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A Rigidity Property for the Novikov Equation and the Asymptotic Stability of Peakons

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Abstract

We consider weak solutions of the Novikov equation that lie in the energy space H^1 with non-negative momentum densities. We prove that a special family of such weak solutions, namely the peakons, is H^1-asymptotically stable. Such a result is based on a rigidity property of the Novikov solutions which are H^1-localized and the corresponding momentum densities are localized to the right, which extends the earlier work of Molinet (Arch Ration Mech Anal 230:185–230, 2018; Nonlinear Anal Real World Appl 50:675–705, 2019) for the Camassa–Holm and Degasperis–Procesi peakons. The main new ingredients in our proof consist of exploring the uniform in time exponential decay property of the solutions from the localization of the H^1 energy and redesigning the localization of the total mass from the finite speed of propagation property of the momentum densities.

1. Introduction

Solitary waves made their scientific debut in Russell's famous horseback observation of Great Wave of Translation moving along the Glasgow–Edinburgh canal [29]. Their relevance has since emerged in a vast area of physical problems, including fluid mechanics, optics, quantum physics, and plasma physics. Understanding their stability under perturbation has significant physical implications. In this work we will consider a quasilinear dispersive equation and establish the asymptotic stability of the solitary waves in this setting.

To be more precise, the equation of interest here reads as

$$u_t - u_{txx} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}. \tag{1.1}$$

This was first proposed by Novikov [26] in classifying integrable equations with cubic nonlinearity, and was recently shown to arise in the modeling of the propagation of shallow water waves where waves are assumed to be of moderately large amplitude [8], which is in sharp contrast with the classical Korteweg–de Vries (KdV) equation. The Novikov equation (1.1) can be reformulated into a more compact form with the introduction of the momentum density $y := u - u_{xx}$,

$$y_t + u^2 y_x + \left(\frac{3}{2}u^2\right)_x y = 0, \tag{1.2}$$

which easily reveals its similarity to the well-known Camassa–Holm (CH) equation [5,11]:

$$y_t + u y_x + 2u_x y = 0. \tag{1.3}$$

Like its CH ancestor, the Novikov equation exhibits strong nonlinear effects as well as nonlinear dispersion, allowing it to support a remarkable class of non-smooth solitary waves, called *peakons* [12,15,16],

$$u(t,x) = \varphi_c(x - ct - x_0) := \sqrt{c}\varphi(x - ct - x_0), \quad \text{where} \quad \varphi(x) := e^{-|x|}, \quad c > 0. \tag{1.4}$$

The study of the stability of solitary waves has a rich history, where the dominant part of the literature lands in the scope of smooth waves. Here we present a short survey with no attempt to be exhaustive. One way to prove stability is to seek invariants of the system that can be combined to give a variational characterization of the solitary waves, and their Lyapunov stability can be concluded by further exploiting the spectrum of the linearized operator. For equations possessing a Hamiltonian structure with symmetries, this approach is very powerful to prove orbital stability [2–4,6,14,30]. The second path to stability is based on a perturbation approach in that one tries to prove linear stability and then propose an approach to bootstrap the nonlinearity to establish stability for the nonlinear flow. This usually requires a direct spectral analysis of the linearized operator. Using this approach, PEGO--WEINSTEIN [28] proved the asymptotic stability of a family of the KdV solitons in some exponentially weighted space. This result was then refined by MIZUMACHI [22] and Germain–Pusateri–Rousset [13] to treat perturbations in polynomially weighted spaces. Another approach was developed in a series of papers of Martel–Merle in the study of a class of generalized KdV equations. Without linearization, the authors either employ a virial inequality to directly prove asymptotic stability [20], or establish a Liouville theorem for solutions around solitary waves [19,21] and use such a property to obtain the asymptotic stability.

All of the above mentioned works require the solitary waves to be sufficiently smooth to allow for the spectral analysis [2–4,6,13,14,22,28,30] or the formulation of the dual problem [19,21]. However, the class of equations we consider here admits solitary waves with peak singularity (1.4), a property shared by many quasilinear integrable systems. We also want to note that the KdV-like equations are semilinear, whereas equations accommodating peakons are often quasilinear. The enhanced nonlinearity would in turn make the stability analysis more delicate. In a recent work [23,24], Molinet introduced a very interesting and deep idea to prove the asymptotic stability for the CH and Degasperis–Procesi (DP) peakons. Utilizing the transport structure for the momentum density, a Liouville type *rigidity* result can be proved and consequently applied to asymptotic stability. In the present work, the idea for treating the Novikov peakons stems from [23,24], and hence we will briefly review Molinet's approach and point out some new challenges.

The CH equation (1.3) possesses the following conservation laws

$$\mathbf{Y}(u) := \int_{\mathbb{R}} y \, \mathrm{d}x, \quad \mathcal{E}(u) := \int_{\mathbb{R}} \left(u^2 + u_x^2\right) \mathrm{d}x, \quad \mathbf{F}(u) := \int_{\mathbb{R}} \left(u^3 + u u_x^2\right) \mathrm{d}x. \tag{1.5}$$

The orbital stability of the CH peakons ensures the weak convergence of the perturbed solution along a time sequence. With the help of the conservation laws $\mathbf{Y}$ and $\mathcal{E}$, this weak limit can be shown to carry some "localized" structure from the initial perturbation and leads to a "localized" solution. Then the rigidity property would imply that such a solution is indeed a peakon. Finally the desired asymptotic stability can be obtained by a standard modulation argument. One of the main new features in [23,24] is the rigidity property in terms of the momentum density y rather than the solution u. The applicability and advantages of this replacement are that: (1) the invertibility of $(1 - \partial_x^2)$ shows that y and u are in one-to-one correspondence, and peakons are solutions whose momentum densities are Dirac delta measures; (2) compared with the *nonlocal* evolution of u, the *local* dynamics of y makes the analysis simpler. Analyzing y also motivates one to consider the function space

$$Y_+ := \left\{u \in H^1(\mathbb{R}) \mid y = u - u_{xx} \in \mathcal{M}_+(\mathbb{R})\right\}, \tag{1.6}$$

where $\mathcal{M}_+(\mathbb{R})$ is the set of finite non-negative Radon measures on $\mathbb{R}$. The non-negativity is needed to guarantee the existence of global solutions. To characterize the "localization" property in a way such that the rigidity result can be built upon, Molinet [23,24] takes advantage of the conservation of the H^1 energy $\mathcal{E}$ and the total mass $\mathbf{Y}$, and considers localization in terms of the densities $(u^2 + u_x^2)\,\mathrm{d}x$ and $y\,\mathrm{d}x$. From the fact that the dispersion travels to the left of solitary waves, one may prove an "$\mathcal{E}+\mathbf{Y}$"-monotonicity in the spirit of Martel–Merle, which is the principal tool to obtain asymptotic stability.

The case for the Novikov equation (1.1) is somewhat similar. The conservation laws include $\mathcal{E}(u)$ as in (1.5), and

$$\mathcal{Y}(u) := \int_{\mathbb{R}} y^{2/3} \, \mathrm{d}x, \quad \mathcal{F}(u) := \int_{\mathbb{R}} \left(u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4\right) \mathrm{d}x. \tag{1.7}$$

However, notice that the total mass $\mathbf{Y}$ is not a conserved quantity anymore. This presents a serious obstacle when applying Molinet's machinery: using the localization in terms of $(u^2 + u_x^2)\,\mathrm{d}x$ and $y\,\mathrm{d}x$ may still produce a corresponding rigidity theorem. However it is not clear whether the ω-limit (in time) of the perturbed solution would carry the same localized structure, since one fails to control the total mass from the left (as in [23, (5.15)]) due to the lack of conservation of $\mathbf{Y}$.

We now explain the ideas and methodology we will use to overcome the above difficulties. As aforementioned, the localization consists of two parts: the H^1-localization and the $\mathbf{Y}$-localization. Treating them separately one can see that the localization of H^1 energy is preserved in the limit because of the conservation of $\mathcal{E}$.

The challenge comes from controlling the total mass $\mathbf{Y}$. In the case of strong solutions, we can only bound $\|y\|_{L^1}$ up to an exponential growing-in-time factor

$$\|y\|_{L^1} \leqslant \mathrm{e}^{t\|u_0\|_{H^1}^2} \|y_0\|_{L^1} . \tag{1.8}$$

Therefore it is not clear whether one can control the total mass uniform and globally in time, let alone the preservation of the $\mathbf{Y}$-localization. On the other hand, the $\mathbf{Y}$-localization in the earlier works [23,24] is mainly used for obtaining the rigidity property. In particular it implies that the support of the momentum density y is bounded from the right (cf. [23, Proposition 4.2]), which is crucial to further characterize the jump of u_x. Moreover, the local dynamics (1.2) indicates that the mass is moving to the right at a finite speed u^2, and thus it is conceivable that one may give up the left localization and only consider the localization of y to the right. As in the case for the CH equation [10], the Novikov solutions that are H^1-localized and move to the right enjoy exponential decay. One consequence of the H^1-localization is that the modulated solution enjoys exponential decay, cf. Proposition 3.1. This in turn provides control of the rate of the focusing of the characteristics in the far field (Lemma 3.2). With this, the total mass of $y(t_1)$ in the far field can be equivalently pushed forward to the mass of $y(t_2)$ in the far field. Then by choosing fast enough modulation one can prove the upper bound for the support of y, cf. Proposition 3.2.

The price to pay for giving up half of the $\mathbf{Y}$-localization constraint is that the proof for the asymptotic stability becomes more difficult. In particular, in characterizing the (time) ω-limit of the perturbed solutions, one needs to check that the total mass for the limit solution is finite. But (1.8) indicates that the total mass can potentially become unbounded as $t \to \infty$. Again, the remedy here is the exponential decay property derived from the H^1-localization. Please refer to Proposition 4.2 for details.

We would like to point out that our relaxed localization condition on y extends the requirement of y in [23,24], and leaves the argument free from the conservation law of y. Thus it could be potentially useful in studying the asymptotic stability of peakons to a wider class of models.

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and notations, state our main results, and also recall some useful estimates and the global well-posedness of H^1 weak solutions. In Section 3, we prove the Liouville property of the Novikov peakons, cf. Theorem 2.1, which follows the adapted strategy in [23]. In Section 4, we apply the result of Section 3 to obtain the asymptotic stability (Theorem 2.2) of the Novikov peakons.

2. Preliminaries and Main Results

In this section, we introduce some notations and definitions, recall some useful estimates and the global well-posedness of H^1 weak solutions, and state our main results.

Let us now introduce some notations following [23]. Set $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}^- := (-\infty, 0)$. We denote $C_b(\mathbb{R})$ to be the set of bounded continuous function on $\mathbb{R}$ and

$C_0(\mathbb{R})$ the set of continuous function on $\mathbb{R}$ vanishing at infinity. For any $\phi \in C_b(\mathbb{R})$ (resp. $\phi \in C_0(\mathbb{R})$), if a sequence $\{v_n\} \subset \mathcal{M}$ satisfies

$$\langle v_n, \phi\rangle \to \langle v, \phi\rangle, \quad v \in \mathcal{M},$$

we say that

$$v_n \overset{*}{\rightharpoonup} v \text{ tightly in } \mathcal{M} \quad (\text{resp. } v_n \overset{*}{\rightharpoonup} v \text{ in } \mathcal{M}).$$

For any interval $I \subset \mathbb{R}$, $y \in C_{ti}(I; \mathcal{M})$ (resp. $y \in C_w(I; \mathcal{M})$) means that

$$\forall\, \phi \in C_b(\mathbb{R}) \text{ (resp. } C_0(\mathbb{R})), \quad t \mapsto \langle y(t), \phi\rangle \text{ is continuous on } I,$$

and $y_n \overset{*}{\rightharpoonup} y$ in $C_{ti}(I; \mathcal{M})$ (resp. $C_w(I; \mathcal{M})$) says

$$\forall\, \phi \in C_b(\mathbb{R}) \text{ (resp. } C_0(\mathbb{R})), \quad \langle y_n(t), \phi\rangle \to \langle y(t), \phi\rangle \text{ in } C(I).$$

Note that $p(x) := \frac{1}{2}e^{-|x|}$ is the Green's function for $1 - \partial_x^2$ in $\mathbb{R}$, that is

$$p * f = (1 - \partial_x^2)^{-1} f \qquad \text{for any} \quad f \in H^{-1}(\mathbb{R}). \tag{2.1}$$

As pointed out in [23, Section 2],

$$\|u\|_{W^{1,1}(\mathbb{R})} = \|p * (u - u_{xx})\|_{W^{1,1}(\mathbb{R})} \leqq \frac{1}{2}\|u - u_{xx}\|_{\mathcal{M}(\mathbb{R})},$$

and the embedding

$$Y := \left\{u \in H^1(\mathbb{R}) \mid y = u - u_{xx} \in \mathcal{M}(\mathbb{R})\right\} \hookrightarrow \left\{u \in W^{1,1}(\mathbb{R}) \text{ such that } u_x \in \mathcal{BV}(\mathbb{R})\right\}$$

holds. Noting that $C_0^\infty(\mathbb{R})$ is dense in Y, we have

$$|u_x| \leqq u \quad \text{for any} \quad u \in Y_+. \tag{2.2}$$

We also define the mollifier $\{\zeta_n\}_{n \geqq 1}$ by

$$\zeta_n(x) = \left(\int_{\mathbb{R}} \zeta(\xi) \mathrm{d}\xi\right)^{-1} n\zeta(nx) \quad \text{with} \quad \zeta(x) = \begin{cases} e^{1/(x^2-1)} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqq 1. \end{cases} \tag{2.3}$$

Let us turn to the wellposedness of the Novikov equation (1.1). The local and global wellposedness of strong solutions is given in [32].

Proposition 2.1. *(Wellposedness of strong solutions [32]) Assume that $u_0 \in H^s$, $s > \frac{3}{2}$, then there exists a unique solution u to equation* (2.4) *and a $T > 0$ such that $u = u(\cdot, u_0) \in C([0, T); H^s) \cap C^1([0, T); H^{s-1})$. Moreover, the solution depends continuously on the initial data. In particular, if $u_0 \in H^\infty$, then $u \in C([0, T); H^\infty)$. In addition, if $y_0 = u_0 - u_{0,xx}$ does not change sign on $\mathbb{R}$, then the local solution can be extended globally.*

However the above theory does not apply to peakons due to the peak singularity, and hence it is natural to work in the weak solution framework. Thus we rewrite the Cauchy problem for (1.1) in the following weak form:

$$\begin{cases} u_t + u^2 u_x + (1-\partial_x^2)^{-1}\partial_x\left(\frac{3}{2}uu_x^2 + u^3\right) + (1-\partial_x^2)^{-1}\left(\frac{1}{2}u_x^3\right) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{2.4}$$

Here we record the following result on the existence and uniqueness of global H^1 weak solutions of the Novikov equation (2.4), which is an improved version of [31, Theorem 3.1]:

Proposition 2.2. (Global existence and uniqueness of weak solutions [9]) *For any $u_0 \in Y_+$, the Cauchy problem (2.4) admits a unique global weak solution $u \in W^{1,\infty}(\mathbb{R}^+ \times \mathbb{R}) \cap C(\mathbb{R}^+; H^1(\mathbb{R}))$ such that $y(t,\cdot) \in \mathcal{M}_+(\mathbb{R})$ for all $t > 0$. Moreover, $\mathcal{E}(u)$ and $\mathcal{F}(u)$ are conservation laws.*

Similar to the argument in [23,24], we can also improve the regularity in time for the above global solution. For brevity we will only state the following adapted proposition without a proof.

