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Technical note

A new method for the representation of articular surfaces using the influence surface theory of plates

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Abstract

The traditional approach to the representation of an articular surface is by using piecewise polynomial functions with a limited continuity to fit the surface from ordered data points. In this study, we introduce a new method, which is based on the influence surface theory of plates, for the representation of articular surfaces. The most significant advantage of this method is that it can effectively represent an articular surface from non-ordered data points. The effectiveness of the present method was shown by reconstruction of a human femoral surface and a mathematical cone. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

To determine an articular surface geometry, it is necessary to represent the articular surface with a mathematical model. In recent years, many surface fitting methods have been used to describe diarthrodial joint surfaces. These include a sphere (Rushfeldt et al., 1981; Blankevoort and Huiskes, 1986; and Soslowsky et al., 1992), and varying degrees of polynomials (Wismans et al., 1980; Blankevoort et al., 1991). While these mathematical descriptions provide good approximation to specific joint surfaces, they are not flexible to model a variety of other diarthrodial joint surfaces. For example, when a polynomial is used for surface fitting, it is necessary to determine its term composition and degree, which are heavily dependent on prior understanding of the surface to be fitted. Therefore, it is difficult to obtain a satisfactory polynomial with high fitting accuracy.

An important surface fitting method is the Coons bicubic spline (Coons, 1967). This method has been widely used in the representation of diarthrodial joint surfaces (Scherrer and Hillberry, 1979; Ateshian et al., 1991, and Hefzy and Yang 1993). The other methods used include the parametric biquintic polynomial splines (Ateshian et al., 1992), and B-spline least-squares surface fitting (Ateshian, 1993). These techniques are remarkably flexible, and hence have been highly successful in modeling a large variety of articular surfaces. However, these methods have some inherent limitations, which include the requirement of ordered data points and the limited continuity of the fitted articular surface.

The purpose of this paper is to introduce a new surface fitting method which, to a great extent, overcomes the above limitations. For example, one significant advantage of the method is its ability to reconstruct a surface from non-ordered data points.

2. Method

2.1. *Influence deflection of the infinite plate*

Under lateral loads, a thin plate bends and forms a deflection surface. The plate deflection obeys the

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governing differential equation (Szilard, 1974):

$$D\nabla^4 w = q, \tag{1}$$

where D denotes the bending rigidity of the plate, w is the lateral deflection, and q is the uniform load.

Consider an infinite plate subjected to concentrated loads, $F_1, F_2, \ldots, F_N$, at points (x_1, y_1), $(x_2, y_2), \ldots$, (x_N, y_N). Then, according to Maxwell's reciprocity law and based on the singular solution of Eq. (1), the deflection surface due to the ith load F_i is

$$w_i(\xi, \eta) = kF_i[(\xi - x_i)^2 + (\eta - y_i)^2] \\ \mathrm{Ln}[(\xi - x_i)^2 + (\eta - y_i)^2], \tag{2}$$

where (ξ, η) is an arbitrary observation point on the plate, (x_i, y_i) represents the location of the individual load $\boldsymbol{F}_i$, and k is a constant representing the bending rigidity of the plate. Without loss of generality, we set $k = 1$ in the following formulation.

The influence deflection under all the concentrated loads, which represents the particular solution of plate Eq. (1), is the sum of the functions $w_i(\xi, \eta)\,(i = 1, 2, \ldots, N)$. That is,

$$w_{\mathrm{p}}(\xi, \eta) = \sum_{i=1}^{N} w_i(\xi, \eta). \tag{3}$$

Suppose that the deflection of the infinite plate tends to be flat at a distance far away from the applied loads. A linear plane function is then chosen as the homogeneous solution of the plate Eq. (1):

$$w_{\mathrm{H}}(\xi, \eta) = A + B\xi + C\eta, \tag{4}$$

where A, B and C are three coefficients.

The entire solution of the plate under all the concentrated loads is the combination of Eqs. (3) and (4). This gives

$$w(\xi, \eta) = A + B\xi + C\eta + \sum_{i=1}^{N} w_i(\xi, \eta). \tag{5}$$

Clearly, the prerequisite condition for Eq. (5) to hold is that the concentrated loads acting on the infinite plate are in equilibrium. Since the loads form a system of spatial parallel forces, the following three equilibrium equations must be satisfied:

$$\sum_{i=1}^{N} F_i = 0, \tag{6}$$

$$\sum_{i=1}^{N} F_i x_i = 0, \tag{7}$$

$$\sum_{i=1}^{N} F_i y_i = 0. \tag{8}$$

2.2. Representation of an articular surface

To describe an articular surface, three-dimensional coordinates of surface points must be determined experimentally. Assume that $w_m(x_i, y_i)\,(i = 1, 2, \ldots, N)$ are the measured height coordinates of the surface points. If, by analogy, the height coordinates of the articular surface are deflection points of an infinite plate under concentrated loads, then according to Eq. (5), we obtain N linear equations

$$A + Bx_i + Cy_i + \sum_{i=1}^{N} w_i(x_i, y_i) = w_m(x_i, y_i), \\ m = 1, 2, \ldots, N \tag{9}$$

with $N + 3$ unknowns A, B, C and $F_i(i = 1, 2, \ldots, N)$ to be determined. Together with Eqs. (6)–(8), a total of $N + 3$ equations are obtained, and thus their $N + 3$ coefficients can be solved. Once the coefficients are determined, $w(\xi, \eta)$ in Eq. (5), which represents the surface, is then determined.

