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$C^{1,\alpha}$-regularity for p-harmonic functions in the Heisenberg group for p near 2

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Abstract. We prove $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group for p in a neighborhood of 2.

1. Introduction

In this note we prove $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group for p in a neighborhood of 2. In the Euclidean case this result is known for $1 < p < \infty$, while in the Heisenberg group there are no answers yet. In the linear case $p = 2$ the C^∞ regularity is a consequence of a result of Hörmander [**10**], while for $p \geq 2$ under the additional assumption of having strictly positive lower and finite upper L^∞ bounds for the horizontal gradient, L. Capogna [**2, 3**] proved C^∞ regularity. The major difficulties in the Heisenberg group are that we have just partial results regarding the second order differentiability of the approximate p-harmonic functions and that the variants of de Giorgi-Moser iteration method for the differentiated equations do not work or do not give uniform estimates. Our result provides the first indication so far that the $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group is possible without any non-degeneracy hypothesis.

We use previous results regarding the Calderón-Zygmund theory in the Heisenberg group (see [**7, 11, 12**], the $HW^{2,2}$ regularity of p-harmonic functions ([**4, 5**]) and the properties of second order PDE operators that are near to the subelliptic Laplacian, to prove $C^{1,\alpha}$ regularity for p-harmonic functions in the Heisenberg group for p in a neighborhood of 2.

Consider the Heisenberg group $\mathbb{H}^n$ as $\mathbb{R}^{2n+1}$ endowed with the group multiplication

$$(x_1, ..., x_{2n}, t)\cdot(y_1, ..., y_{2n}, u) = (x_1+y_1, ..., x_{2n}+y_{2n}, t+u-\frac{1}{2}\sum_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i))\,.$$

For $i \in \{1, ..., n\}$ consider the left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t}\ ,\ \ X_{n+i} = \frac{\partial}{\partial x_{n+i}} + \frac{x_i}{2}\frac{\partial}{\partial t}\ ,\ \ T = \frac{\partial}{\partial t}\,.$$

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The nontrivial commutators are $[X_i, X_{n+i}] = T$, otherwise we have $[X_i, X_j] = 0$. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain, $1 < s < \infty$ and $k \in \mathbb{N}$. Let us consider the following Sobolev space with respect to the horizontal vector fields X_i:

$$HW^{k,s}(\Omega) = \Big\{ u \in L^s(\Omega) \ : \ X_{i_1}...X_{i_k} u \in L^s(\Omega)\,, \text{ for all } \{i_1, ..., i_k\} \subset \{1, ..., 2n\} \Big\} .$$

Let $HW_0^{k,s}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $HW^{k,s}(\Omega)$.

We denote by X^2u the matrix of second order horizontal derivatives and by $\Delta_X u = \sum_{i=1}^{2n} X_i X_i u$ the subelliptic Laplacian associated to the horizontal gradient $Xu = (X_1 u, ..., X_{2n} u))$.

The Calderón-Zygmund theory gives the following lemma (see the theorem on page 917 in [**7**]).

Lemma 1.1. *For all $1 < s < \infty$ there exists a positive constant C_s such that for all $u \in HW_0^{2,s}(\Omega)$ we have*

$$||X^2 u||_{L^s(\Omega)} \leq C_s ||\Delta_X u||_{L^s(\Omega)} .$$

Remark 1.1. In the case $s = 2$ we may take

$$C_2 = \sqrt{1 + \frac{2}{n}} .$$

See [**5**].

Throughout this paper we will keep the notation C_s just for the constant from Lemma 1.1. Let us consider now

$$\mathcal{A}u = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u$$

where the coefficients $a_{ij} \in L^\infty(\Omega)$. Let us denote by $A = (a_{ij})$ the $2n \times 2n$ matrix of these coefficients.

Theorem 1.1. *Let $1 < s < \infty$ and $0 < \varepsilon \leq 1$, such that $\varepsilon \cdot C_s < 1$, where C_s is the constant given by Lemma 1.1. Let us suppose that*

$$|\Delta_X u(x) - \mathcal{A}u(x)| \leq \varepsilon \left| X^2 u(x) \right| \tag{1.1}$$

for a.e. $x \in \Omega$ and for all $u \in HW_0^{2,s}(\Omega)$. Then $\mathcal{A} : HW_0^{2,s}(\Omega) \to L^s(\Omega)$ is a Banach space isomorphism and there exists $c > 0$ such that

$$||X^2 u||_{L^s(\Omega)} \leq c ||\mathcal{A}u||_{L^s(\Omega)} \tag{1.2}$$

for all $u \in HW_0^{2,s}(\Omega)$.