Proposition 2.3. *Let $u_0 \in Y_+$ and $n \in \mathbb{N}$, then*

- (Global existence and Uniqueness) *There exists a unique solution $u \in C^1(\mathbb{R}; L^2(\mathbb{R})) \cap C(\mathbb{R}; H^1(\mathbb{R}))$ of (2.4) such that $y \in C_{ti}(\mathbb{R}; \mathcal{M}_+)$. Moreover, $\mathcal{E}(u)$ and $\mathcal{F}(u)$ are conserved.*
- (Continuity with respect to u_0 in $H^1(\mathbb{R})$) *For any sequence $\{u_{0,n}\}$ bounded in Y_+ such that*

  $$u_{0,n} \to u_0 \text{ in } H^1(\mathbb{R}) \quad \text{and} \quad \left(1-\partial_x^2\right)u_{0,n} \overset{*}{\rightharpoonup} (1-\partial_x^2)u_0 \text{ tightly in } \mathcal{M},$$

  then for any $T > 0$, the sequence of solutions $\{u_n\} \subset C^1\left(\mathbb{R}^+; L^2(\mathbb{R})\right) \cap C\left(\mathbb{R}^+; H^1(\mathbb{R})\right)$ emanating from $\{u_{0,n}\}$ satisfies that as $n \to \infty$,

  $$u_n \to u \text{ in } C\left([-T,T]; H^1(\mathbb{R})\right), \tag{2.5}$$

  and

  $$\left(1-\partial_x^2\right)u_n \overset{*}{\rightharpoonup} y \text{ in } C_{ti}([-T,T]; \mathcal{M}). \tag{2.6}$$

- (Continuity with respect to u_0 in Y equipped with the weak topology) *For any sequence $\{u_{0,n}\}$ bounded in Y_+ such that $u_{0,n} \overset{*}{\rightharpoonup} u_0$ in Y, then for any $T > 0$, the sequence of solutions $\{u_n\} \subset C^1\left(\mathbb{R}^+; L^2(\mathbb{R})\right) \cap C\left(\mathbb{R}^+; H^1(\mathbb{R})\right)$ emanating from $\{u_{0,n}\}$ satisfies that as $n \to \infty$,*

  $$u_n \rightharpoonup u \text{ in } C_w\left([-T,T]; H^1(\mathbb{R})\right),$$

  and

  $$\left(1-\partial_x^2\right)u_n \overset{*}{\rightharpoonup} y \text{ in } C_w([-T,T]; \mathcal{M}).$$

Now we turn to the main results of the paper. First we give the definition of localized solutions.

Definition 2.1. A solution $u \in C\left(\mathbb{R}; H^1(\mathbb{R})\right)$ with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$ of (1.1) is said to be H^1-localized if there exist $c_0 > 0$ and a C^1 function $\rho(t)$ with $\rho_t \geqq c_0 > 0$ such that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ so that

$$\left\langle (u^2 + u_x^2)(t, \cdot), \Phi_1(\cdot - \rho(t)) \right\rangle \leqq \varepsilon \tag{2.7}$$

holds for all $t \in \mathbb{R}$ and all $\Phi_1 \in C(\mathbb{R})$ with $0 \leqq \Phi_1 \leqq 1$ and $supp\Phi_1 \subset [-R_\varepsilon, R_\varepsilon]^c$.

We say the solution u is **Y**-localized to the right if

$$\langle (u - u_{xx})(t, \cdot), \Phi_r(\cdot - \rho(t)) \rangle \leqq \varepsilon \tag{2.8}$$

for all $t \in \mathbb{R}$ and all $\Phi_r \in C(\mathbb{R})$ with $0 \leqq \Phi_r \leqq 1$ and $supp\Phi_r \subset [R_\varepsilon, \infty)$.

Remark 2.1. The notion of H^1-localized solution is similar to that of the CH equation [10]. As is pointed out in [24, Remark 1.1], such a characterization is natural for equations admitting the H^1 conservation law. Otherwise (2.7) can be replaced by a uniform in time exponential decay characterization. Compared with the localized solutions defined in [23, Definition 1.1] and [24, Definition 1.1], here in (2.8) we only consider semi-localization to the right.

Our first main theorem states a rigidity property of the solution to the Novikov equation that is H^1-localized and **Y**-localized to the right.

Theorem 2.1. *If $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ is a solution to (1.1) that is H^1-localized and **Y**-localized to the right, and is not identically vanishing, then there exist $c^* > 0$ and $x_0 \in \mathbb{R}$ such that*

$$u(t) = \sqrt{c^*}\varphi(\cdot - x_0 - c^* t), \quad \text{for all } t \in \mathbb{R}.$$

With the help of the above rigidity theorem, we can prove the following theorem on the asymptotic stability of the Novikov peakons:

Theorem 2.2. *For $c > 0$, if for each $u_0 \in Y_+$ and $\theta \in (0, \sqrt{c})$ satisfying $\theta^4 < \frac{1}{2}$, there exists a constant η_0 such that*

$$\|u_0 - \varphi_c\|_{H^1} \leqq \eta_0 \sqrt{c}\theta^4, \tag{2.9}$$

then there exist $c^* > 0$ and a function $\rho(t) \in C^1(\mathbb{R})$ satisfying $\lim\limits_{t\to\infty} \rho_t = c^*$ such that

$$u(t, \cdot + \rho(t)) \rightharpoonup \varphi_{c^*} \ \ \text{in} \ \ H^1(\mathbb{R}) \ \ \text{as} \ \ t \to \infty, \tag{2.10}$$

where $u \in C(\mathbb{R}; H^1)$ solves (2.4) with initial data u_0. Moreover,

$$\lim_{t\to\infty} \|u(t) - \varphi_{c^*}(\cdot - \rho(t))\|_{H^1(\theta t, \infty)} = 0. \tag{2.11}$$

We remark that during the refereeing process of this paper we found a work by Palacios [27] where similar results were obtained independently.

3. Liouville Property for the Novikov Peakon

The goal of this section is to prove the Liouville type rigidity property for the Novikov peakons, cf. Theorem 2.1, in the space of H^1-localized and $\mathbf{Y}$-localized to the right functions. As explained in the Introduction, we will consider the H^1-localization and the $\mathbf{Y}$-localization (to the right) separately.

For the H^1-localized solutions, there is a well established theory about the asymptotics in the context of the CH equation; see, for example, [10,23] and the references therein. Here we will first follow the idea of [23] to prove an almost monotonicity of the H^1 energy at both sides of an H^1-localized Novikov solution, from which an easy adaptation of the method of [10] leads to an exponential decay property for the H^1-localized Novikov solutions.

3.1. Exponential Decay of H^1-Localized Solutions

The aim of this subsection is to prove the following decay result:

Proposition 3.1. *Let $u \in C\left(\mathbb{R}; H^1\right)$ with $y = \left(1-\partial_x^2\right)u \in C_w(\mathbb{R}; \mathcal{M}_+)$ be an H^1-localized solution of (1.1) with $\inf_{\mathbb{R}} \rho_t \geqq c_0 > 0$. Then for any $\alpha \in (0,1)$, $K > 1$, there exists an $R_0 > 0$ such that*

$$\|u(t)\|^2_{L^\infty(|x-\rho(t)|>R_0)} \leqq \frac{\alpha c_0}{2L}, \quad \textit{where } L := \left(6+\frac{1}{K}\right)\left(\frac{K^2}{K^2-1}\right). \tag{3.1}$$

Moreover, there exists a constant $C > 0$ depending on c_0, K, $\mathcal{E}(u)$, and R_0, such that for all $t \in \mathbb{R}$, $R \geqq R_0$, and $\Phi \in C(\mathbb{R})$ with $0 \leqq \Phi \leqq 1$ and $\operatorname{supp}\Phi \subset [-R,R]^c$, we have

$$\int_{\mathbb{R}} \left(u^2(t)+u_x^2(t)\right)\Phi(\cdot-\rho(t))\mathrm{d}x \leqq Ce^{-\frac{R}{K}}. \tag{3.2}$$

Remark 3.1. From the Sobolev embedding we find from (3.2) that

$$\|u(t)\|^2_{L^\infty(|x-\rho(t)|>R)} \leqq Ce^{-\frac{R}{K}}. \tag{3.3}$$

A main tool for proving this proposition is the H^1-monotonicity of a quantity (3.6) related to the H^1 norm of the solution. This means that the localized energy of the solution is almost decreasing in time. Such a strategy was first introduced by MARTEL--MERLE [19], and later was widely applied to other models and viewed as a crucial prerequisite for showing the rigidity theorem under different settings. One common feature of these arguments is that all quantities are related to the nontrivial conservation laws of the Novikov equation, which is showed in the lemma below.

Following the similar strategy in [23], consider an even function $\zeta \in C_c^\infty([-1,1])$ such that $\zeta \geqq 0$ and $\int_{\mathbb{R}} \zeta = 1$. For $K > 1$, set

$$g(x) := p * \zeta \quad \text{and} \quad \Psi_K(x) := \frac{1}{K}\int_{-\infty}^{x} g\left(\frac{y}{K}\right)\mathrm{d}y, \tag{3.4}$$

where we recall that $p = \frac{1}{2}e^{-|x|}$. Then $\Psi_K \geqq 0$ and is increasing, and $\Psi_K \to 1$ as $x \to \infty$. Moreover

$$\begin{aligned} &|\Psi_K| + |\Psi_K'| + |\Psi_K''| + |\Psi_K'''| \leqq C_K e^{x/K}, \quad \text{for } x \leqq 0, \\ &p * \Psi_K' \leqq \frac{K^2}{K^2-1}\Psi_K'. \end{aligned} \tag{3.5}$$

Now we state the following lemma(for the sake of completeness, its proof will be given in Appendix A.1):

Lemma 3.1. *Let $u \in C\left(\mathbb{R}; H^1\right)$ with $y \in C_w(\mathbb{R}; \mathcal{M}_+)$ be a solution of (2.4) such that there exists a C^1 function $\rho(t)$ with $\inf_{\mathbb{R}} \rho_t \geqq c_0 > 0$. Let $\alpha \in (0,1)$ and $K > 1$. Consider a constant R_0 such that (3.1) holds. Furthermore, for $R \geqq R_0 > 0$, define*

$$I_{t_0}^{\pm R}(t) = \left\langle u^2(t) + u_x^2(t), \Psi_K(\cdot - \rho(t_0) \mp R - \alpha(\rho(t) - \rho(t_0)))\right\rangle, \tag{3.6}$$

where Ψ_K is a smooth function given in (3.4). Then we have

$$I_{t_0}^{+R}(t_0) - I_{t_0}^{+R}(t) \leqq Ce^{-R/K}, \quad \textit{for all } t \leqq t_0, \tag{3.7}$$

and

$$I_{t_0}^{-R}(t) - I_{t_0}^{-R}(t_0) \leqq Ce^{-R/K}, \quad \textit{for all } t \geqq t_0, \tag{3.8}$$

where C depends on K, α, R_0, $\mathcal{E}(u)$ and c_0.

Remark 3.2. It turns out that in the proof of this lemma, the regularity assumption on y is not necessary. In fact the arguments still work if one only assumes $u(t,\cdot) \in W^{1,\infty}$ and $|u_x| \leqq u$.

Now Proposition 3.1 follows easily from the above lemma.

Proof of Proposition 3.1. The proof of (3.1) can be easily deduced from Definition 2.1 and the Sobolev embedding $H^1 \subset L^\infty$.

For $I_{t_0}^{+R}(t)$, we fix $\alpha = \frac{1}{2}$ and introduce $\phi \in C_c^\infty([-1,1])$ satisfying $\phi \equiv 1$ on $[-1/2, 1/2]$ and $0 \leqq \phi \leqq 1$ on $[-1,1]$. By decomposition, we have

$$\begin{aligned} I_{t_0}^{+R}(t) &= \left\langle u^2(t) + u_x^2(t), \left(1 - \phi\left(\frac{\cdot - \rho(t)}{R_\varepsilon}\right)\right)\Psi_K(\cdot - z_{t_0}^{\pm R}(t))\right\rangle \\ &\quad + \left\langle u^2(t) + u_x^2(t), \phi\left(\frac{\cdot - \rho(t)}{R_\varepsilon}\right)\Psi_K(\cdot - z_{t_0}^{\pm R}(t))\right\rangle =: I_1(t) + I_2(t), \end{aligned}$$

where $z_{t_0}^{\pm R}(t) = \rho(t_0) \pm R + \frac{1}{2}(\rho(t) - \rho(t_0))$, and $R_\varepsilon > 0$ depends on ε. From the H^1-localization of u, for any $\varepsilon > 0$, there exists an R_ε such that $I_1(t) < \varepsilon$. As for $I_2(t)$, we know that for $|x - \rho(t)| \leqq R_\varepsilon$,

$$x - z_{t_0}^{\pm R}(t) = x - \rho(t_0) \mp R - \frac{1}{2}(\rho(t) - \rho(t_0)) \leqq R_\varepsilon \mp R - \frac{1}{2}c_0(t_0 - t) \to -\infty,$$
$$\text{as } t \to -\infty,$$

which yields

$$I_2(t) \leqq \|u_0\|_{H^1}^2 \, \Psi_K\left(x - z_{t_0}^{\pm R}(t)\right) \to 0, \quad \text{as } t \to -\infty,$$

since $\lim_{x\to-\infty} \Psi_K(x) = 0$. Thus with (3.7) it follows that

$$I_{t_0}^{+R}(t_0) \leqq Ce^{-R/K}.$$

Since $g(x)$ is even from (3.4), it is easy to see that for all $t_0 \in \mathbb{R}$,

$$\frac{1}{2}\int_R^\infty \left(u^2(t_0, x + \rho(t_0)) + u_x^2(t_0, x + \rho(t_0))\right) \mathrm{d}x \leqq I_{t_0}^{+R}(t_0).$$

Therefore for all $t \in \mathbb{R}$, $R \geqq R_0$ and all $\Phi \in C(\mathbb{R})$ satisfying $0 \leqq \Phi \leqq 1$ and $\mathrm{supp}\Phi \subset (R, \infty)$, we derive

$$\int_{\mathbb{R}} \left(u^2(t) + u_x^2(t)\right) \Phi(\cdot - \rho(t))\, \mathrm{d}x \leqq Ce^{-R/K}.$$

Under the transformation $(t, x) \mapsto (-t, -x)$, the desired result for $\mathrm{supp}\Phi \subset (-\infty, -R)$ is also obtained. □

A consequence of the decay property is the control of the rate of the focusing of characteristics in the far field. For a smooth solution u of (2.4), let us define its characteristics by the following:

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t}(t, x) = u^2(t, q(t, x)), & (t, x) \in \mathbb{R}^2, \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \tag{3.9}$$

Then it is easily seen that

$$q_x(t, x) = \exp\left(2\int_0^t uu_x(s, q(s, x))\mathrm{d}s\right), \tag{3.10}$$

$$y(0, x) = y(t, q(t, x))\, q_x(t, x)^{\frac{3}{2}}. \tag{3.11}$$

Lemma 3.2. *Let the assumptions of Proposition 3.1 hold, and assume further that $u \in H^\infty$. Then there exists a constant $R_0 > 0$ such that, for any $K > 1$ and any $t_* \in \mathbb{R}$,*

$$\frac{1}{C_0} \leqq q_x(t - t_*, \rho(t_*) + R_0 + x) \leqq C_0 \quad \text{for all} \quad t \leqq t_*, \ x \geqq 0, \tag{3.12}$$

where C_0 depends on K, $\mathcal{E}(u)$, c_0, and R_0.