2.3. Implementation of the present method

To implement the present method, a computer program was written to solve linear Eqs. (6)–(9) numerically. The computation algorithm of the program was based on the method of LU decomposition (Pizer, 1975) using the PV-WAVE built-in subroutines (Precision-Visuals, PV ∼ WAVE Technical Reference Manual, 1990). The entire computation was performed on a SUN computer. After a fitting surface, which is represented by $w(\xi, \eta)$ in Eq. (5), was determined, numerical computation based on an integral method for surface area (Wylie and Barrett, 1982) was also performed to determine surface area of the fitting surface. Specially, volumes of the mathematical cone and reconstructed one were computed for comparison (see next section).

2.4. Verification of the effectiveness of the present method

Two surfaces were reconstructed using the present method to show its effectiveness. The first was a femoral surface. Briefly, a commercially available plastic femur model was mounted on the frame of a custom-made three-dimensional digitizer. By moving the stylus of the digitizer on the distal femoral surface, three-dimensional coordinates of the surface points, with respect to a fixed global coordinate system of the digitizer, were obtained. A total of 396 surface points, which were arbitrarily selected, were digitized on the femoral surface (Fig. 1). The coordinates of these data points were used to reconstruct the femoral surface and to determine the femoral surface area.

The second was a mathematical cone surface. The mathematical cone was defined to be $Z = 1 - \sqrt{X^2 + Y^2}$. Clearly, if $Z = 0$, then $X^2 + Y^2 = 1$, which represents a circle on the XY plane. To represent the cone surface with the present method, we chose 81 points by dividing the circular area into a meshed pattern, and because of the regularity of the meshed patterns, the X and Y coordinates of mesh node points were readily determined. Height coordinates of the cone surface were determined from $Z = 1 - \sqrt{X^2 + Y^2}$. With the obtained three-dimensional coordinates, the mathematical cone surface was reconstructed. The surface area and volume of the reconstructed cone was then computed and compared with those of the mathematical cone.

3. Results

The femoral surface was effectively represented with a total of 396 data points (Fig. 2). With half of the data points, which were chosen alternately from the 396 data points, the surface area was still well represented. For comparison, the femoral surface area was 32.4 for the 396 points, whereas it was 31.4 for the 198 data points.

The mathematical cone was also effectively reconstructed with the present method (Fig. 3). The reconstructed cone resembles the mathematical cone. In fact, both the surface area and volume of the reconstructed cone were close to those of the mathematical cone. For the surface area, 4.41 vs. 4.44; and for the volume, 1.02 vs. 1.05. These results indicated that the cone surface reconstructed using the present method was sufficiently close to the mathematical cone.

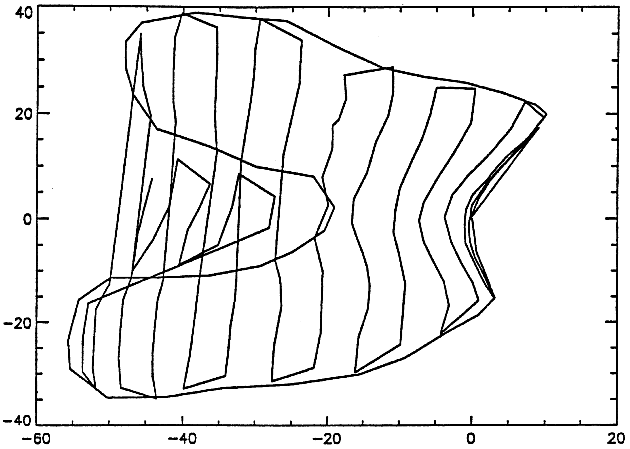

Fig. 1. The arbitrary track of digitizing surface data points on the human femoral surface. A total of 396 surface data points were digitized. The surface points were arbitrarily selected on the surface which includes the femoral condyles and patella groove area. The coordinates of the surface data points were used for the reconstruction of the femoral surface.

