Math 1470: Review Questions for Midterm I

You should not assume that the first midterm exam will resemble this review sheet. The problems below were chosen because they seemed interesting or because they illustrated an important concept. They are not meant to be typical exam problems.

1. Solve the following PDEs for $u(t, x)$.
   
   (1) $u_{tt} = 0$.
   
   
   
   $u(t, x) = tf(x) + g(x)$ where $f$ and $g$ are arbitrary $C^2$ functions.
   
   (2) $u_x + tu = 0$.
   
   
   
   $u(x, t) = e^{-tx}f(t)$ where $f$ is an arbitrary $C^1$ function.
   
   (3) $u_t + u_x = 0$, $u(1, x) = x/(1 + x^2)$, for $t, x \in \mathbb{R}$.
   
   
   
   $u(t, x) = (x - t + 1)/[1 + (x - t + 1)^2]$.
   
   (4) $u_t + 2u_x = 0$, $u(-1, x) = x/(1 + x^2)$, for $t, x \in \mathbb{R}$.
   
   
   
   $u(t, x) = (x - 2t - 2)/[1 + (x - 2t - 2)^2]$.
   
   (5) $u_t - 4u_x + u = 0$, $u(0, x) = 1/(1 + x^2)$, for $t, x \in \mathbb{R}$.
   
   
   
   $u(t, x) = e^{-t}/[1 + (x + 4t)^2]$.
   
   (6) $u_t - xu_x = 0$, $u(0, x) = 1/(1 + x^2)$, for $t, x \in \mathbb{R}$. What is $\lim_{t \to +\infty} u(t, x)$?
   
   
   
   $u(t, x) = \left\{ \begin{array}{ll}
   \frac{1}{1 + (x e^t)^2}, & x \geq t \\
   \frac{1}{2(t-x)}, & x < t
   \end{array} \right.$
   
   (7) $u_t + u_x = 0$, $u(0, x) = x/(1 + x^2)$, $u(t, 0) = 2t$, for $t, x > 0$.
   
   
   
   $u(t, x) = \left\{ \begin{array}{ll}
   \frac{x-t}{1+(x-t)^2}, & x \geq t \\
   \frac{2(t-x)}{2t-x}, & x < t
   \end{array} \right.$

2. Graph the characteristic curves of the PDEs in Problem 1 (3)–(6).

3. Write the following solutions to the wave equation $u_{tt} = u_{xx}$ in d’Alembert form (2.1.8). (Hint: What is the appropriate initial data?)

   (1) $\cos x \cos t$.
   
   
   
   $\frac{1}{2} [\cos(x + t) + \cos(x - t)]$.
   
   (2) $e^{x+t}$.
   
   
   
   $\frac{e^{x+t} + e^{x-t}}{2} + \frac{e^{x+t} - e^{x-t}}{2}$.
   
   (3) $t^2 + x^2$.
   
   
   
   $\frac{1}{2} [(x + t)^2 + (x - t)^2]$.

4. Consider the wave equation $u_{tt} = 4u_{xx}$, $u(0, x) = \sin x$, $u_t(0, x) = \cos x$, for $t > 0, x \in \mathbb{R}$.

   (1) Find the explicit formula for the solution to the above IVP.
   
   $u = \frac{1}{4} \sin(x - 2t) + \frac{3}{4} \sin(x + 2t)$.
(2) True or False: the solution is periodic in \( t \).
True.

(3) If there is an external force of the form \( \cos 2t \), find the corresponding solution.
\[
u = \frac{1}{4} \sin(x - 2t) + \frac{3}{4} \sin(x + 2t) + \frac{1}{4} - \frac{1}{4} \cos 2t.
\]

(4) For the same initial data, solve the equation \( u_{tt} - 2u_{xt} - 3u_{xx} = 0 \) for \( x \in \mathbb{R}, \ t > 0 \).
Set \( \xi = x + 3t, \ \eta = x - t \) then the equation is transformed to \(-16v_{\xi\eta} = 0\) and so the d’Alembert formula now becomes
\[
u(x,t) = \frac{\phi(x+3t) + 3\phi(x-t)}{4} + \frac{1}{4} \int_{x-t}^{x+3t} \psi(s) \, ds
\]
\[
= \frac{\sin(x+3t) + \sin(x-t)}{2}.
\]

5. Find the solution to the heat equation \( u_t = u_{xx} \) on the entire real line subject to the following initial conditions at time \( t = 0 \).

(1) \( e^{-|x|} \).
\[
u = \frac{1}{2} \cosh(t - x) + \frac{1}{2} e^{-x} \text{Erf} \left( \frac{x - 2t}{2\sqrt{t}} \right) - \frac{1}{2} e^{t+x} \text{Erf} \left( \frac{x + 2t}{2\sqrt{t}} \right).
\]

(2) the Heaviside function.
\[
u = \frac{1}{2} + \frac{1}{2} \text{Erf} \left( \frac{x}{2\sqrt{t}} \right).
\]

(3) \( e^{-x^2} \).
\[
u = \frac{1}{\sqrt{4t+1}} e^{-x^2/(4t+1)}.
\]

6. Solve the following IBVP.

(1) \( u_t = 4u_{xx} + x \) for \( t, x > 0, u(0, x) = e^{-x^2}, u(t, 0) = t \).
\[
u = \frac{e^{-\frac{x^2}{16t+1}}}{\sqrt{16t+1}} \text{Erf} \left( \frac{x}{\sqrt{16t(16t+1)}} \right) + xt + \int_0^t \text{Erf} \left( \frac{x}{\sqrt{4(t-\tau)}} \right) \, d\tau.
\]