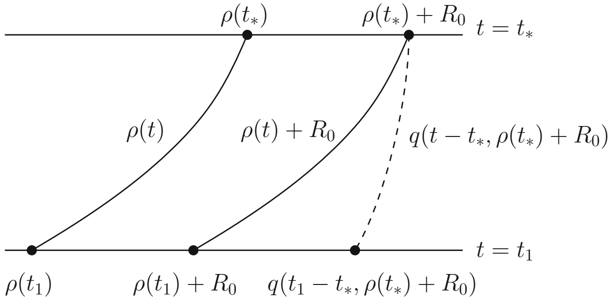

Fig. 1. Separation between the modulation and the characteristics

Proof. From (3.1) we confirm the existence of $R_0 > 0$ such that

$$\|u(t)\|^2_{L^\infty(|x-\rho(t)|>R_0)} \leqq \frac{c_0}{20}.$$

Therefore from (3.9) we know that

$$q_t(t, x) \leqq \frac{c_0}{20}, \quad \text{for } x > \rho(t) + R_0.$$

Given any characteristics $q(t - t_*, \rho(t_*) + R_0)$ that passes through $(t_*, \rho(t_*) + R_0)$, since $\rho_t \geqq c_0$, together with the above we have

$$q\,(t - t_*, \rho(t_*) + R_0) - (\rho(t) + R_0) \geqq \frac{19}{20} c_0(t_* - t), \quad \text{for } t \leqq t_*; \tag{3.13}$$

see Fig. 1 for an illustration. Therefore from (3.3) we know that for any $t \leqq t_*$, $x \geqq 0$,

$$u^2\,(t - t_*, q\,(t - t_*, \rho(t_*) + R_0 + x)) \leqq C \exp\left(-\frac{1}{K}\left(R_0 + \frac{19}{20}c_0(t_* - t)\right)\right). \tag{3.14}$$

By (2.2), the above estimate also holds for u_x^2. Therefore from (3.10) we know that for $t \leqq t_*$ and $x \geqq 0$,

$$\exp\left(-2C\int_{t-t_*}^0 e^{-\frac{R_0}{K}} e^{-\frac{19c_0}{20K}|s|}\,\mathrm{d}s\right) \leqq q_x\,(t - t_*, \rho(t_*) + R_0 + x)$$
$$\leqq \exp\left(2C\int_{t-t_*}^0 e^{-\frac{R_0}{K}} e^{-\frac{19c_0}{20K}|s|}\,\mathrm{d}s\right).$$

Denote $C_0 := \exp\left(\frac{40KC}{19c_0}e^{-R_0/K}\right)$. Then the above estimate yields

$$\frac{1}{C_0} \leqq q_x\,(t - t_*, \rho(t_*) + R_0 + x) \leqq C_0,$$

which completes the proof of the lemma. $\square$

3.2. Finite Upper Bound of the Support of the Momentum Density

With the decay property and the lower bound of the deformation rate of the characteristics established in Proposition 3.1 and Lemma 3.2, the goal of this subsection is to show that the **Y**-localization indeed forces the support of the momentum density to be bounded above. As a consequence, this leads to the bound of u_x, which is a key step for the forthcoming discussion.

Proposition 3.2. *Let $u \in C(\mathbb{R}; Y_+)$ be a solution to (2.4) that is both H^1-localized and* **Y***-localized to the right, with $\rho_t \geqq c_0 > 0$. There exists an $r_0 > 0$ such that for all $t \in \mathbb{R}$, it holds that*

$$\operatorname{supp} y(t, \cdot + \rho(t)) \subset (-\infty, r_0], \tag{3.15}$$

and for any $r > r_0$,

$$u(t, \rho(t) + r) = -u_x(t, \rho(t) + r) \geqq \alpha_r, \tag{3.16}$$

where $\alpha_r := \frac{e^{-2r}}{4\sqrt{r}}\sqrt{\mathcal{E}(u)}$.

Proof. Since the equation (2.4) is invariant under time translation, it suffices to prove (3.15) at time $t = 0$. So we only need to prove

$$\langle y(0), \phi\left(\cdot - (\rho(0) + r_0)\right)\rangle = 0, \tag{3.17}$$

where $\phi \in C^\infty(\mathbb{R})$ with $\phi \equiv 0$ on $\mathbb{R}^-$, $\phi' \geqq 0$ and $\phi \equiv 1$ on $[1, \infty)$.

Through the proof by contradiction, we suppose that (3.17) does not hold, that is, there exists an $\varepsilon_0 > 0$ such that for all $r_0 > 0$,

$$\langle y(0), \phi\left(\cdot - (\rho(0) + r_0)\right)\rangle \geqq \varepsilon_0. \tag{3.18}$$

We will use a density argument to apply the method of characteristics. Define $u_{0,n} := \zeta_n * u_0 \in H^\infty(\mathbb{R})$, we have a smooth function sequence, $\{u_{0,n}\} \subset H^\infty(\mathbb{R}) \cap Y_+$, which emanates the solution sequence $\{u_n\}$ satisfying (2.5) and (2.6). Recalling the Proposition 2.1, we know that $u_n \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ and $y_n \in C_w\left(\mathbb{R}; L^1(\mathbb{R})\right)$. Hence for each fixed $T > 0$ and $n \geqq n_0$ large enough, we have

$$\max_{t\in(-T,T)} \|u_n - u\|_{H^1} < \frac{\sqrt{c_0}}{10}, \qquad \text{and} \qquad \left\|y_{0,n} - y_0\right\|_{\mathcal{M}} < \frac{\varepsilon_0}{2}. \tag{3.19}$$

From (3.18) we know that for $n \geqq n_0$,

$$\int_{\rho(0)+r_0}^{\infty} y_n(0, x)\mathrm{d}x \geqq \frac{9}{10}\varepsilon_0, \tag{3.20}$$

Consider the characteristics associated with u_n:

$$\begin{cases} \frac{\mathrm{d}q_n}{\mathrm{d}t}(t, x) = u_n^2\left(t, q_n(t, x)\right), & (t, x) \in \mathbb{R}^2, \\ q_n(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.21}$$

and

$$q_{n,x}(t,x) = \exp\left(\int_0^t (2u_n u_{n,x})(s, q_n(s,x))\,\mathrm{d}s\right),$$
$$y_n(0,x) = y_n\,(t, q_n(t,x))\, q_{n,x}^{3/2}(t,x). \tag{3.22}$$

Applying Lemma 3.2 on q_n with $t_* = 0$ together with Proposition 2.3, we conclude that an estimate of the form (3.12) holds for q_n when $n \geqq n_0$ is sufficiently large. That is,

$$\frac{1}{C_0} \leqq q_{n,x}\,(t, \rho(0) + R_0 + x) \leqq C_0, \quad \text{for all } t \in [-T, 0], \quad x \geqq 0. \tag{3.23}$$

Substituting (3.22) into (3.20) with $r_0 \geqq R_0$, using (3.23) and changing variables $z = q_n(t,x)$, we have

$$\int_{\rho(0)+r_0}^{\infty} y_n\,(t, q_n(t,x))\, q_{n,x}(t,x)\mathrm{d}x = \int_{q_n(t,\rho(0)+r_0)}^{\infty} y_n\,(t,z)\;\mathrm{d}z \geqq \frac{9}{10}\sqrt{C_0}\varepsilon_0.$$

The convergence of $u_n \to u$ implies the estimate of (3.13) with q replaced by q_n. Thus we have for $t \in [-T, 0]$,

$$\int_{\rho(t)+r_0-c_0t/2}^{\infty} y_n(t,z)\mathrm{d}z \geqq \frac{9}{10}\sqrt{C_0}\varepsilon_0.$$

Therefore, for sufficiently large $n \geqq n_0$ and $T \to \infty$, it follows from (2.6) that

$$\langle y(t), \phi\,(\cdot - \rho(t) - r_0 + c_0 t/2)\rangle \geqq \frac{9}{10}\sqrt{C_0}\varepsilon_0, \text{ for all } t \leqq 0,$$

which contradicts the fact that u is **Y**-localized to the right. Thus (3.15) is proved.

We now prove (3.16). For the same r_0 in (3.15), the H^1 localization, the conservation of $\mathcal{E}(u)$ and $|u_x| \leqq u$ ensure that for all $t \in \mathbb{R}$ and $r \geqq r_0$,

$$\sqrt{2}\|u(t, x - \rho(t))\|_{L^2(-r,r)} \geqq \|u(t, x - \rho(t))\|_{H^1(-r,r)} \geqq \frac{1}{2}\sqrt{\mathcal{E}(u)},$$

which shows

$$\max_{x\in[-r,r]} u^2(t, x - \rho(t)) \geqq \frac{1}{16r}\mathcal{E}(u). \tag{3.24}$$

On the other hand, again from $u_x \geqq -u$, we have $\int_x^{x_0} (ue^s)_s\,\mathrm{d}s \geqq 0$ for all $x \leqq x_0$, which naturally yields

$$u\,(t, x_0) \geqq u(t,x)e^{x-x_0}, \quad \text{for all } x \leqq x_0.$$

Then by setting $x_0 = \rho(t) + r$, for all $t \in \mathbb{R}$ and $r \geqq r_0$ we get

$$u\,(t, \rho(t) + r) \geqq u(t,x)e^{-2r}, \quad \text{for all } \rho(t) - r \leqq x \leqq \rho(t) + r,$$

which gives (3.16) by (3.24). This completes the proof. □

3.3. The First Jump of u_x from the Right

Anticipating a rigidity property, one is naturally led to consider the dynamics of u_x and look to show that it only experiences a one-time jump discontinuity. At the level of the second derivative u_{xx}, and thus y, such a point discontinuity should correspond to the boundary of the support of the measure. To this end, we define

$$\rho_+(t) := \inf\{x \in \mathbb{R} : \ \operatorname{supp} y(t, \cdot) \subset (-\infty, \rho(t) + x]\}.$$

As Proposition 3.2 shows, for each $t \in \mathbb{R}$, the function $t \mapsto \rho_+(t)$ takes values in $(-\infty, r_0]$ and

$$u\,(t, \rho(t) + \rho_+(t)) = -u_x\,(t, \rho(t) + \rho_+(t)) \geqq \alpha_{r_0}. \tag{3.25}$$

It turns out that $t \mapsto \rho(t) + \rho_+(t)$ is indeed a characteristics of u. The unique solvability of the characteristics equation (3.9) is guaranteed from the $W^{1,1}$ regularity and global boundedness of u [1]. The proof of the fact that this characteristics coincides with the boundary of the support of y follows almost identically as in [23, Lemma 4.3], and hence we omit it.

Lemma 3.3. *For all $t \in \mathbb{R}$, we have*

$$\rho(t) + \rho_+(t) = q\,(t, \rho(0) + \rho_+(0))\,, \tag{3.26}$$

where $q(\cdot, \cdot)$ is defined as in (3.9).

Let us record the following lemma in [17] which will be very useful in characterizing y:

Lemma 3.4. *Let μ be a finite nonnegative measure on $\mathbb{R}$. Then μ is the sum of a nonnegative non-atomic measure ν and a countable sum of positive Dirac measures (the discrete part of μ). Moreover, for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if I is an interval of length less than δ, then $\nu(I) \leqq \varepsilon$.*

Next we study the jump of u_x across the characteristics $q^* : t \mapsto \rho(t) + \rho_+(t)$ defined by

$$q^*(t) := q(t, \rho(0) + \rho_+(0)) = \rho(t) + \rho_+(t) \ \text{ for all } t \in \mathbb{R}. \tag{3.27}$$

As in [23], we denote $a(t)$ to be the strength of the jump of u_x across q^* as

$$a(t) := u_x(t, q^*(t)-) - u_x(t, q^*(t)+), \ \ \textit{for all } t \in \mathbb{R}. \tag{3.28}$$

Proposition 3.3. *Let $a : \mathbb{R} \to \mathbb{R}$ be a function defined as in* (3.28)*. Then $0 < a(t) \leqq 2\sqrt{\mathcal{E}(u)}$.*

Proof. The upper bound of $a(t)$ follows directly from $a(t) < 2|u_x| \leqq 2u \leqq 2\sqrt{\mathcal{E}(u)}$.

Now we turn to show that $a(t) > 0$. Arguing by contradiction, we assume that there exists $t_0 \in \mathbb{R}$ such that $a(t_0) = 0$. Due to the fact that $y(t_0) \in \mathcal{M}$ satisfies $\operatorname{supp} y(t_0) \subset (-\infty, q^*(t_0)]$, Lemma 3.4 implies that

$$\lim_{x \to q^*(t_0)^-} \|y(t_0)\|_{\mathcal{M}(q^*(t_0),\infty)} = a(t_0) = 0.$$

Without loss of generality, we again choose $t_0 = 0$. This indicates that one may find a small $\beta_0 > 0$ and $0 < \varepsilon_1 \ll \alpha_{r_0}$ such that

$$\|y(0)\|_{\mathcal{M}(q^*(0)-\beta_0,\infty)} < \varepsilon_1 \ll \alpha_{r_0}. \tag{3.29}$$

Next we approximate u_0 by smooth functions $\tilde{u}_0 := \zeta_n * u_0 \in Y_+ \cap H^\infty(\mathbb{R})$. Picking a large enough n, there must exist $\tilde{\rho}_+ > \rho_+(0)$ close to $\rho_+(0)$ satisfying

$$\tilde{y}_0 \equiv 0 \text{ on } [\tilde{q}_2(0), \infty), \qquad \text{and} \qquad \|\tilde{y}_0\|_{L^1(\tilde{q}_1(0),\infty)} \leqq 2\varepsilon_1, \tag{3.30}$$

where $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$ are defined as

$$\tilde{q}_1(t) := \tilde{q}\left(t, \rho(0) + \tilde{\rho}_+ - \beta_0\right), \quad \text{and} \quad \tilde{q}_2(t) := \tilde{q}\left(t, \rho(0) + \tilde{\rho}_+\right),$$

where $\tilde{q}$ is the characteristics given by (3.9) with u replaced by $\tilde{u}$.