4. Discussion

We have presented a new surface fitting method. This method, by analogy, considers an articular surface to be the deflection surface of an infinite plate under lateral concentrated loads. Mathematically, the deflection surface is described by a linear plane function plus a series of modified logarithmic functions. This combination of function can effectively represent any articular surface, since the infinite plate can have a deflection surface that is sufficiently close to the articular surface, as long as individual concentrated loads (i.e., the coefficients in Eq. (5)) at individual deflection points have proper magnitudes. The effectiveness of the present method was shown by the two examples of reconstructing a femoral surface and a mathematical cone. Specifically, the example of fitting the femur surface showed that by properly choosing points on a surface, a good fit for a complex surface can still be achieved with fewer data points. Moreover, the second example showed that this method provides an accurate fit even for a surface with a singular point like the mathematical cone.

The most significant advantage of the present method, however, is that it can use non-ordered data points for surface representation. This was shown by the example of

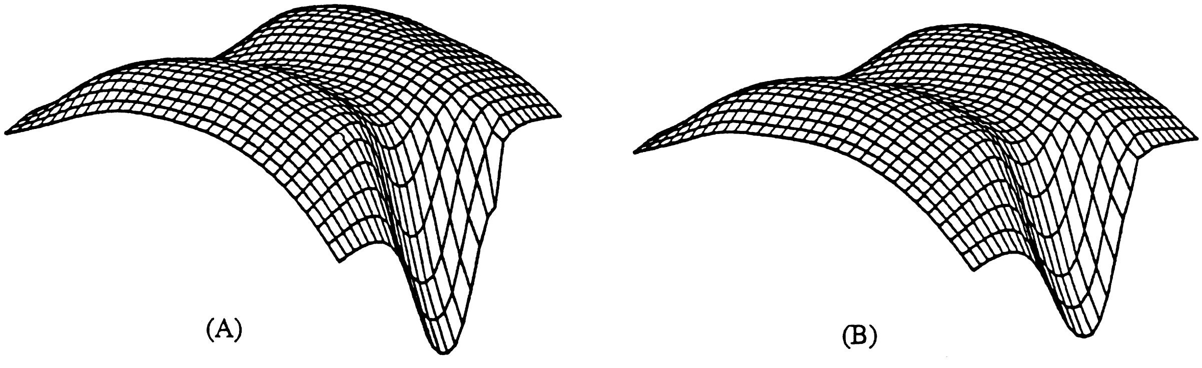

Fig. 2. The reconstructed femoral surfaces by the present method. Fig. 2A was created with 396 surface data points, whereas Fig. 2B was obtained using half of these surface data points (that is, 198 points), which were selected alternately from the 396 data points. It is evident that both graphs were similar. In fact, two surface areas of the two reconstructed surfaces were close: they are 32.4 for 396 data points and 31.4 for 198 data points.

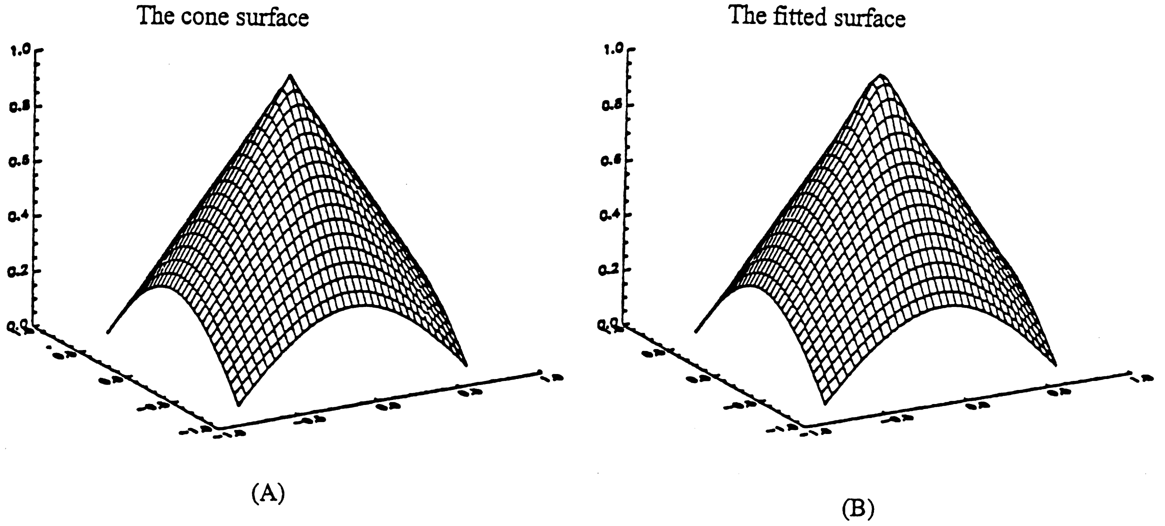

Fig. 3. Comparison between the mathematical cone surface (A) (defined by $Z = 1 - \sqrt{X^2 + Y^2}$) and the reconstructed cone surface (B) by the present method. A total of 81 data points were used for the reconstruction. It is apparent that the reconstructed cone is similar to the mathematical cone. In fact, the surface area of the reconstructed cone was 4.41, compared with 4.44 for the mathematical cone. And the volume of the reconstructed cone was 1.02, whereas the mathematical cone was 1.05.