Proof. The proof follows from the fact that Lemma 1.1 and formula (1.1) shows that $\mathcal{A} : HW_0^{2,s}(\Omega) \to L^s(\Omega)$ satisfies the relation

$$||\Delta_X u - \mathcal{A}u||_{L^s(\Omega)} \leq \varepsilon \cdot C_s \, ||\Delta_X u||_{L^s(\Omega)}$$

which proves that $\mathcal{A}$ is near to Δ_X and hence it is an isomorphism. For the properties inherited by operators that are near to each other we quote Theorems 1 and 2 from [**1**]. □

We need the following result which involves interpolation inequalities and higher order extensions of functions over the boundaries of homogeneous balls (see [**13**]).

LEMMA 1.2. *Let $u \in HW^{2,s}_{\mathrm{loc}}(\Omega)$, $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, 2r) \subset \Omega$. Then for all $\delta 0$ there exists $c(\delta) > 0$ such that*

$$||Xu||_{L^s(B(x_0,r))} \le \delta ||X^2u||_{L^s(B(x_0,r))} + c(\delta)||u||_{L^s(B(x_0,r))} \,.$$

We can use now Theorem 1.1, Lemma 1.2 and a method similar to the proof of Theorem 9.11 in [**9**] and Lemma 3.3 in [**5**] to get the following result.

THEOREM 1.2. *Let us suppose that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 1.1 and that $B(x_0, 3r) \subset \Omega$. Then*

$$||X^2u||_{L^s(B(x_0,r))} \le c\Big(||\mathcal{A}u||_{L^s(B(x_0,2r))} + ||u||_{L^s(B(x_0,2r))}\Big)$$

for all $u \in HW^{2,s}_{\mathrm{loc}}(\Omega)$.

2. $C^{1,\alpha}$-interior regularity for p-harmonic functions in $\mathbb{H}^n$

Let $\Omega \in \mathbb{H}^n$ be a domain, $h \in HW^{1,p}(\Omega)$ and $p > 1$. Consider the problem of minimizing the functional

$$\Phi(u) = \frac{1}{p} \int_\Omega |Xu(x)|^p \, dx$$

over all $u \in HW^{1,p}(\Omega)$ such that $u - h \in HW^{1,p}_0(\Omega)$.

The Euler-Lagrange equation for this problem is the p-Laplace equation

$$\sum_{i=1}^{2n} X_i \left(|Xu|^{p-2} \, X_i u\right) = 0 \,, \text{ in } \Omega \,. \tag{2.1}$$

A p-harmonic function $u \in HW^{1,p}(\Omega)$ is a weak solution for (2.1) and therefore satisfies

$$\sum_{i=1}^{2n} \int_\Omega |Xu(x)|^{p-2} X_i u(x) \; X_i\varphi(x) dx = 0 \,, \text{ for all } \varphi \in HW^{1,p}_0(\Omega) \,. \tag{2.2}$$

For $m \in \mathbb{N}$ let us define now the approximating problems of minimizing the functionals

$$\Phi_m(u) = \frac{1}{p} \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2\right)^{\frac{p}{2}}$$

and the corresponding Euler-Lagrange equations

$$\sum_{i=1}^{2n} X_i \left(\left(\frac{1}{m} + |Xu|^2\right)^{\frac{p-2}{2}} X_i u \right) = 0 \,, \text{ in } \Omega \,. \tag{2.3}$$

The weak form of this equation is

$$\sum_{i=1}^{2n} \int_\Omega \left(\frac{1}{m} + |Xu(x)|^2\right)^{\frac{p-2}{2}} X_i u(x) \; X_i\varphi(x) dx = 0 \,, \text{ for all } \varphi \in HW^{1,p}_0(\Omega) \,. \tag{2.4}$$

In the case of a weak solution u_m of (2.3) let us consider, at this moment just formally, the differentiated version of equation (2.3):

$$\sum_{i,j=1}^{2n} a^m_{ij} \, X_i X_j u_m = 0 \,, \text{ in } \Omega \tag{2.5}$$

where

$$a_{ij}^m(x) = \delta_{ij} + (p-2)\frac{X_i u_m(x)\, X_j u_m(x)}{\frac{1}{m} + |Xu_m(x)|^2}\,.$$

Define the mapping $L_m : W_0^{2,s}(\Omega) \to L^s(\Omega)$ by

$$L_m(v)(x) \;=\; \sum_{i,j=1}^{2n} a_{ij}^m(x) X_i X_j v(x)\,. \tag{2.6}$$

We remark that

$$|L_m v(x) - \Delta_X v(x)| \le |p-2|\; n\; |X^2 v(x)|$$

for a.e. $x \in \Omega$ and for all $v \in HW_0^{2,s}(\Omega)$.