We would like to show that no matter how far apart $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$ are from each other, $\tilde{u}(t, \tilde{q}_1(t))$ and $\tilde{u}(t, \tilde{q}_2(t))$ are stay bounded below uniformly in time, which contradicts the H^1 localization of u (cf. (3.37)). From (2.5) we know that for any $T > 0$, there exists $\varepsilon_2 \ll \alpha_{r_0}$ such that the emanating solution $\tilde{u}$ satisfies that, for all $t \in [-T, T]$,

$$\|u(t) - \tilde{u}(t)\|_{H^1} \leqq \frac{\varepsilon_2}{3\sqrt{\mathcal{E}(u)}} \ll \alpha_{r_0}. \tag{3.31}$$

Therefore, we have

$$\begin{aligned}\left|q_t^*(t) - \tilde{q}_{2,t}(t)\right| &= \left|u(t, q^*(t)) - \tilde{u}(t, \tilde{q}_2(t))\right| \\ &\leqq \|u^2(t) - \tilde{u}^2(t)\|_{L^\infty} + \left|(\tilde{u}^2)_x(t,\xi)\right| \left|q^*(t) - \tilde{q}_2(t)\right| \\ &\leqq \varepsilon_2 + 2\mathcal{E}(u)\left|q^*(t) - \tilde{q}_2(t)\right|,\end{aligned}$$

leading to

$$\left|q^*(t) - \tilde{q}_2(t)\right| \leqq e^{2\mathcal{E}(u)t}\left|\rho_+(0) - \tilde{\rho}_+\right| + \frac{e^{2\mathcal{E}(u)t}}{2\mathcal{E}(u)}\varepsilon_2 =: \varepsilon_3 \ll \alpha_{r_0}, \tag{3.32}$$

by choosing n large enough so that $\tilde{\rho}_+ - \rho_+(0)$ and ε_2 are sufficiently small. Hence by (3.25), $|u_x| \leqq u \leqq \sqrt{\mathcal{E}(u)}$ and the mean-value theorem, we conclude that

$$\begin{aligned}\tilde{u}\left(t, \tilde{q}_2(t)\right) &= u(t, q^*(t)) + \left[\tilde{u}\left(t, q^*(t)\right) - u\left(t, q^*(t)\right)\right] + \left[\tilde{u}\left(t, \tilde{q}_2(t)\right) - \tilde{u}\left(t, q^*(t)\right)\right] \\ &\geqq \alpha_{r_0} - \frac{\varepsilon_2}{3\sqrt{\mathcal{E}(u)}} - \left|\tilde{u}_x\left(t, z\right)\right| \left|q^*(t) - \tilde{q}_2(t)\right| \geqq \frac{3\alpha_{r_0}}{4},\end{aligned} \tag{3.33}$$

where $z \in (q^*(t), \tilde{q}_2(t))$. Also (3.30) ensures that

$$\tilde{u}_x\left(t, \tilde{q}_2(t)\right) = -\tilde{u}\left(t, \tilde{q}_2(t)\right) \leqq -\frac{3\alpha_{r_0}}{4} < 0. \tag{3.34}$$

Next we show that,

$$\text{for all } \ t \in [-T, 0], \quad \tilde{u}_x(t, x) < -\frac{\alpha_{r_0}}{2} \quad \text{on } \ [\tilde{q}_1(t), \tilde{q}_2(t)]. \tag{3.35}$$

To this end, for $\gamma > 0$ we define

$$\mathcal{A}_\gamma := \left\{t \in \mathbb{R}^- \mid \text{for all } \tau \in [t, 0],\ u_x(\tau, x) < -\gamma \text{ on } \left[\tilde{q}_1(\tau), \tilde{q}_2(\tau)\right]\right\}.$$

Thus (3.35) is equivalent to $[-T, 0] \subset \mathcal{A}_{\alpha_{r_0}/2}$. By (3.30) and (3.34), for $0 \leqq \beta \leqq \beta_0$ we have

$$\tilde{u}_x\left(0, \tilde{q}_1(0) + \beta\right) \leqq \tilde{u}_x\left(0, \tilde{q}_2(0)\right) + \int_{\tilde{q}_1(0)+\beta}^{\tilde{q}_2(0)} \tilde{y}_0(s) \mathrm{d}s \leqq -\frac{3\alpha_{r_0}}{4} + 2\varepsilon_1, \tag{3.36}$$

and hence $0 \in \mathcal{A}_{\alpha_{r_0}/2}$. By a continuity argument there exists some $t_1 < 0$ such that $[t_1, 0] \subset \mathcal{A}_{\alpha_{r_0}/2}$. Suppose that (3.35) is not true. Then there exist some $t_* \in [-T, 0]$ and $x_* \in [\tilde{q}_1(t_*), \tilde{q}_2(t_*)]$ such that $\tilde{u}_x(t, x) < -\alpha_{r_0}/2$ for $t \in (t_*, 0]$ and $x \in [\tilde{q}_1(t), \tilde{q}_2(t)]$, $\tilde{u}_x(t_*, x) \leqq -\alpha_{r_0}/2$ for $x \in [\tilde{q}_1(t_*), \tilde{q}_2(t_*)]$, but $\tilde{u}_x(t_*, x_*) = -\alpha_{r_0}/2$. This way from (3.22) we see that for $t \in [t_*, 0]$,

$$\tilde{q}_x(t, z) = \exp\left(-\int_t^0 (2\tilde{u}\tilde{u}_x)(s, \tilde{q}(s, z))\, \mathrm{d}s\right) \geqq 1, \quad \text{for all } \ z \in [\tilde{q}_1(t), \tilde{q}_2(t)].$$

Hence

$$\begin{aligned}
\tilde{u}_x(t_*, x_*) &= \tilde{u}_x(t_*, \tilde{q}_2(t_*)) - \int_{x_*}^{\tilde{q}_2(t_*)} \tilde{u}_{xx}(t_*, s)\, \mathrm{d}s \leqq -\frac{3\alpha_{r_0}}{4} + \int_{\tilde{q}_1(t_*)}^{\tilde{q}_2(t_*)} \tilde{y}(t, s)\, \mathrm{d}s \\
&= -\frac{3\alpha_{r_0}}{4} + \int_{\rho(0)+\tilde{\rho}_+-\beta_0}^{\rho(0)+\tilde{\rho}_+} \tilde{y}(t, \tilde{q}(t_*, z))\tilde{q}_x(t_*, z)\, \mathrm{d}z \\
&\leqq -\frac{3\alpha_{r_0}}{4} + \int_{\rho(0)+\tilde{\rho}_+-\beta_0}^{\rho(0)+\tilde{\rho}_+} \tilde{y}(t, \tilde{q}(t_*, z))\tilde{q}_x^{3/2}(t_*, z)\, \mathrm{d}z \\
&= -\frac{3\alpha_{r_0}}{4} + \int_{\rho(0)+\tilde{\rho}_+-\beta_0}^{\rho(0)+\tilde{\rho}_+} \tilde{y}_0(t, s)\, \mathrm{d}s \leqq -\frac{3\alpha_{r_0}}{4} + 2\varepsilon_1 < -\frac{\alpha_{r_0}}{2},
\end{aligned}$$

which is a contradiction. Therefore (3.35) holds true.

For all $t \in [-T, 0]$, it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{q}_2(t) - \tilde{q}_1(t)\right) &= \tilde{u}^2\left(\tilde{q}_2(t)\right) - \tilde{u}^2\left(\tilde{q}_1(t)\right) \\
&= \int_{\tilde{q}_1(t)}^{\tilde{q}_2(t)} 2\tilde{u}\tilde{u}_x(t, s) \mathrm{d}s \leqq -\alpha_{r_0}^2 \left(\tilde{q}_2(t) - \tilde{q}_1(t)\right).
\end{aligned}$$

Hence,

$$\left(\tilde{q}_2 - \tilde{q}_1\right)(t) \geqq \left(\tilde{q}_2 - \tilde{q}_1\right)(0)\, e^{-\alpha_{r_0}^2 t} = \beta_0 e^{-\alpha_{r_0}^2 t}.$$

According to (3.33), $\tilde{u}\,(t, \tilde{q}_2(t)) \geqq 3\alpha_{r_0}/4$ and $\tilde{u}_x < 0$ on $(\tilde{q}_1(t), \tilde{q}_2(t))$, which implies

$$\tilde{u}\,(t, \tilde{q}_1(t)) > \tilde{u}\,(t, \tilde{q}_2(t)) \geqq 3\alpha_{r_0}/4 \quad \text{on } [-T, 0]\,.$$

Consequently, by (3.31) one has, for all $t \in [-T, 0]$,

$$\min(u(t, \tilde{q}_1(t)), u(t, \tilde{q}_2(t))) \geqq \alpha_{r_0}/2 \quad \text{with} \quad (\tilde{q}_2 - \tilde{q}_1)(t) \geqq \beta_0 e^{-\alpha_{r_0}^2 t}. \tag{3.37}$$

Taking $T > 0$ large enough, it is clear that (3.37) contradicts the H^1-localization of u, which implies $a(t) > 0$ and therefore $u_x(t, x)$ has a jump at $q^*(t)$. □

Applying Lemma 3.4, the measure $y(t)$ can be decomposed as

$$y(t) = \nu(t) + a(t)\delta_{\rho(t)+\rho_+(t)} + \sum_{i=1}^{\infty} a_i(t)\delta_{x_i(t)}, \tag{3.38}$$

where $\nu(t)$ is a non-negative, non-atomic measure with $\nu(t) \equiv 0$ on $(\rho(t) + \rho_+(t), \infty)$, $a_i \geqq 0$ with $\sum_{i=1}^{\infty} a_i(t) < \infty$ and $x_i(t) < \rho(t) + \rho_+(t)$ for all $i \in \mathbb{N}$.

As suggested by [23], the next goal is to show that $a(t)$ is non-decreasing and differentiable.

Proposition 3.4. *The function $a(t)$ defined in (3.28) is non-decreasing and differentiable, and*

$$a'(t) = \frac{1}{2}u(u^2 - u_x^2)(t, q^*(t)-), \quad \textit{for all } t \in \mathbb{R}. \tag{3.39}$$

Proof. We will show that for any $t_1 < t_2$,

$$a\,(t_2) - a\,(t_1) = \frac{1}{2}\int_{t_1}^{t_2} u\left(u^2 - u_x^2\right)\left(\tau, q^*(\tau)-\right) \mathrm{d}\tau. \tag{3.40}$$

Since $|u_x| \leqq u$ and $u \in C\left(\mathbb{R}^2\right)$, $a(t)$ must be a non-decreasing continuous function. Next, $u(t, q^*(t)-) = u(t, q^*(t)+) = -u_x(t, q^*(t)+)$ leads to

$$\begin{aligned} u\left(u^2 - u_x^2\right)\left(t, q^*(t)-\right) &= a(t)u\,(u - u_x)\left(t, q^*(t)-\right) \\ &= a(t)u\big(2u(t, q^*(t)-) - a(t)\big) \\ &= a(t)u\big(2u(t, q^*(t)) - a(t)\big). \end{aligned}$$

Since $t \mapsto u(t, q^*(t)) \in C(\mathbb{R})$, $a(t)$ is differentiable on $\mathbb{R}$. In fact $a \in C^1$.

Now consider a sequence $\{\phi_\epsilon\}$ approximating the Heaviside function. That is, $\phi_\epsilon = \phi(\frac{\cdot}{\epsilon})$ where $\phi \in C^\infty$ is non-decreasing satisfying supp$\phi \subset [-1, \infty)$ and $\phi \equiv 1$ on $\mathbb{R}^+$. Then (3.38) gives

$$a(t) = \lim_{\epsilon \to 0^+} \left\langle y(t), \phi_\epsilon\left(\cdot - q^*(t)\right)\right\rangle.$$

Hence without loss of generality, for any $t \in (0, 1)$, showing (3.40) is equivalent to proving the following equality:

$$\lim_{\epsilon\to 0+} \left\langle y(t), \phi_\epsilon\left(\cdot - q^*(t)\right)\right\rangle - \left\langle y(0), \phi_\epsilon\left(\cdot - q^*(0)\right)\right\rangle \\ = \frac{1}{2}\int_0^t u\left(u^2 - u_x^2\right)\left(\tau, q^*(\tau)-\right) \mathrm{d}\tau. \tag{3.41}$$

Again we approximate u_0 by smooth functions $u_{0,n} = \zeta_n * u_0 \in Y_+ \cap H^\infty(\mathbb{R})$. Therefore by (2.6), to prove (3.41) one only needs to show that for any fixed positive number δ, there exists $\epsilon_0 > 0$ and sufficiently large n such that for all $0 < \epsilon < \epsilon_0$, it holds that

$$\left|\langle y(t) - y_n(t), \phi_\epsilon(\cdot - q^*(t))\rangle - \langle y(0) - y_n(0), \phi_\epsilon(\cdot - q^*(0))\rangle\right| \\ + \Big|\langle y_n(t), \phi_\epsilon(\cdot - q^*(t))\rangle - \langle y_n(0), \phi_\epsilon(\cdot - q^*(0)) \\ - \frac{1}{2}\int_0^t\int_{\mathbb{R}} u\left(u^2 - u_x^2\right)\left(\tau, q^*(\tau) + \epsilon z\right)\phi'(z)\mathrm{d}z\mathrm{d}\tau\rangle\Big| \leqq \delta \ \text{ for all } t \in (0, 1), \tag{3.42}$$

where $y_{0,n} = u_{0,n} - \partial_x^2 u_{0,n}$. Moreover, we denote by u_n the solution to the Novikov equation emanating from $u_{0,n}$ and $y_n = u_n - u_{n,xx}$. It directly follows from (2.6) that there exist $\epsilon_0 > 0$ and sufficiently large n_0 such that, for all $0 < \epsilon < \epsilon_0$ and $n > n_0$,

$$\left|\langle y(t) - y_n(t), \phi_\epsilon(\cdot - q^*(t))\rangle - \langle y(0) - y_n(0), \phi_\epsilon(\cdot - q^*(0))\rangle\right| \leqq \frac{\delta}{2}.$$

From the decomposition of $y(t)$ in (3.38), for any $\alpha > 0$ there exists $\gamma(\alpha) > 0$ such that

$$\|y(0)\|_{\mathcal{M}(q^*(0)-\gamma(\alpha), q^*(0))} < \alpha,$$

and for large enough $n > n_1$, $\left\|y_0 - y_{0,n}\right\|_{\mathcal{M}(\mathbb{R})} \leqq \alpha$ and $\left\|u_{0,n}\right\|_{H^1} \leqq 2\left\|u_0\right\|_{H^1}$. Then (3.22) implies

$$|u_{n,x}| \leqq 2\left\|u_0\right\|_{H^1} \quad \text{and} \quad e^{-8\|u_0\|_{H^1}^2} \leqq q_{n,x}(t, z) \leqq e^{8\|u_0\|_{H^1}^2}, \tag{3.43}$$

where $(t, z) \in (-1, 1) \times \mathbb{R}$ and q_n is defined as in (3.21) satisfying $q_n^*(t) = q_n\left(t, q^*(0)\right)$. In order to show

$$\Big|\langle y_n(t), \phi_\epsilon(\cdot - q^*(t))\rangle - \langle y_n(0), \phi_\epsilon(\cdot - q^*(0)) \\ - \frac{1}{2}\int_0^t\int_{\mathbb{R}} u\left(u^2 - u_x^2\right)\left(\tau, q^*(\tau) + \epsilon z\right)\phi'(z)\mathrm{d}z\mathrm{d}\tau\rangle\Big| \leqq \frac{\delta}{2},$$

we first use (1.2), integration by parts and changing of variables to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} y_n\phi_\epsilon\left(\cdot - q_n^*(t)\right) \\ = -u_n^2\left(t, q_n^*(t)\right)\int_{\mathbb{R}} y_n\phi_\epsilon' - \int_{\mathbb{R}}\partial_x\left(y_n u_n^2\right)\phi_\epsilon - \int_{\mathbb{R}} u_n u_{n,x} y_n\phi_\epsilon$$

$$\begin{aligned}
&= \frac{1}{\epsilon}\int_{\mathbb{R}} \left(u_n^2(t,\cdot) - u_n^2\left(t, q^*(t)\right)\right) y_n(t,\cdot)\phi'\left(\frac{\cdot - q_n^*(t)}{\epsilon}\right) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}} u_{n,x}\left(u_n^2 - u_{n,x}^2\right)(t,\cdot)\,\phi_\epsilon \\
&\quad + \frac{1}{2}\int_{\mathbb{R}} u_n\left(u_n^2 - u_{n,x}^2\right)(t,\cdot)\,\phi'\left(\frac{\cdot - q_n^*(t)}{\epsilon}\right) =: A_t^{\epsilon,n} + B_t^{\epsilon,n} + C_t^{\epsilon,n},
\end{aligned} \tag{3.44}$$

which indicates that

$$\langle y_n(t), \phi_\epsilon(\cdot - q^*(t))\rangle - \langle y_n(0), \phi_\epsilon(\cdot - q^*(0))\rangle = \int_0^t A_\tau^{\epsilon,n} + B_\tau^{\epsilon,n} + C_\tau^{\epsilon,n}\mathrm{d}\tau.$$

Thanks to (3.22), (3.43) and $\mathrm{supp}\phi' \subset [-1, 0]$, we have

$$\begin{aligned}
\left|A_t^{\epsilon,n}\right| &\leqq \frac{8\left\|u_0\right\|_{H^1}^2}{\epsilon}\int_{\mathbb{R}} \left|x - q_n^*(t)\right| y_n(t,x)\phi'\left(\frac{x - q_n^*(t)}{\epsilon}\right)\mathrm{d}x \\
&\leqq 8\left\|u_0\right\|_{H^1}^2 \int_{\mathbb{R}} y_n(t,x)\phi'\left(\frac{x - q_n^*(t)}{\epsilon}\right)\mathrm{d}x \\
&\leqq 8\left\|u_0\right\|_{H^1}^2 e^{4\|u_0\|_{H^1}} \int_{\mathbb{R}} y_n(0,z)\phi'\left(\frac{q_n(t,z) - q_n^*(t)}{\epsilon}\right)\mathrm{d}z.
\end{aligned}$$