fitting the femoral surface. In the example, the femoral surface was reconstructed with the surface data points arbitrarily selected on the surface (see Fig. 1). In contrast, traditional techniques (Coons, 1967; Ateshian et al., 1992; 1993) require ordered data sets to perform surface fitting. This means that small regular meshes must be divided on an articular surface in order to establish the surface. In addition, the B-spline method requires a complete rectangular net, and therefore requires an additional interpolation method (e.g., Akima's method) to implement the technique.

Another advantage of the present method is that it offers unlimited continuity. This is in contrast to traditional techniques which only offer the second-order continuity if a cubic polynomial is used. Higher orders of polynomials are hardly useful because of the "wiggling" problem between data points (Ateshian, 1993). The limited continuity may cause large errors in estimating surface curvatures — the high-order spatial derivatives, and thus may cause problems in determining correct joint contact stress, which is known to be surface geometry dependent.

One limitation of the present method is that it requires the fitting surface to pass through all measured surface data points. On one hand, this means that the fitting accuracy of the present method is partially controlled by the measurement instrument: An instrument with high accuracy yields a fitting surface closer to the actual surface. On the other hand, because of the noise nature of measured data, forcing the fitting surface to pass through the measured data points may not produce an optimal fitting surface. But it should be cautioned that the optimum is achieved usually under the assumption of "white noise" [i.e., the errors follow the standard normal distribution $N(0, \sigma^2)$] at each data point, and under some criterion. Although the white noise may often be a reasonable assumption, the criterion selection is highly subjective (Ateshian, 1993). Thus, in order to obtain accurate representation of an articular surface, surface data points with a high measurement accuracy must be used, regardless of whether the fitting surface passes through the surface data points or not.

The present method is especially useful in joint mechanics. Because it does not require ordered data sets, so that it is possible to use more data points in an interested region, such as a joint contact area, to increase fitting accuracy. Since contact stress is geometry shape dependent, this method may increase the accuracy of estimating the joint contact stress. Also, since this method uses biharmonic functions, which are smooth and stable (just like a plate deflection surface, which is smooth and stable), it can be used for interpolation of data points. For example, using similar functions, Ruan successfully solved collocation problems in surveying (Ruan, 1987). We also used the functions to correct lens distortion effectively (Han et al., 1995).

A couple of points need to be further clarified in this study. First, we used the present method to fit the surface of a plastic femur model instead of a real femur. Ideally, it would be preferred to use a real articular surface for the surface fitting, since others may use the real surface data for modeling. However, our purpose of using this fitting example was to show the effectiveness of the present method in fitting a complex surface. Hence, for this purpose, nothing would be lost or gained by using an artificial surface for surface fitting. In other words, the effectiveness of the present method is independent of the selection of a sample for surface fitting. In addition, because the surface data (coordinates) in this example were obtained from the plastic femur model and measured by a less accurate digitizer (0.5 mm), the data points themselves were not presented, since they would not be useful to others who are interested in accurately modelling a real femur surface.

Second, it should be noted that the advantages of the present method in surface fitting are largely due to the fact that it uses a series of modified logarithmic functions (see Eq. (2)). To represent a surface, these individual functions do not require ordered data points, since they cover the entire surface instead of a partial, or patched, surface, as many existing surface fitting techniques do. Furthermore, these functions are smoother than commonly used polynomial functions, because they are smooth biharmonic functions (so called, since they satisfy the biharmonic equation of the plate, Eq. (1)) and have an unlimited continuity mathematically. Physically, the smoothness is obvious, since these functions represent the smooth deflection surface of the plate. In addition, these functions do not have the "wiggling" problem (Ateshian, 1993), as high orders (> 3) of polynomial functions do. This would be best explained when one considers the deflection surface of a plate under concentrated loads. Obviously, the deflection surface of the plate cannot change unpredictably (i.e., wiggle) in any part of the surface, since it results from all individual concentrated loads that must counterbalance each other (Eq. (6)–(8)).

In summary, we have presented a new surface fitting method for the representation of articular surfaces. The present method has the following advantages: (1) it can model articular surfaces from non-ordered data points; (2) it has an unlimited continuity and no "wiggling" problem; and (3) it fits any articular surfaces in a uniform manner. The effectiveness of the present method was verified by reconstructing a femoral surface and a human femoral surface. The method may be particularly useful in joint mechanics, where accurate representation of the geometric surface is essential for determining correct joint contact stress.

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