(2) \( u_{tt} = u_{xx} \) for \( t, x > 0, u(0, x) = 0, u_t(0, x) = x^2, u(t, 0) = 1 \).
\[
u = \begin{cases} x^2 t + \frac{t^3}{3} & x > t, \\ xt^2 + \frac{t^3}{3} + 1, & 0 < x < t. \end{cases}
\]

7. By a separable eigensolution to the heat equation \( u_t = u_{xx} \) we mean a solution \( o \) the form \( u(t, x) = e^{\lambda t} X(x) \). Find all separable eigensolutions to this heat equation on the interval \( x \in [0, \pi] \) subject to

(1) homogeneous Dirichlet boundary conditions \( u(t, 0) = u(t, \pi) = 0 \).
\( \exp(-n^2t) \sin nx, \ n \in \mathbb{N} \).

(2) mixed boundary conditions \( u(t, 0) = u_x(t, \pi) = 0 \).
\( \exp \left[ - (n + \frac{1}{2})^2 t \right] \sin \left( n + \frac{1}{2} \right), \ n = 0, 1, 2, \ldots. \)
Neumann boundary conditions $u_x(t,0) = u_x(t,\pi) = 0.$
\[ \exp(-n^2t) \cos nx, \ n = 0, 1, 2, \ldots \]

8. Find the (real) Fourier series of the following functions (on $[-\pi, \pi]$).

(1) $\sin^2 x$.
\[ \frac{1}{2} - \frac{1}{2} \cos 2x. \]

(2) $|x|$.
\[ \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^\infty \frac{\cos((2k+1)x)}{(2k+1)^2}. \]

(3) $|\sin x|$.
\[ \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=0}^\infty \frac{\cos(2kx)}{4k^2 - 1}. \]

(4) $x \cos x$.
\[ -\frac{1}{2} \sin x + 2 \sum_{n=2}^\infty \frac{(-1)^n n \sin nx}{n^2 - 1}. \]

9. Find the Fourier Sine and Cosine series of the functions in Problem 8 (on $[0, \pi]$).

10. Graph the $2\pi$ periodic extension of $x^2$, $e^x$, and $\frac{1}{x}$.

11. Find the complex Fourier series of the following functions (on $[-\pi, \pi]$).

(1) $\sin x$.
\[ \frac{1}{2} i e^{-ix} - \frac{1}{2} i e^{ix}. \]

(2) $x$.
\[ i \sum_{n=-\infty}^\infty \frac{(-1)^n e^{inx}}{n}. \]

(3) $|\sin x|$.
\[ \frac{2}{\pi} \sum_{k=-\infty}^\infty \frac{e^{2ikx}}{1 - 4k^2}. \]

(4) $|x|$.
\[ \frac{\pi}{2} - \frac{2}{\pi} \sum_{k=-\infty}^\infty \frac{e^{i(2k+1)x}}{(2k+1)^2}. \]

12. Consider the heat equation $u_t = u_{xx}$ on $0 < x < 1$ with initial temperature $u(0,x) = f(x)$. Find the series solution to the initial-boundary value problem when

(1) the left end of the bar is held at 0 degree and the other end is insulated.

Answer: $u(t,x) = \sum_{n=1}^\infty d_n \exp \left[ -\left(n + \frac{1}{2}\right)^2 \pi^2 t \right] \sin \left(n + \frac{1}{2}\right) \pi x,$
where $d_n = 2 \int_0^1 f(x) \sin \left(n + \frac{1}{2}\right) \pi x \, dx.$
13. The diffusion equation $u_t = ku_{xx} - \alpha u$ with $k, \alpha > 0$ models heat flow in a medium with heat loss (say, through radiation) that is proportional to temperature. Find a Fourier series solution of the heat loss equation on $0 < x < 1$, $t > 0$ with the following initial-boundary conditions

$$u(0, x) = f(x), \quad u(t, 0) = u(t, 1) = 0.$$ 

Answer: $u(t, x) = e^{-\alpha t} \sum_{n=1}^{\infty} b_n e^{-(\alpha + kn^2\pi^2)t} \sin n\pi x$, where $b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx$.

14. Write down the solutions to the following initial-boundary value problems for the wave equation on $[0, \pi]$, in the form of a Fourier series:

(1) $u_{tt} = u_{xx}$, $u(t, 0) = u(t, \pi) = 0$, $u(0, x) = 0$, $u_t(0, x) = 1$.

Answer: $u(t, x) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin \sqrt{2}(2j + 1)t \sin (2j + 1)x}{\sqrt{2}(2j + 1)^2}$.

(2) $u_{tt} = 3u_{xx}$, $u(t, 0) = u(t, \pi) = 0$, $u(0, x) = \sin^3 x$, $u_t(0, x) = 0$.

Answer: $u(t, x) = \frac{3}{4} \cos \sqrt{3} \sin x - \frac{1}{4} \cos 3\sqrt{3} \sin 3x$.

(3) $u_{tt} = 4u_{xx}$, $u(t, 0) = u(t, 1) = 0$, $u(0, x) = x$, $u_t(0, x) = -x$.

Answer: $u(t, x) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \left( \cos 2nt - \frac{\sin 2nt}{2n} \right) \frac{\sin nx}{n}$.

15. Find the condition for the parameters $\alpha, \beta, \gamma, \delta$ so that the following boundary conditions on $(a, b)$ are symmetric.

$$X(b) = \alpha X(a) + \beta X'(a)$$

$$X'(b) = \gamma X(a) + \delta X'(a).$$

Answer: $\alpha \delta - \beta \gamma = 1$. 

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