Again using the fact that $\mathrm{supp}\phi' \subset [-1, 0]$ and the mean value theorem that

$$\epsilon > \left|q_n(t,z) - q_n^*(t)\right| = |q_{n,x}(t,\tilde{z})||z - q_n^*(0)| \geqq e^{-8\|u_0\|_{H^1}^2}|z - q_n^*(0)|,$$

we can take

$$\alpha = \frac{\delta}{32\left\|u_0\right\|_{H^1}^2 e^{4\|u_0\|_{H^1}}}, \quad \text{and} \quad \epsilon_0 = e^{-8\|u_0\|_{H^1}^2}\gamma(\alpha).$$

This way, for any $0 < \epsilon < \epsilon_0$, it follows that

$$|z - q^*(0)| < e^{8\|u_0\|_{H^1}^2}\epsilon < \gamma(\alpha),$$

and then

$$8\left\|u_0\right\|_{H^1}^2 e^{4\|u_0\|_{H^1}}\left(\|y(0)\|_{\mathcal{M}(q^*(0)-\gamma(\alpha),q^*(0))} + \left\|y_0 - y_{0,n}\right\|_{\mathcal{M}(\mathbb{R})}\right) < \frac{\delta}{2},$$

that is, for all $0 < \epsilon < \epsilon_0$, $t \in (0, 1)$ and $n > n_1$,

$$\int_0^t \left|A_\tau^{\epsilon,n}\right|\mathrm{d}\tau < \frac{\delta}{2}. \tag{3.45}$$

Next we estimate $\int_0^t B_\tau^{\epsilon,n}\mathrm{d}\tau$ and $\int_0^t C_\tau^{\epsilon,n}\mathrm{d}\tau$. Recalling (2.5) and Helly's theorem, one may deduce that for each $t \in [-1, 1]$, there exists a set $\Omega_t \subset \mathbb{R}$ with measure zero such that $u_x(t)$ is continuous on $\mathbb{R}\backslash\Omega_t$ and

$$u_{n,x}(t,x) \to u_x(t,x) \quad \text{for all } x \in \mathbb{R}\backslash\Omega_t,$$

which naturally leads to

$$u_{n,x}\left(t, q_n^*(t)+x\right) \to u_x\left(t, q^*(t)+x\right), \quad \text{for all } x \in \mathbb{R}\backslash\tau_{q^*(t)}\left(\Omega_t\right),$$

where $\tau_a(\Omega) := \{x-a, a \in \Omega\}$ with $\Omega \subset \mathbb{R}$, since $q_n^*(t) \to q^*(t)$. Note that $\|u_x\|_{L^\infty} \leqq \|u\|_{L^\infty} \leqq \|u_0\|_{H^1}$, then the Lebesgue dominated convergence theorem ensures $\forall\, t \in (0,1)$, as $n \to \infty$,

$$\int_0^t B_\tau^{\epsilon,n}\mathrm{d}\tau \to \frac{1}{2}\int_0^t\int_{\mathbb{R}} u_x\left(u^2-u_x^2\right)(\tau,\cdot)\,\phi_\epsilon(\cdot - q^*(\tau))\mathrm{d}\tau, \tag{3.46}$$

$$\int_0^t C_\tau^{\epsilon,n}\mathrm{d}\tau \to \frac{1}{2}\int_0^t\int_{\mathbb{R}} u\left(u^2-u_x^2\right)\left(\tau, q^*(\tau)+\epsilon z\right)\phi'(z)\mathrm{d}z\mathrm{d}\tau. \tag{3.47}$$

In addition, by direct computation, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}} u_x\left(u^2-u_x^2\right)(\tau,\cdot)\,\phi_\epsilon(\cdot - q^*(\tau))\mathrm{d}\tau \\
&= \int_0^t\int_{\mathbb{R}} u_x\left(u^2-u_x^2\right)(\tau,x)\,\phi\left(\frac{x-q^*(\tau)}{\epsilon}\right)\mathrm{d}x\mathrm{d}\tau \\
&= \epsilon\int_0^t\int_{\mathbb{R}} u_x\left(u^2-u_x^2\right)(\tau, q^*(\tau)+\epsilon z)\phi(z)\mathrm{d}z\mathrm{d}\tau \to 0 \text{ as } \epsilon \to 0^+. \quad (3.48)
\end{aligned}$$

Combining (3.45)–(3.48), the desired result (3.42) holds for large enough $n > \max\{n_0, n_1\}$ and all $t \in (0,1)$ and $0 < \epsilon < \epsilon_0$.

Now from $u_x \in BV(\mathbb{R})$ and $\phi' \equiv 0$ on $\mathbb{R}^+$, it follows that $u\left(u^2-u_x^2\right)(\tau,\cdot) \in BV(\mathbb{R})$ and for each fixed (τ, z), as $\epsilon \to 0^+$,

$$u\left(u^2-u_x^2\right)\left(\tau, q^*(\tau)+\epsilon z\right)\phi'(z) \to u\left(u^2-u_x^2\right)\left(\tau, q^*(\tau)-\right)\phi'(z).$$

Again the Lebesgue dominated convergence theorem shows that, as $\epsilon \to 0^+$,

$$\int_0^t\int_{\mathbb{R}} u\left(u^2-u_x^2\right)\left(\tau, q^*(\tau)+\epsilon z\right)\phi'(z)\mathrm{d}z\mathrm{d}\tau \to \int_0^t u\left(u^2-u_x^2\right)\left(\tau, q^*(\tau)-\right)\mathrm{d}\tau.$$

Finally this together with (3.42) shows that (3.41), which completes the proof. □

Next we state the asymptotic property of $a(t)$. The proof follows along the same lines as to [23, Lemma 4.5], and hence we omit it.

Lemma 3.5. *There exist two constants a_- and $a_+ \in (0, 2\mathcal{E}(u)]$ with $a_- \leqq a_+$ satisfying*

$$\lim_{t\to\infty} u\left(t, \rho(t)+\rho_+(t)\right) = \lim_{t\to\infty} a(t)/2 = a_+/2,$$

$$\lim_{t\to-\infty} u\left(t, \rho(t)+\rho_+(t)\right) = \lim_{t\to-\infty} a(t)/2 = a_-/2.$$

For later use, we compute $a'(t)$ in the following way:

$$a'(t) = \frac{1}{2}u\left(u^2-u_x^2\right)\left(t, \rho(t)+\rho_+(t)-\right) = \frac{1}{2}a(t)u\big(2u(t,\rho(t)+\rho_+(t)) - a(t)\big). \tag{3.49}$$

3.4. Localization of the Space-Time Reflected Solutions

A very useful observation is that the solution of equation (1.1) under the transformation $(t, x) \mapsto (-t, -x)$ is also a solution. Thus when $u(t, x)$ is a solution to the Novikov equation, so is the reflected one $\hat{u}(t, x) := u(-t, -x)$.

It is obvious to check that

$$\begin{aligned}\left\langle (u^2 + u_x^2)(-t, x),\ \Phi_1(x - \rho(-t))\right\rangle &= \left\langle (u^2 + u_x^2)(-t, -x),\ \Phi_1(-x - \rho(-t))\right\rangle \\ &= \left\langle (\hat{u}^2 + \hat{u}_x^2)(t, x),\ \Phi_2(x - \hat{\rho}(t))\right\rangle,\end{aligned}$$

where $\Phi_2(\cdot) := \Phi_1(-\cdot)$ and the modulation $\hat{\rho}(t) := -\rho(-t)$. Hence $\hat{\rho}_t = \rho_t \geqq c_0$. This means that the H^1-localization of u is equivalent to the H^1-localization of $\hat{u}$.

Similarly,

$$\begin{aligned}&\langle y(-t, x),\ \Phi_r(x - \rho(-t))\rangle = \langle y(-t, -x),\ \Phi_r(-x - \rho(-t))\rangle \\ &\quad = \left\langle \hat{y}(t, x),\ \Phi_l(x - \hat{\rho}(t))\right\rangle,\end{aligned}$$

where $\hat{y}(t, x) := y(-t, -x)$ and $\Phi_l(\cdot) := \Phi_r(-\cdot)$. Therefore if the solution u is $\mathbf{Y}$-localized to the right then $\hat{u}$ becomes $\mathbf{Y}$-localized to the left.

Repeating the arguments in Section 3.2 and 3.3 for the solutions that are $\mathbf{Y}$-localized to the left allows us to define $\rho_-(t) := \sup\left\{x \in \mathbb{R} :\ \operatorname{supp}\hat{y}(t, \cdot) \subset [x + \hat{\rho}(t), \infty)\right\}$. Then

$$\hat{u}(t, \hat{\rho}(t) + \rho_-(t)) = -\hat{u}_x(t, \hat{\rho}(t) + \rho_-(t)) \geqq \alpha_{r_0} > 0.$$

We can further show that the jump of $\hat{u}_x$ across $\hat{\rho}(t) + \rho_-(t)$,

$$\hat{a}(t) := \hat{u}_x(t, \hat{\rho}(t) + \rho_-(t)+) - \hat{u}_x(t, \hat{\rho}(t) + \rho_-(t)-),$$

is non-decreasing and differentiable, and satisfies

$$\begin{aligned}&\hat{a}(t) > 0, \qquad \hat{a}'(t) = \frac{1}{2}\hat{u}(\hat{u}^2 - \hat{u}_x^2)(t, \hat{\rho}(t) + \rho_-(t)+), \\ &\lim_{t\to\pm\infty} \hat{u}(t, \hat{\rho}(t) + \rho_-(t)) = \lim_{t\to\pm\infty} \hat{a}(t)/2 = \hat{a}_\pm/2.\end{aligned} \tag{3.50}$$

3.5. Proof of Theorem 2.1

From (3.50) and the relation between u and $\hat{u}$ it follows that

$$\begin{aligned}\lim_{t\to\infty} u(t, \rho(t) - \rho_-(-t)) &= \lim_{t\to\infty} \hat{u}(-t, -\rho(t) + \rho_-(-t)) \\ &= \lim_{s\to-\infty} \hat{u}(s, -\rho(-s) + \rho_-(s)) = \hat{a}_-/2.\end{aligned} \tag{3.51}$$

Note that

$$\begin{aligned}\rho_-(-t) &= \sup\left\{x \in \mathbb{R} :\ \operatorname{supp}\hat{y}(-t, \cdot) \subset [x + \hat{\rho}(-t), \infty)\right\} \\ &= \sup\left\{x \in \mathbb{R} :\ \operatorname{supp}y(t, -\cdot) \subset [x - \rho(t), \infty)\right\} \\ &= \sup\left\{x \in \mathbb{R} :\ \operatorname{supp}y(t, \cdot) \subset (-\infty, -x + \rho(t)]\right\}\end{aligned}$$

$$= -\inf\{x \in \mathbb{R} : \operatorname{supp} y(t,\cdot) \subset (-\infty, x+\rho(t)\} = -\rho_+(t).$$

Therefore (3.51) implies that

$$\hat{a}_-/2 = \lim_{t\to\infty} u(t, \rho(t) - \rho_-(-t)) = \lim_{t\to\infty} u(t, \rho(t) + \rho_+(t)) = a_+/2.$$

Then, by the monotonicity property: $a_- \leqq a_+$ and $\hat{a}_- \leqq \hat{a}_+$, one has

$$a_- = a(t) = a_+ = \hat{a}_- = \hat{a}(t) = \hat{a}_+.$$

In particular we have $a'(t) \equiv 0$. Therefore from (3.49) we know that $u(t, \rho(t) + \rho_+(t)) = \frac{a_+}{2}$. From (3.26) and (3.9) we further obtain that, for all $t \in \mathbb{R}$,

$$\rho(t) + \rho_+(t) = \rho(0) + \rho_+(0) + \frac{a_+^2}{4}t, \qquad u\left(t, \rho(0) + \rho_+(0) + \frac{a_+^2}{4}t\right) = \frac{a_+}{2}.$$

Then applying the decomposition (3.38) yields that

$$y(t) = \nu(t) + \sum_{i=1}^{\infty} a_i(t)\delta_{x_i(t)} + a_+\delta_{\rho(t)+\rho_+(t)}.$$

Now applying $\left(1-\partial_x^2\right)^{-1}$ to $y(0)$ we get that

$$\frac{a_+}{2} = u(t, \rho(t) + \rho_+(t)) = (1-\partial_x^2)^{-1} y(t, \rho(t) + \rho_+(t)) \geqq \frac{a_+}{2},$$

where the last equality holds if and only if $\nu(t) \equiv 0$ and $a_i(t) = 0$ for all $i \in \mathbb{N}$. Therefore $y(t) = a_+\delta_{\rho(t)+\rho_+(t)}$, which gives

$$u(t,x) = \frac{a_+}{2}\exp\left(-\left|x - \rho(0) - \rho_+(0) - \left(\frac{a_+}{2}\right)^2 t\right|\right).$$

This completes the proof of Theorem 2.1.

4. Asymptotic Stability of the Novikov Peakons

In this section, we apply the result of Section 3 to prove the asymptotic stability of the Novikov peakons.

4.1. Orbital Stability

Let us recall some of the relevant orbital stability results. The first H^1-orbital stability of a single Novikov peakon is established in [18] for smooth initial perturbations $u_0 \in Y_+ \cap H^s$ with $s \geqq 3$. Such a result was later refined for smooth data $u_0 \in H^s$, $s > 5/2$ without constraint on the sign of y_0 in [7], and further improved for rough data $u_0 \in H^1 \cap W^{1,\infty}$ in [9]. Note that by removing the sign condition on y_0, the corresponding solution may not exist for all time. First, a straightforward consequence of [9, Theorem 3.9] gives the following H^1-orbital stability in Y_+ space:

Proposition 4.1. *Let $u_0 \in Y_+$ such that*

$$\left\| u_0 - \sqrt{c}\varphi \right\|_{H^1} < \frac{1}{4}\sqrt{c}\varepsilon^4, \quad 0 < \varepsilon^4 < \frac{1}{2}, \tag{4.1}$$

then the corresponding global solution $u \in C(\mathbb{R}; Y_+)$ of the Cauchy problem (2.4) satisfies

$$\sup_{t\in\mathbb{R}} \left\| u(\cdot, t) - \sqrt{c}\varphi\left(\cdot - \xi(t)\right) \right\|_{H^1} < M\varepsilon, \tag{4.2}$$

where M depends on c and $\|u_0\|_{H^1}$, and $\xi(t)$ is the point where u takes its maximum.

Following the strategy of [23], the next result indicates that as long as u remains in some neighborhood of a peakon, we can decompose u as the sum of a modulated peakon plus a small H^1 error satisfying certain orthogonality condition (cf. (4.5)).