We recall the following result from [**4**], which makes rigorous the previously introduced formal calculations:

THEOREM 2.1. *Let $u_m \in HW^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of (2.3).*

(i) *In the case $2 \le p < 4$ we have $u_m \in HW^{2,2}_{\text{loc}}(\Omega)$ with bounds depending on m and $L_m(u_m)(x) = 0$ for a.e. $x \in \Omega$.*
(ii) *In the case $\frac{\sqrt{17}-1}{2} \le p \le 2$ we have $u_m \in HW^{2,p}_{\text{loc}}(\Omega)$with bounds depending on m and $L_m(u_m)(x) = 0$ for a.e. $x \in \Omega$.*

For $\gamma > 0$ small but fixed, let us denote by

$$\tilde{c} = \sup\left\{ C_s\,,\; s \in \left[\frac{\sqrt{17}-1}{2}\;,\; 2n+2+\gamma\right]\right\}.$$

REMARK 2.1. Let us remark that by Lemma 1.1 the constants $C_{\frac{\sqrt{17}-1}{2}}$ and $C_{2n+2+\gamma}$ are finite, and the convexity theorem of M. Riesz and G.O. Thorin (see for example page 16 in [**14**]) implies that if

$$\frac{1}{s} = \frac{1-\theta}{\frac{\sqrt{17}-1}{2}} + \frac{\theta}{2n+2+\gamma}$$

then

$$C_s \le C^{1-\theta}_{\frac{\sqrt{17}-1}{2}}\; C^{\theta}_{2n+2+\gamma}\,.$$

This shows that $\tilde{c} < +\infty$.

THEOREM 2.2. *For*

$$\max\left\{\frac{\sqrt{17}-1}{2}\,,\, 2 - \frac{1}{\tilde{c}\, n}\right\} \le p \le 2 + \frac{1}{\tilde{c}\, n}$$

and a p-harmonic function u there exists $0 < \alpha < 1$ such that we have the interior regularity $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.

PROOF. **The case $\mathbf{2 \le p}$.**
Theorems 1.2 and 2.1 show that $Xu_m \in HW^{1,2}_{\text{loc}}(\Omega)$ with uniform bounds in m. We use the embedding (see for example [**6, 7, 8**])

$$HW^{1,2}_{\text{loc}}(\Omega) \hookrightarrow L^{q_0}_{\text{loc}}(\Omega)$$

where

$$q_0 = \frac{(2n+2)\cdot 2}{2n+2-2} = \frac{2n+2}{n}\,.$$

For a corresponding cut-off function η between the homogeneous balls B_r and B_{2r} we have

$$\begin{aligned}
&||L_m(\eta^2 u_m)||_{L^{q_0}(B_{3r})}\\
&= c\left\| u_m\, L_m(\eta^2) + \sum_{i,j=1}^{2n} a^m_{ij}(x)\Big(X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m\Big)\right\|_{L^{q_0}(B_{3r})}\\
&\le c\Big(||u_m||_{L^{q_0}(\mathrm{supp}\eta)} + ||Xu_m||_{L^{q_0}(\mathrm{supp}\eta)}\Big) < +\infty
\end{aligned} \tag{2.7}$$

Therefore, by Theorems 1.1 and 1.2 we have that $u_m \in HW^{2,q_0}_{\text{loc}}(\Omega)$ with locally uniform bounds. Repeating this procedure k times we get that $u_m \in HW^{2,q_k}_{\text{loc}}(\Omega)$ for

$$q_k = \frac{2n+2}{n-k}\,.$$

We stop after $n-1$ steps and get $q_{n-1} = 2n+2$ which is the homogeneous dimension of $\mathbb{H}^n$ and obtain $u_m \in HW^{2,2n+2}_{\text{loc}}(\Omega)$. Let us choose now $\beta 1$ enough close to 1 such that

$$(2n+2)\frac{\beta}{2-\beta} \le 2n+2+\gamma\,.$$

Then we use the embedding

$$HW^{1,\frac{2n+2}{2}\beta}_{\text{loc}}(\Omega) \hookrightarrow L^{(2n+2)\frac{\beta}{2-\beta}}_{\text{loc}}(\Omega)$$

and inequalities similar to (2.7) to conclude that

$$u_m \in HW^{2,(2n+2)\frac{\beta}{2-\beta}}_{\text{loc}}(\Omega)\,.$$

The embedding

$$HW^{1,(2n+2)\frac{\beta}{2-\beta}}_{\text{loc}}(\Omega) \hookrightarrow C^{\frac{2\beta-2}{\beta}}$$

shows that u_m has interior regularity $C^{1,\alpha}$ where

$$\alpha = \frac{2\beta-2}{\beta}\,.$$

Because of the estimates for u_m are uniform in m, we let $m \to +\infty$ and conclude that $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$.

The case $\mathbf{p \le 2}$.

Theorems 1.2 and 2.1 implies that $Xu_m \in HW^{1,p}_{\text{loc}}(\Omega)$ with uniform bounds in m. Then we can start with $q_0 = p$ and follow the proof of the previous case until we get the first k with

$$q_k = \frac{(2n+2)p}{2n+2-kp} > \frac{2n+2}{2}\,.$$

Let us choose now $\beta 1$ enough close to 1 such that

$$(2n+2)\frac{\beta}{2-\beta} \le 2n+2+\gamma$$

and

$$(2n+2)\frac{\beta}{2} \le q_k\,.$$

The rest is similar to the last part of the previous case. □

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