ and φ_0 satisfying the compatibility conditions up to order s_0 (see [6]), and $(U_0^\pm - \bar{U}^\pm, \varphi_0) \in H^{s_0+1/2}(\mathbb{R}_+^2) \times H^{s_0+1}(\mathbb{R})$ having a compact support together with

$$\|U_0^\pm - \bar{U}^\pm\|_{H^{s_0+1/2}(\mathbb{R}_+^2)} + \|\varphi_0\|_{H^{s_0+1}(\mathbb{R})} \le \varepsilon,$$

the vortex sheet problem admits a solution $(U^\pm, \Phi^\pm, \varphi)$ on the time interval $[0, T_0]$ satisfying

$$(U^\pm - \bar{U}^\pm, \Phi^\pm - \bar{\Phi}^\pm) \in H^{s_0-8}((0, T_0) \times \mathbb{R}_+^2), \quad \varphi \in H^{s_0-7}((0, T_0) \times \mathbb{R}).$$

Proof. Again we will only sketch the idea of the proof. As is explained earlier, Theorem 4.2 can be viewed as a perturbative result from [6, Theorem 1.1]. Recall that the linear stability hinges heavily upon the root distribution of the Lopatinskiĭ determinant. More explicitly, it requires that all the roots be purely imaginary. From the discussion in Section 3, we know that this is the same as asking that all roots of equation (3.11) are real, which is a property that can persist under perturbation of the parameters, except when the root is repeated. In the isentropic case, it is shown in [4] that the roots are all simple when the background state satisfies

$$\bar{v}^2 > 2\bar{c}_+^2 + \bar{F}_{11}^2 + \bar{F}_{12}^2, \quad \text{or} \quad 0 < \bar{v}^2 < \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2}{4}.$$

Thus for $[S]$ and C_ε small enough, (4.4) or (4.5) will ensure the linear stability of the piecewise constant vortex sheets.

The next step is to establish linear stability of the compact perturbation of the piecewise constant vortex sheets, which is referred to as the variable-coefficient linear stability. Such a problem for isentropic elastic vortex sheets has been dealt with in [5] with additional requirement on the spectrum that points where the eigenvalues $\omega^{r,l}$ vanish (which are called the poles) do not coincide with the roots of the Lopatinskiĭ determinant. Such a requirement induces further restrictions on the background state in the subsonic bubble, namely a few speed nodes need to be excluded, among which two slowest nodes being (see [5, Theorem 2.1])

$$\bar{v}^2 \neq \frac{\bar{F}_{11}^2 + \bar{F}_{12}^2}{4}, \quad \text{and} \quad \bar{v}^2 \neq \frac{\left(\sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2 + \bar{c}_+^2} - \sqrt{\bar{F}_{11}^2 + \bar{F}_{12}^2}\right)^2}{4}.$$

Notice that this non-collapsing property is also stable under perturbations, and hence we conclude that for $[S]$ and C_ε small enough, (4.4) or (4.5) further proves variable-coefficient linear stability.

Developing from linear analysis to structurally stability analysis can be achieved by a routine iteration technique. The main step is to obtain the tame estimate for the linearized problem after which, a modified version of Nash–Moser iteration can be implemented. We emphasize that in the nonlinear analysis, we need to avoid the possibility of double roots. Due to the lack of exact information of the root distribution of the Lopatinskiĭ determinant, we will limit ourselves to working only in the smallest "subsonic" bubble. Following the similar argument of [6, 28], the estimate for the normal derivatives of the characteristic variables can be recovered from the linearized vorticities and divergences. Compared with 2D isentropic elastic vortex sheets [6], the missing normal derivatives needed in deriving the higher energy estimates can be compensated through the equations of the linearized entropy, which is the same as the linearized vorticity. This allows us to follow a similar argument as [24] to extend the stability for the isentropic elastic vortex sheets [6] to the nonisentropic (thermoelastic) case. Finally, we make the conclusion that the two dimensional thermoelastic vortex sheets are weakly stable in both the supersonic and subsonic zones under the elasticity stabilization and a small jump of the entropy. □

Remark 4.5. Another type of free boundary problem, namely the thermoelastic contact discontinuity, is considered in [3], where the velocity is continuous across the interface. As is proved in [3, Proposition 2.1], such a configuration does not persists in the isentropic frame $[S] = 0$. On the other hand, when considering the vortex sheets structure, we can recover the isentropic process as $[S] \to 0$.

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