Lemma 4.1. (Modulation) *Assume that $u \in C(\mathbb{R}; H^1)$ with $y \in C_w(\mathbb{R}; Y_+)$ is the solution to (2.4) and that there exists a small enough $0 < \varepsilon_0 < 1$ such that*

$$\sup_{t\in\mathbb{R}} \|u(t) - \sqrt{c}\varphi(\cdot - z(t))\|_{H^1} < \sqrt{c}\varepsilon_0, \tag{4.3}$$

with some function $z(t) : \mathbb{R} \to \mathbb{R}$, then there exist $\kappa_0 > 0$, $n_0 \in \mathbb{N}$ and a unique function $\rho(t) \in C^1(\mathbb{R})$ such that

$$\sup_{t\in\mathbb{R}} |\rho(t) - z(t)| < \kappa_0 \quad \textit{and} \quad \sup_{t\in\mathbb{R}} |\rho_t(t) - c| \leqq \frac{c}{8}, \tag{4.4}$$

$$\int_{\mathbb{R}} u(t)\left(\zeta_{n_0} * \varphi'\right)(\cdot - \rho(t)) = 0, \quad \textit{for all } \ t \in \mathbb{R}, \tag{4.5}$$

where ζ_n is defined in (2.3) and n_0 is large enough to ensure that for all $y \in [-1/2, 1/2]$,

$$\int_{\mathbb{R}} \varphi(\cdot - y)\left(\zeta_{n_0} * \varphi'\right) = 0 \Leftrightarrow y = 0. \tag{4.6}$$

If, additionally,

$$\sup_{t\in\mathbb{R}} \|u(t) - \sqrt{c}\varphi(\cdot - z(t))\|_{H^1} < M\varepsilon, \ \textit{where } \varepsilon \in (0, \sqrt{c}\varepsilon_0), \tag{4.7}$$

there exists $N > 1$ such that

$$\sup_{t\in\mathbb{R}} \|u(t) - \sqrt{c}\varphi(\cdot - \rho(t))\|_{H^1} < N\sqrt{\varepsilon}. \tag{4.8}$$

Proof. We postpone the proof to Appendix A.2. □

For $0 < \theta < \sqrt{c}$ satisfying $\theta^4 < \frac{1}{2}$, we set

$$\sqrt{\varepsilon} = \frac{1}{\sqrt{LN}} \min\left\{\frac{\theta}{2}, \frac{\sqrt{c}\varepsilon_0}{M}\right\}. \tag{4.9}$$

One may easily check that ε satisfies (4.1), (4.3) and (4.7). Notice that the function $z(t)$ in (4.7) and (4.3) coincides with $\xi(t)$ in (4.2). Furthermore it follows from (4.4) that $\inf_{t\in\mathbb{R}} \rho_t \geqq \frac{7c}{8} > \frac{c}{2}$. Recalling the assumptions in Lemma 3.1, for ε in (4.9), we set $c_0 = \frac{c}{2}$ and $\alpha = \frac{\theta^2}{c}$ such that for a suitable $R_0 > 0$, it holds that

$$\|u(t)\|^2_{L^\infty(|x-\rho(t)|>R_0)} \leqq \|u(t) - \sqrt{c}\varphi(\cdot - \rho(t))\|^2_{H^1} \leqq N^2\varepsilon = \frac{\theta^2}{4L} = \frac{\alpha c_0}{2L}.$$

Remark 4.1. When the three parameters ε, α and c_0 are suitably chosen above, we set

$$\eta_0 := \frac{1}{4\sqrt{LN}} \min\left\{\frac{1}{2}, \frac{\varepsilon_0}{M}\right\}. \tag{4.10}$$

Then the assumption in Theorem 2.2 shows that all conclusions in Lemma 3.1, Proposition 4.1 and Lemma 4.1 hold true.

4.2. The ω-Limit Set

As we pointed out in the Introduction, one of the key ingredients in the approach of [23] for the CH peakons is to apply the almost monotonicity properties together with the conservation laws $\mathbf{Y}$ and $\mathcal{E}$ to show that as the initial data u_0 is close enough to a peakon, the ω-limit set in the weak H^1-topology of the orbit of u_0 consists of initial data of the *fully* $\mathbf{Y}$-localized solution. However for the Novikov equation, $\mathbf{Y}$ is not conserved. In fact it can grow exponentially in time, which presents a serious obstacle to proving even the H^1-localization. Nevertheless we can still characterize the ω-limit set using the new notion of $\mathbf{Y}$-localization as in Definition 2.1. What we do is to first use the almost monotonicity for the H^1 energy density to prove the localization of the H^1 energy, and then take advantage of the fact that the Novikov solutions lying in $W^{1,\infty}$ with localized H^1 energy enjoy (uniform in time) exponential decay, as in Proposition 3.1. This in turn implies that the corresponding momentum density has a finite total mass. Therefore the limit solution is indeed H^1-localized. Further applying the exponential decay of the limit allows one to ensure the semi-$\mathbf{Y}$-localization as well. This new route seems to require less on $\mathbf{Y}$ and hence it is conceivable that it can be applied to treat other quasilinear models admitting peakons.

Proposition 4.2. *(ω-limit set) Let $u_0 \in Y_+$ satisfy* (4.1) *with ε given in* (4.9). *Let u be the solution to* (2.4) *as in Proposition* 2.3 *with initial data u_0. Then*

(i) *for each increasing sequence* $\{t_n\}$ *going to infinity, there exists a subsequence* $\{t_{n_k}\} \subset \{t_n\}$ *and* $\tilde{u}_0 \in Y_+$ *such that, as* $k \to \infty$*,*

$$u\left(t_{n_k}, \cdot + \rho\left(t_{n_k}\right)\right) \rightharpoonup \tilde{u}_0 \text{ in } H^1(\mathbb{R}), \tag{4.11}$$

where the function $\rho(t)$ *is in* $C^1(\mathbb{R})$ *and satisfies* (4.4), (4.5) *and* (4.8). *Moreover the solution* $\tilde{u}$ *of (2.4) emanating from* $\tilde{u}_0$ *is* H^1*-localized;*

(ii) $\tilde{u}$ *is* **Y***-localized to the right.*

Proof. (i) Notice that η_0 in (4.10) ensures that assumptions (4.3) and (4.7) hold. The properties of $\rho(t)$ follow from Lemma 4.1.

For a sequence $\{t_n\}$ increasing to ∞ and $t \in [-T, T]$ with $T > 0$, the sequence $\{\rho(t_n + t) - \rho(t_n)\}$ is bounded and equicontinuous from the properties of $\rho(\cdot)$ in Lemma 4.1. Hence by using Arzela–Ascoli theorem, we derive that there is a subsequence $\{t_{n_k}\}$ such that, as $k \to \infty$,

$$\rho(t_{n_k} + t) - \rho(t_{n_k}) \to \tilde{\rho}(t) \text{ in } C([-T, T]) \tag{4.12}$$

for some $\tilde{\rho}(t) \in C(\mathbb{R})$. Now by utilizing (4.12) and the fact that $\{u(t_{n_k} + t, \cdot + \rho(t_{n_k} + t))\}$ is uniformly bounded in $H^1(\mathbb{R})$ we have up to a subsequence that, as $k \to \infty$,

$$u(t_{n_k} + t, \cdot + \rho(t_{n_k} + t)) \rightharpoonup \tilde{u}(t, \cdot + \tilde{\rho}(t)) \text{ in } H^1(\mathbb{R}), \tag{4.13}$$

for some $\tilde{u} \in L^\infty(\mathbb{R}; H^1(\mathbb{R}))$. Again because $\{u(t_{n_k} + t, \cdot + \rho(t_{n_k} + t))\}$ is bounded in $H^1(\mathbb{R})$, we know that it is also bounded in $L^\infty(\mathbb{R})$ and that for any $t \in \mathbb{R}$, it converges, up to a further subsequence (still denoted by n_k), pointwise almost everywhere to $\tilde{u}(t, \cdot + \tilde{\rho}(t))$. Recall from (2.2) that $|u_x| \leqq u$, we also have for any $t \in \mathbb{R}$, as $k \to \infty$,

$$u_x(t_{n_k} + t, \cdot + \rho(t_{n_k} + t)) \to \tilde{u}_x(t, \cdot + \tilde{\rho}(t)) \quad \text{almost everywhere } \mathbb{R}.$$

Therefore $|\tilde{u}_x(t, \cdot + \tilde{\rho}(t))| \leqq \tilde{u}(t, \cdot + \tilde{\rho}(t))$, and by passing to the limit of the Novikov equation (2.4) we see that $\tilde{u}(t, \cdot + \tilde{\rho}(t))$ is a weak solution with initial data $\tilde{u}_0 := \tilde{u}(0, \cdot + \tilde{\rho}(0))$. The conservation of $\mathcal{E}(\tilde{u})$ allows us to improve the regularity to $\tilde{u} \in C(\mathbb{R}; H^1(\mathbb{R}))$. Moreover, from the convergence (4.13) we know that $(\tilde{u}, \tilde{\rho})$ satisfy (4.6) and (4.8), and hence (4.3). Therefore from Lemma 4.1, uniqueness implies that $\tilde{\rho} \in C^1(\mathbb{R})$.

Following the idea as in [23, Proposition 5.2], for $v \in H^1$ we introduce the two functionals

$$J_r^R(v) := \left\langle v^2 + v_x^2, \Psi_K(\cdot - R)\right\rangle, \qquad J_l^R(v) := \left\langle v^2 + v_x^2, (1 - \Psi_K(\cdot + R))\right\rangle.$$

It is easily deduced from (3.7) and (3.8) that

$$J_r^R\left(u\left(t_0, \cdot + \rho\left(t_0\right)\right)\right) \leqq J_r^R(u(t, \cdot + \rho(t))) + Ce^{-R/K}, \quad \forall\, t \leqq t_0, \tag{4.14}$$

$$J_l^R(u(t, \cdot + \rho(t))) \geqq J_l^R\left(u\left(t_0, \cdot + \rho\left(t_0\right)\right)\right) - Ce^{-R/K}, \quad \forall\, t \geqq t_0. \tag{4.15}$$

Denote

$$G_o^R(v) := J_r^R(v) + J_l^R(v), \qquad G_i^R(v) := \mathcal{E}(v) - G_o^R(v).$$

To prove that the H^1 energy of $\tilde{u}$ is localized, that is, (2.7) holds for $(\tilde{u}, \tilde{\rho})$, it suffices to prove that for all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $\forall\, t \in \mathbb{R}$,

$$G_o^{R_\varepsilon}\big(\tilde{u}(t, \cdot + \tilde{\rho}(t))\big) < \varepsilon.$$

If not, then there exists some $\varepsilon_0 > 0$ such that for any $R > 0$ there exists $t_R > 0$ such that

$$G_o^R\big(\tilde{u}(t_R, \cdot + \tilde{\rho}(t_R))\big) \geqq \varepsilon_0.$$

Then (4.13) implies

$$\liminf_{k\to\infty} G_o^R\big(u(t_{n_k} + t_R, \cdot + \rho(t_{n_k} + t_R))\big) \geqq G_o^R\big(\tilde{u}(t_R, \cdot + \tilde{\rho}(t_R))\big) \geqq \varepsilon_0,$$

and thus for k sufficiently large

$$G_o^R\big(u(t_{n_k} + t_R, \cdot + \rho(t_{n_k} + t_R))\big) \geqq G_o^R\big(\tilde{u}(t_R, \cdot + \tilde{\rho}(t_R))\big) \geqq \varepsilon_0.$$

Applying (4.8) and shrinking ε if needed, we can choose R large enough such that

$$G_o^R\big(u(t_{n_k}, \cdot + \rho(t_{n_k}))\big) \leqq \frac{\varepsilon_0}{10}.$$

Then the conservation of $G_o^R(u) + G_i^R(u)$ implies that

$$G_i^R\big(u(t_{n_k} + t_R, \cdot + \rho(t_{n_k} + t_R))\big) \leqq G_i^R\big(u(t_{n_k}, \cdot + \rho(t_{n_k}))\big) - \frac{9}{10}\varepsilon.$$

Then the rest of the argument follows the same way as in [23, Proposition 5.2], and hence we conclude that the H^1 energy of $\tilde{u}$ is localized.

When the H^1 energy localization is established, we may revisit the proof of Lemma 3.1 and from Remark 3.2 it follows that under the condition that $|\tilde{u}_x| \leqq \tilde{u}$ one can still obtain the almost monotonicity result applied to $\tilde{u}$ with $\inf_{\mathbb{R}} \tilde{\rho}_t(t) \geqq \tilde{c}_0 > 0$. Therefore $\tilde{u}$ enjoys the exponential decay property (3.2) as well, with ρ replaced by $\tilde{\rho}$.

To finish the proof of part (i), we are left to show that $\tilde{y} := \tilde{u} - \tilde{u}_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$. We will first prove that, as $k \to \infty$,

$$\big\langle y(t_{n_k} + t, \cdot + \rho(t_{n_k} + t)), \phi\big\rangle \to \langle \tilde{y}(t, \cdot + \tilde{\rho}(t)), \phi\rangle \text{ for every } \phi \in C_c(\mathbb{R}). \quad (4.16)$$

As usual, we will work with the mollified solutions $u_n = \zeta_n * u$ and $y_n = \zeta_n * y$. Since $y_n \geqq 0$, for any compact set $\Omega \subset \mathbb{R}$, we have, for n large enough,

$$\begin{aligned}\|y_n(t_{n_k} + t, \cdot + \rho(t_{n_k} + t))\|_{L^1(\Omega)} &\leqq \|u_n\|_{L^\infty}|\Omega| + 2\|u_{n,x}\|_{L^\infty}\\ &\leqq 2\,(|\Omega| + 2)\,\|u_0\|_{H^1}.\end{aligned}$$

The above uniform bound implies (4.16), and hence we know that $\tilde{y}$ is a non-negative Radon measure.

Mollifying $\tilde{u}$ and $\tilde{y}$ by $\tilde{u}_n = \zeta_n * \tilde{u}$ and $\tilde{y}_n = \zeta_n * \tilde{y}$. The sign property of $\tilde{y}$ leads to the non-negativity of $\tilde{y}_n$. Then the weak lower semi-continuity implies that

$$\|\tilde{y}\|_{\mathcal{M}(\mathbb{R})} \leqq \liminf_{n\to\infty} \|\tilde{y}_n\|_{L^1(\mathbb{R})} = \liminf_{n\to\infty} \|\tilde{u}_n\|_{L^1(\mathbb{R})} \leqq \|\tilde{u}\|_{L^1(\mathbb{R})},$$

provided that $\tilde{u} \in L^1(\mathbb{R})$. However this is true because of the exponential decay of $\tilde{u}$. As a result $\tilde{y} \in \mathcal{M}_+$, and hence we complete the proof of part (i).
(ii) Next we turn to the **Y**-localization. Arguing by contradiction, we assume that $\tilde{u}$ is not **Y**-localized to the right, that is, there exists an ε_0 such that for all $R > 0$ one can find a t_R such that

$$\langle \tilde{y}(t_R, \cdot), \Phi_r(\cdot - \tilde{\rho}(t_R)\rangle \geqq \varepsilon_0, \tag{4.17}$$

for some $\Phi_r \in C(\mathbb{R})$ with $0 \leqq \Phi_r \leqq 1$ and supp$\Phi_r \subset (R, \infty)$. Now approximating y by $y_n = \zeta_n * y$ and using (2.6) and (4.16), it follows from (4.17) that there exist large enough n_0 and k_0 such that, for all $n \geqq n_0$ and $k \geqq k_0$, we have

$$\int_{\rho(t_{n_k}+t_R)+R}^{\infty} y_n(t_{n_k} + t_R, x)\, \mathrm{d}x \geqq \frac{9}{10}\varepsilon_0. \tag{4.18}$$

The H^1-localization of $\tilde{u}$ and (4.13) ensure that one may apply Lemma 3.2 to u_n so that there exists some R_0 such that for sufficiently large n and k (still denoted by $n \geqq n_0$ and $k \geqq k_0$),

$$\begin{aligned}
&T_k := t_{n_k} + t_{R_0} > 0,\\
&u_n^2(T_k, \rho(T_k) + R_0 + x) \leqq \frac{c_0}{10}, \quad \text{for} \quad x \geqq 0,\\
&q_n\,(t - T_k, \rho(T_k) + R_0) - (\rho(t) + R_0) \geqq \frac{9}{10} c_0(T_k - t), \quad \text{for} \quad t \leqq T_k,\\
&\frac{1}{C_0} \leqq q_{n,x}\,(t - T_k, \rho(T_k) + R_0 + x) \leqq C_0, \quad \text{for all} \quad t \in [0, T_k],\ x \geqq 0,
\end{aligned} \tag{4.19}$$

where C_0 is independent of n and k.

Now take $R = R_0$ in (4.17) and (4.18). For each $k \geqq k_0$ and $n \geqq n_0$, there exists an $R_k > R_0$ such that

$$\int_{\rho(T_k)+R_0}^{\rho(T_k)+R_k} y_n(T_k, x)\, \mathrm{d}x \geqq \frac{4}{5}\varepsilon_0. \tag{4.20}$$

From (4.18) and $(4.19)_4$ it follows that

$$\begin{aligned}
\frac{4}{5}\varepsilon_0 &\leqq \int_{\rho(T_k)+R_0}^{\rho(T_k)+R_k} y_n(T_k, x)\, \mathrm{d}x = \int_{q_n(-T_k,\rho(T_k)+R_0)}^{q_n(-T_k,\rho(T_k)+R_k)} y_n(T_k, q_n(T_k, z)) q_{n,x}(T_k, z)\, \mathrm{d}z\\
&\leqq C_0^{-1/2} \int_{q_n(-T_k,\rho(T_k)+R_0)}^{q_n(-T_k,\rho(T_k)+R_k)} y(0, z)\, \mathrm{d}z,
\end{aligned}$$

and then, for all $k \geqq k_0$,

$$\int_{X_k}^{Z_k} y_n(0, z)\, \mathrm{d}z \geqq \frac{4}{5}\sqrt{C_0}\, \varepsilon_0, \quad \text{where} \quad \begin{cases} X_k := q_n(-T_k, \rho(T_k) + R_0),\\ Z_k := q_n(-T_k, \rho(T_k) + R_k). \end{cases} \tag{4.21}$$

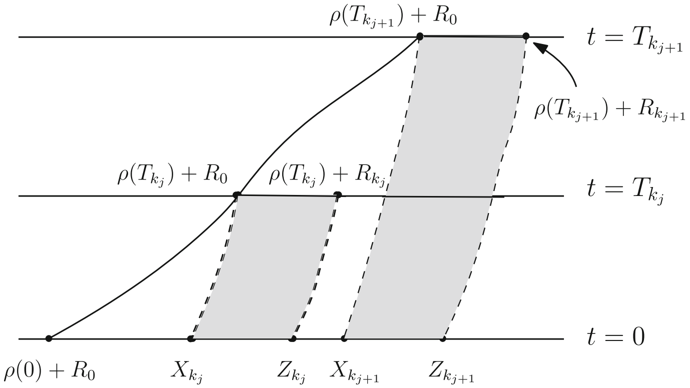

Fig. 2. Infinite accumulation of mass: the dashed curves are characteristics

Now we can apply (4.21) to a sequence of $\{k_j\}$ with the property that $X_{k_{j+1}} > Z_{k_j}$ (this is doable because $T_k \to \infty$, $\rho_t \geqq c_0$, and $(4.19)_2$). See Fig. 2 for an illustration. This way we have

$$\int_{\rho(0)+R_0}^{\infty} y_n(0, z)\,\mathrm{d}z \geqq \sum_{j=1}^{\infty} \int_{X_{k_j}}^{Z_{k_j}} y_n(0, z)\,\mathrm{d}z \to \infty,$$

which in turns leads to

$$\langle y(0, \cdot),\ \phi(\cdot - (\rho(0) + R_0))\rangle = \infty,$$

contradicting the fact that $y_0 \in \mathcal{M}_+(\mathbb{R})$. Therefore we complete the proof of the proposition. □

4.3. Proof of Asymptotic Stability

With the above results, now we turn to the proof of Theorem 2.2. The general strategy follows similarly from [23]. For the sake of completeness we will provide the necessary details, with an emphasis on some differences when establishing the convergence of ρ_t.

Proof of Theorem 2.2. Given that $\tilde{u}$ is both H^1-localized and **Y**-localized to the right in Proposition 4.2, by Theorem 2.1 we can see that there exist $x_0 \in \mathbb{R}$ and $c^* > 0$ such that

$$\tilde{u} = \sqrt{c^*}\varphi\left(\cdot - x_0 - c^* t\right). \tag{4.22}$$

Hence by Lemma 4.1 and Proposition 4.2, it follows that

$$|\sqrt{c} - \sqrt{c^*}| \leqq N\varepsilon \leqq \frac{\theta}{2\sqrt{L}} \quad \text{and} \quad \tilde{u}_0 = \sqrt{c^*}\varphi\left(\cdot - x_0\right). \tag{4.23}$$

Further applying orthogonality (4.5) and (4.6) to $\tilde{u}_0$ we conclude that $x_0 = 0$. Let $\lambda(t) = \max_{\mathbb{R}} u(t) \geqq 0$. Thus (4.13) and Rellich's Theorem gives that $u(t_{n_k}, \cdot + \rho(t_{n_k})) - \sqrt{c^*}\varphi \to 0$, which implies that

$$\lim_{k\to\infty} \lambda(t_{n_k}) = \sqrt{c^*}. \tag{4.24}$$

Notice that for every subsequence $\{t_{n_k}\}$, $\sqrt{c^*}$ is the only possible limit for $\lambda(t_{n_k})$, then it follows from (4.11) that, as $t \to \infty$,

$$u(t, \cdot + \rho(t)) - \lambda(t)\varphi \rightharpoonup 0 \ \text{ in } \ H^1(\mathbb{R}). \tag{4.25}$$

For any compact set $\Omega \subset \mathbb{R}$, we have as $k \to \infty$, $u(t_{n_k}, \cdot + \rho(t_{n_k})) \to \tilde{u}(0)$ in $L^2(\Omega)$, and $u_x(t_{n_k}, \cdot + \rho(t_{n_k})) \rightharpoonup \tilde{u}_x(0)$ in $L^2(\Omega)$. Also recall that $|u_x| \leqq u$. Therefore from (4.22) we have

$$\begin{aligned}\liminf_{k\to\infty} \|u(t_{n_k}, \cdot + \rho(t_{n_k}))\|_{H^1(\Omega)} &\leqq 2 \liminf_{k\to\infty} \|u(t_{n_k}, \cdot + \rho(t_{n_k}))\|_{L^2(\Omega)} \\ &= 2\|\tilde{u}(0)\|_{L^2(\Omega)} = \|\tilde{u}(0)\|_{H^1(\Omega)}.\end{aligned}$$

Thus $u(t_{n_k}, \cdot + \rho(t_{n_k})) \to \tilde{u}(0)$ in $H^1(\Omega)$, and hence

$$u(t, \cdot + \rho(t)) - \lambda(t)\varphi \to 0 \ \text{ in } \ H^1_{loc}(\mathbb{R}). \tag{4.26}$$

Moveover, setting $t = 0$ in (4.14), for a fixed $\delta > 0$ choosing a large enough R such that $Ce^{-R/K} < \delta$ and $J_r^R(u(0, \cdot + \rho(0))) < \delta$, we may deduce that $J_r^R(u(t, \cdot + \rho(t))) < \delta$ for all $t \geqq 0$. Then it follows from (4.26) that, as $t \to \infty$,

$$u(t, \cdot + \rho(t)) - \lambda(t)\varphi \to 0 \quad \text{in } H^1(-A, \infty) \text{ for each } A > 0. \tag{4.27}$$

Now for any fixed $\epsilon > 0$ and each pair $(t_1, t_2) \in \mathbb{R}^2$ satisfying $t_1 \geqq t_2$, we may pick a sufficiently large R so that (4.15) holds, and deduce

$$\left\langle (u^2 + u_x^2)(t_1, \cdot), \Psi_K(\cdot - \rho(t_1) + R)\right\rangle \leqq \left\langle (u^2 + u_x^2)(t_2, \cdot), \Psi_K(\cdot - \rho(t_2) + R)\right\rangle + \epsilon. \tag{4.28}$$

In addition, by (4.27), there exists a $T > 0$ such that for all $t_i > T$,

$$\left|\left\langle (u^2 + u_x^2)(t_i, \cdot), \Psi_K(\cdot - \rho(t_i) + R)\right\rangle - \lambda^2(t_i)\mathcal{E}(\varphi)\right| \leqq \epsilon, \quad i = 1, 2. \tag{4.29}$$

Consequently, combining (4.28) and (4.29), we have

$$\lambda^2(t_1)\mathcal{E}(\varphi) - \lambda^2(t_2)\mathcal{E}(\varphi) \leqq 3\epsilon, \ \text{ for all } t_1 > t_2 > T.$$

This implies that $\lim_{t\to\infty} \lambda(t)$ exists, which, together with (4.24), implies

$$\lim_{t\to\infty} \lambda(t) = \sqrt{c^*}. \tag{4.30}$$

Therefore (2.10) follows from (4.25), and (4.27) becomes, as $t \to \infty$,

$$u(t, \cdot + \rho(t)) \to \sqrt{c^*}\varphi \quad \text{in } H^1(-A, \infty) \text{ for every } A > 0. \tag{4.31}$$

Now we show the convergence of $\rho_t(t)$. We define

$$w(t,\cdot) := \sqrt{c^*}\varphi(\cdot-\rho(t)), \quad \eta(t,\cdot) := u(t,\cdot) - \sqrt{c^*}\varphi(\cdot-\rho(t)) = u(t,\cdot) - w(t,\cdot),$$

so it is not hard to see that $w_t = -\rho_t w_x$. Substituting $w(t,\cdot)$ and $\eta(t,\cdot)$ into (2.4) we obtain

$$\begin{aligned}
&-\eta_t + (\rho_t - c^*)w_x \\
&\quad = \frac{1}{2}(1-\partial_x^2)^{-1}(\eta_x^3 + 3\eta_x^2 w_x + 3\eta_x w_x^2) \\
&\quad\quad + (\eta^2\eta_x + 2\eta\eta_x w + w^2\eta_x + \eta^2 w_x + 2\eta w w_x) \\
&\quad\quad + (1-\partial_x^2)^{-1}\partial_x\left(\frac{3}{2}\left(\eta\eta_x^2 + 2\eta\eta_x w_x + w\eta_x^2 + \eta w_x^2 + 2\eta_x w w_x\right)\right. \\
&\quad\quad \left.+(\eta^3 + 3\eta^2 w + 3\eta w^2)\right).
\end{aligned} \tag{4.32}$$

Taking derivative of (4.5) with respect to time and recalling $\varphi - \varphi'' = 2\delta_0$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \eta_t \left(\zeta_{n_0} * \varphi\right)'(\cdot - \rho(t)) &= \rho_t(t)\int_{\mathbb{R}} \eta\left(\zeta_{n_0} * \varphi\right)''(\cdot - \rho(t)) \\
&= \rho_t(t)\int_{\mathbb{R}} \eta\left(\zeta_{n_0} * \varphi\right)(\cdot - \rho(t)) \\
&\quad - 2\rho_t(t)\int_{\mathbb{R}} \eta(x)\zeta_{n_0}(x-\rho(t))\,\mathrm{d}x.
\end{aligned}$$

Then,

$$\begin{aligned}
\left|\int_{\mathbb{R}} \eta_t \left(\zeta_{n_0} * \varphi\right)'(\cdot - \rho(t))\right| &\leqq 3|\rho_t - c^*|\|\eta\|_{H^1} + c^*\left|\int_{\mathbb{R}} \eta\left(\zeta_{n_0} * \varphi\right)(\cdot - \rho(t))\right| \\
&\quad + 2c^*\left|\int_{\mathbb{R}} \eta(x)\zeta_{n_0}(x-\rho(t))\,\mathrm{d}x\right|.
\end{aligned} \tag{4.33}$$

The idea is to take the L^2-inner product of (4.32) with $\left(\zeta_{n_0} * \varphi\right)'(\cdot - \rho(t))$ to show that all terms on the right-hand side vanish as $t \to \infty$.

From (4.31) and the exponential decay of φ and φ' it follows that, as $t \to \infty$,

$$\left|\int_{\mathbb{R}} \eta(x)\zeta_{n_0}(x-\rho(t))\,\mathrm{d}x\right| + \left\|\partial_x^i\eta\left(\zeta_{n_0} * \partial_x^j\varphi\right)(\cdot - \rho(t))\right\|_{L^1\cap L^2} \to 0, \tag{4.34}$$

for $i, j = 0$ or 1.

Also from (2.1) we have, for $i = 0$ or 1,

$$\begin{aligned}
&\int_{\mathbb{R}} \left[(1-\partial_x^2)^{-1}\partial_x^i \eta\right] \left(\zeta_{n_0} * \varphi\right)' (\cdot - \rho(t)) \\
&\quad = \int_{\mathbb{R}} \partial_x^i \eta \; p * \left(\zeta_{n_0} * \varphi\right)' (\cdot - \rho(t)) \\
&\quad = \left(\int_{-\infty}^{-A+\rho(t)} + \int_{-A+\rho(t)}^{\infty} \right) \partial_x^i \eta p \; * \left(\zeta_{n_0} * \varphi\right)' (\cdot - \rho(t)) \\
&\quad = \frac{1}{2} \int_{-\infty}^{-A} \partial_x^i \eta(t, x+\rho(t)) \left(\int_{\mathbb{R}} e^{-|y|} \left(\zeta_{n_0} * \varphi\right)' (x-y) \, \mathrm{d}y \right) \mathrm{d}x \\
&\qquad + \int_{-A}^{\infty} \partial_x^i \eta(t, x+\rho(t)) \; p * \left(\zeta_{n_0} * \varphi\right)' (x) \, \mathrm{d}x \to 0, \quad \text{as } \; t, A \to \infty.
\end{aligned} \tag{4.35}$$

From (4.34), (4.35) and the fact that $\eta, w \in W^{1,\infty}(\mathbb{R})$ we conclude that, as $t \to \infty$,

$$\int_{\mathbb{R}} \left(-\eta_t + (\rho_t - c^*) w_x\right) \left(\zeta_{n_0} * \varphi\right)' (\cdot - \rho(t)) \to 0. \tag{4.36}$$

Substituting (4.34), (4.8), (4.9) and (4.23) into (4.33) we have, as $t \to \infty$,

$$\left| \int_{\mathbb{R}} \eta_t \left(\zeta_{n_0} * \varphi\right)' (\cdot - \rho(t)) \right| < \frac{3\sqrt{c^*}}{2\sqrt{L}-1} |\rho_t - c^*|.$$

On the other hand, as $n_0 \to \infty$,

$$\left| \int_{\mathbb{R}} (\rho_t - c^*) w_x \left(\zeta_{n_0} * \varphi\right)' (\cdot - \rho(t)) \right| \to \sqrt{c^*} |\rho_t - c^*|.$$

Substituting the above two into (4.36) we see that

$$\rho_t(t) \to c^* \qquad \text{as} \qquad t \to \infty. \tag{4.37}$$

Having established the above convergence results, we then follow the steps of [23, Section 5.4] by using the almost monotonicity (4.15) and (4.31), as well as the properties of Ψ_K and φ, and finally we have, for any $\delta > 0$ and $\theta \in (0, \sqrt{c})$ satisfying $\theta^4 < \frac{1}{2}$,

$$\int_{\mathbb{R}} (\eta^2 + \eta_x^2)(t, \cdot) \Psi_K(\cdot - \theta t) \lesssim \delta,$$

which implies (2.11).

This completes the proof of Theorem 2.2. □

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Appendix A. Proofs of Lemma 3.1 and Lemma 4.1

A.1. Proof of Lemma 3.1

Proof. For a fixed $t_0 \in \mathbb{R}$, we approximate $u(t_0)$ by $u_{0,n} = \zeta_n * u(t_0)$ which converges to $u(t_0)$ in Y. Thanks to the continuity result in Proposition 2.3, the solution sequence $\{u_n\}$ induced by $\{u_{0,n}\}$ to (2.4) lies in $C\left(\mathbb{R}; H^\infty(\mathbb{R})\right)$. Hence for an arbitrary positive T,

$$u_n \to u \text{ in } C\left([t_0 - T, t_0 + T]\,; H^1(\mathbb{R})\right). \tag{A.1}$$

For this T, there exists an $n_0 \geqq 0$ such that, for all $n \geqq n_0$,

$$\|u_n^2 - u^2\|_{L^\infty((t_0 - T, t_0 + T)\times\mathbb{R})} \leqq \frac{\alpha c_0}{2L},$$

which combines with (3.1) implying

$$\sup_{[t_0 - T, t_0 + T]} \|u_n^2\|_{L^\infty(|x-\rho(t)|>R_0)} \leqq \frac{\alpha c_0}{L}. \tag{A.2}$$

For the sake of convenience, in the following arguments, we leave out the subscript n of u_n. For $t \in [t_0 - T, t_0]$ and $R > R_0$, differentiating $I_{t_0}^{+R}(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{t_0}^{+R}(t) = -\alpha\rho_t(t) \int_{\mathbb{R}} \Psi_K' \left(u^2 + u_x^2\right) + \int_{\mathbb{R}} \Psi_K \frac{\mathrm{d}}{\mathrm{d}t}\left(u^2 + u_x^2\right). \tag{A.3}$$

Next we estimate the second term on the right-hand side. A direct computation yields that

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\left(u^2 + u_x^2\right)\Psi_K = 2\int_{\mathbb{R}} u u_t \Psi_K + 2\int_{\mathbb{R}} u_x u_{xt} \Psi_K.$$

Using (2.4), we have

$$2\int_{\mathbb{R}} u u_t \Psi_K = -2\int_{\mathbb{R}} u^3 u_x \Psi_K + 2\int_{\mathbb{R}} u \Psi_K' h_1 + 2\int_{\mathbb{R}} u_x \Psi_K h_1 - \int_{\mathbb{R}} u \Psi_K h_2, \tag{A.4}$$

where $h_1 := p * \left(\frac{3}{2} u u_x^2 + u^3\right)$ and $h_2 := p * u_x^3$. Furthermore,

$$
\begin{aligned}
2\int_{\mathbb{R}} u_x u_{xt}\Psi_K =& 2\int_{\mathbb{R}} \partial_x(u_x\Psi_K)u^2u_x - 2\int_{\mathbb{R}} u_x\Psi_K h_1 + 2\int_{\mathbb{R}} u_x\Psi_K\left(\frac{3}{2}uu_x^2+u^3\right)\\
&+\int_{\mathbb{R}} u\left(\Psi_K\partial_x^2h_2+\Psi_K'\partial_xh_2\right)\\
=&\int_{\mathbb{R}} u^2u_x^2\Psi_K' + 2\int_{\mathbb{R}} u_x\Psi_K h_1 + 2\int_{\mathbb{R}} u^3u_x\Psi_K\\
&+\int_{\mathbb{R}} u\left(h_2\Psi_K+\Psi_K'\partial_xh_2\right).
\end{aligned}
\tag{A.5}
$$

Now combining (A.4) and (A.5), we arrive at

$$
\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}t}\left(u^2+u_x^2\right)\Psi_K = \int_{\mathbb{R}} u^2u_x^2\Psi_K' + \int_{\mathbb{R}} u\Psi_K'\partial_xh_2 + 2\int_{\mathbb{R}} uh_1\Psi_K', \tag{A.6}
$$

which, together with (A.3) and (A.6), implies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I_{t_0}^{+R}(t) =& -\alpha\rho_t(t)\int_{\mathbb{R}}\Psi_K'\left(u^2+u_x^2\right) + \int_{\mathbb{R}} u^2u_x^2\Psi_K' + \int_{\mathbb{R}}\left(u\partial_xh_2+2uh_1\right)\Psi_K'\\
=:& -\alpha\rho_t(t)\int_{\mathbb{R}}\Psi_K'\left(u^2+u_x^2\right) + J_1 + J_2.
\end{aligned}
$$

Next we estimate J_1 and J_2. We write J_1 as

$$
J_1 = \int_{|x-\rho(t)|<R_0} u^2u_x^2\Psi_K' + \int_{|x-\rho(t)|>R_0} u^2u_x^2\Psi_K' =: J_{11} + J_{12}.
$$

From $R \geqq R_0$, for $|x-\rho(t)| < R_0$ and $t \in [t_0-T, t_0]$, we have

$$
\begin{aligned}
x-\rho(t_0)-R-\alpha(\rho(t)-\rho(t_0)) =& x-\rho(t)\\
&-R+(\rho(t)-\alpha\rho(t))-(\rho(t_0)-\alpha\rho(t_0))\\
\leqq& R_0-R-(1-\alpha)c_0\left(t_0-t\right),
\end{aligned}
$$

which, together with (3.5) and the non-negativity of Ψ_K, shows that

$$
\begin{aligned}
J_{11}(t) \leqq& 2R_0C_0\|u(t)\|_{L^\infty}^2\|u_x(t)\|_{L^2}^2 e^{R_0/K}e^{-R/K}e^{-\frac{(1-\alpha)}{K}c_0(t_0-t)}\\
\lesssim& \|u(t)\|_{H^1}^4 e^{R_0/K}e^{-R/K}e^{-\frac{(1-\alpha)}{K}c_0(t_0-t)}.
\end{aligned}
\tag{A.7}
$$

As for J_{12}, it holds that

$$
J_{12} \leqq \|u(t)\|_{L^\infty(|x-\rho(t)|>R_0)}^2\int_{|x-\rho(t)|>R_0} u_x^2\Psi_K'. \tag{A.8}
$$

From $|\Psi_K''| < \frac{1}{K}\Psi_K'$ and (3.5), for J_2 we have

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}} (u\partial_x h_2 + 2uh_1)\, \Psi_K' = -\int_{\mathbb{R}} u_x h_2 \Psi_K' - \int_{\mathbb{R}} u h_2 \Psi_K'' + \int_{\mathbb{R}} 2u h_1 \Psi_K' \\
&\leqq \int_{\mathbb{R}} u(p * u^3)\Psi_K' + \int_{\mathbb{R}} \frac{u}{K}(p * u^3)\Psi_K' + \int_{\mathbb{R}} 2u \left(p * \left(\frac{3}{2}u^3 + u^3\right)\right) \Psi_K' \\
&\leqq \left(6 + \frac{1}{K}\right) \|u(t)\|_{L^\infty} \int_{\mathbb{R}} (p * u^3)\Psi_K' \leqq\ L\|u(t)\|_{L^\infty}^2 \int_{\mathbb{R}} u^2 \Psi_K',
\end{aligned}
$$

where $L := \left(6 + \frac{1}{K}\right)\left(\frac{K^2}{K^2-1}\right)$. Therefore applying similar treatment to J_2, we also have

$$
\begin{aligned}
J_2 \leqq & L\|u(t)\|_{L^\infty(|x-\rho(t)|<R_0)}^2 \int_{|x-\rho(t)|<R_0} u^2\Psi_K' + L\|u(t)\|_{L^\infty(|x-\rho(t)|>R_0)}^2 \\
& \int_{|x-\rho(t)|>R_0} u^2 \Psi_K' \\
=: & J_{21} + J_{22}.
\end{aligned}
$$

Similarly, we have

$$
J_{21} \lesssim \|u(t)\|_{H^1}^4 e^{R_0/K} e^{-R/K} e^{-\frac{(1-\alpha)}{K} c_0 (t_0 - t)}, \tag{A.9}
$$

$$
J_{22} \leqq L\|u(t)\|_{L^\infty(|x-\rho(t)|>R_0)}^2 \int_{|x-\rho(t)|>R_0} u^2 \Psi_K'. \tag{A.10}
$$

Combining (A.7)–(A.10) and (A.2) we see that there exists a sufficiently large $R > R_0$ such that

$$
\begin{aligned}
& -\alpha \rho_t(t) \int_{\mathbb{R}} \Psi_K' \left(u^2 + u_x^2\right) + J_{12} + J_{22} \\
& \quad \leqq \left(-\alpha c_0 + L\|u(t)\|_{L^\infty(|x-\rho(t)|>R_0)}^2\right) \int_{\mathbb{R}} \Psi_K' \left(u^2 + u_x^2\right) < 0,
\end{aligned}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t} I_{t_0}^{+R}(t) \leqq \tilde{C} e^{-R/K} e^{-\frac{(1-\alpha)}{K} c_0 (t_0 - t)} \ \text{ for } R \geqq R_0 \text{ and } t \in [t_0 - T, t_0],
$$

where $\tilde{C}$ depends on K, R_0 and $\mathcal{E}(u)$. Since T is an arbitrary positive number, integrating from t to t_0 and by (A.1), we have

$$
I_{t_0}^{+R}(t_0) - I_{t_0}^{+R}(t) \leqq C e^{-R/K}, \quad \forall\, t \leqq t_0,
$$

where C depends on K, α, R_0, $\mathcal{E}(u)$ and c_0. One may follow the same steps to obtain

$$
I_{t_0}^{-R}(t) - I_{t_0}^{-R}(t_0) \leqq C e^{-R/K}, \quad \forall\, t \geqq t_0,
$$

which completes the proof. □

A.2. Proof of Lemma 4.1

Proof. As in the proof of Lemma 4.1 in [23], utilizing the implicit function theorem, similarly we can derive (4.3). And as for the Novikov equation (2.4), each solution $u \in C\left(\mathbb{R}; H^1(\mathbb{R})\right) \cap C^1\left(\mathbb{R}; L^2(\mathbb{R})\right)$ indicates that the mapping $t \mapsto \rho(t)$ is C^1. Setting $R(t,x) := \sqrt{c}\varphi(x-\rho(t))$ and $w := u - R$, we can check that R satisfies

$$\partial_t R + (\rho_t - c)\partial_x R + R^2 \partial_x R + \left(1-\partial_x^2\right)^{-1} \partial_x \left(\frac{3}{2} R R_x^2 + R^3\right) + \frac{1}{2}\left(1-\partial_x^2\right)^{-1} R_x^3 = 0,$$

which gives

$$\begin{aligned} w_t - (\rho_t - c)\partial_x R = & -(w+R)^2 \partial_x w - \left((w+R)^2 - R^2\right)\partial_x R \\ & - \frac{1}{2}\left(1-\partial_x^2\right)^{-1}\left[(w_x + R_x)^3 - R^3\right] \\ & - \left(1-\partial_x^2\right)^{-1} \partial_x \left(\frac{3}{2}(w+R)(w_x+R_x)^2 - \frac{3}{2} R R_x^2 + (w+R)^3 - R^3\right). \end{aligned}$$

Taking the L^2-scalar product of this last equality with $(\zeta_{n_0} * \varphi)'(\cdot - \rho(t))$, similarly we have

$$\left|(\rho_t - c)\left(\int_{\mathbb{R}} \partial_x R \partial_x \left(\zeta_{n_0} * \varphi\right)(\cdot - \rho(t)) + \|w\|_{H^1}\right)\right| \lesssim K c \varepsilon_0.$$

The rest of the proof is similar to [23]. □

References

1. Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* **158**, 227–260, 2004
2. Benjamin, T.B.: The stability of solitary waves. *Proc. Roy. Soc. London Ser. A* **328**, 153–183, 1972
3. Bona, J.: On the stability theory of solitary waves. *Proc. Roy. Soc. London Ser. A* **344**, 363–374, 1975
4. Bona, J.L., Souganidis, P.E., Strauss, W.A.: Stability and instability of solitary waves of Korteweg-de Vries type. *Proc. Roy. Soc. London Ser. A* **411**, 395–412, 1987
5. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664, 1993
6. Cazenave, T., Lions, P.-L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.* **85**, 549–561, 1982
7. Chen, R.M., Di, H., Liu, Y.: Stability of peakons for a class of cubic quasilinear shallow-water equations. submitted, 2019
8. Chen, R.M., Hu, T., Liu, Y.: The integrable shallow-water models with cubic nonlinearity. submitted, 2019
9. Chen, R.M., Pelinovsky, D.E.: $W^{1,\infty}$ instability of H^1-stable peakons in the Novikov equation (2019). arXiv preprint arXiv:1911.08440
10. El Dika, K., Molinet, L.: Exponential decay of H^1-localized solutions and stability of the train of N solitary waves for the Camassa-Holm equation. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **365**, 2313–2331, 2007

11. FUCHSSTEINER, B., FOKAS, A.S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D* **4**, 47–66, 1981/82
12. GENG, X., XUE, B.: An extension of integrable peakon equations with cubic nonlinearity. *Nonlinearity* **22**, 1847–1856, 2009
13. GERMAIN, P., PUSATERI, F., ROUSSET, F.: Asymptotic stability of solitons for mKdV. *Adv. Math.* **299**, 272–330, 2016
14. GRILLAKIS, M., SHATAH, J., STRAUSS, W.: Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.* **74**, 160–197, 1987
15. HONE, A.N.W., LUNDMARK, H., SZMIGIELSKI, J.: Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa-Holm type equation. *Dyn. Partial Differ. Equ.* **6**, 253–289, 2009
16. HONE, A.N.W., WANG, J.P.: Integrable peakon equations with cubic nonlinearity. *J. Phys. A* **41**, 372002, 2008. 10
17. IFTIMIE, D.: Large time behavior in perfect incompressible flows. *Séminaires & Congrés* **15**, 119–179, 2007
18. LIU, X., LIU, Y., QU, C.: Stability of peakons for the Novikov equation. *J. Math. Pures Appl.* **9**(101), 172–187, 2014
19. MARTEL, Y., MERLE, F.: Asymptotic stability of solitons for subcritical generalized KdV equations. *Arch. Ration. Mech. Anal.* **157**, 219–254, 2001
20. MARTEL, Y., MERLE, F.: Asymptotic stability of solitons of the subcritical gkdv equations revisited. *Nonlinearity* **18**, 55–80, 2005
21. MARTEL, Y., MERLE, F.: Asymptotic stability of solitons of the gKdV equations with general nonlinearity. *Math. Ann.* **341**, 391–427, 2008
22. MIZUMACHI, T.: Large time asymptotics of solutions around solitary waves to the generalized Korteweg-de Vries equations. *SIAM J. Math. Anal.* **32**, 1050–1080, 2001
23. MOLINET, L.: A Liouville property with application to asymptotic stability for the Camassa-Holm equation. *Arch. Ration. Mech. Anal.* **230**, 185–230, 2018
24. MOLINET, L.: A rigidity result for the Holm-Staley b-family of equations with application to the asymptotic stability of the Degasperis-Procesi peakon. *Nonlinear Anal. Real World Appl.* **50**, 675–705, 2019
25. NATALI, F., PELINOVSKY, D.E.: Instability of H^1-stable peakons in the Camassa-Holm equation. *J. Differential Equations* **268**, 7342–7363, 2020
26. NOVIKOV, V.: Generalizations of the Camassa-Holm equation. *J. Phys. A* **42**, 342002, 2009. 14
27. PALACIOS, J.M.: Asymptotic stability of peakons for the Novikov equation. *J. Differential Equations* **269**, 7750–7791, 2020
28. PEGO, R.L., WEINSTEIN, M.I.: Asymptotic stability of solitary waves. *Comm. Math. Phys.* **164**, 305–349, 1994
29. RUSSELL, J.S.: *Report on waves, made to the meetings of the British Association in 1842–43.*, London 1845
30. WEINSTEIN, M.I.: Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.* **39**, 51–67, 1986
31. WU, X., YIN, Z.: Global weak solutions for the Novikov equation. *J. Phys. A* **44**, 055202, 2011. 17
32. WU, X., YIN, Z.: Well-posedness and global existence for the Novikov equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5**(11), 707–727, 2012

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