Asset Markets in General Equilibrium

Econ 3030 Fall 2020

Lecture 12, November 17

Outline

1. Asset Markets and Equilibrium
2. Linear Asset Pricing
3. Complete Markets
4. Span
5. No Arbitrage and Asset Pricing
Asset Markets in General Equilibrium

- An asset is completely characterized by its return vector $r_k \in \mathbb{R}^S$
  - $r_s$ is the dividend paid to the holder of a unit of $r_k$ if and only if state $s$ occurs.
- The return matrix $R$ is an $S \times K$ matrix whose $k$th column is the return vector of asset $k$.
- $q \in \mathbb{R}_+^K$ are the asset prices.
- $z_i \in \mathbb{R}^K$ are consumer $i$’s holdings of each asset.
- Consumer $i$’s budget constraints are

  \[
  \begin{aligned}
  \text{asset markets} & \quad \sum_{k=1}^K q_k z_{ki} \leq 0 \\
  \text{spot markets} & \quad \sum_{l=1}^L p_{ls} x_{lsi} \leq \sum_{l=1}^L p_{ls} \omega_{lsi} + p_{1s} \sum_{k=1}^K z_{ki} r_{sk} \quad \text{for each } s
  \end{aligned}
  \]

  or

  \[
  \begin{aligned}
  q \cdot z_i \leq 0 & \quad \text{and} \quad \sum_{l=1}^L p_{ls} x_{lsi} \leq p \cdot \omega_i + R z_i
  \end{aligned}
  \]

  where we normalized $p_{1s} = 1$. 
What do budget sets look like?

- Normalizing $p_1 = 1$, one can write $i$’s budget set as

$$B_i(p, q, R) = \left\{ x_i \in \mathbb{R}_{+}^L : \begin{array}{c} \text{there is } z_i \in \mathbb{R}^K \\
\text{such that} \\
q \cdot z_i \leq 0 \\
\text{and} \\
p \cdot (x_i - \omega_i) \leq Rz_i \end{array} \right\}$$

- The second part of the budget set defines $S$ inequalities

$$\begin{pmatrix} p_1 \cdot (x_{1i} - \omega_{1i}) \\ \vdots \\ p_s \cdot (x_{si} - \omega_{si}) \\ \vdots \\ p_S \cdot (x_{Si} - \omega_{Si}) \end{pmatrix} \leq \begin{pmatrix} z_{i1}r_{11} + \ldots + z_{ki}r_{k1} + \ldots + z_{Ki}r_{K1} \\ \vdots \\ z_{i1}r_{1s} + \ldots + z_{ki}r_{ks} + \ldots + z_{Ki}r_{Ks} \\ \vdots \\ z_{i1}r_{1S} + \ldots + z_{ki}r_{kS} + \ldots + z_{Ki}r_{KS} \end{pmatrix}.$$ 

where each $p_s \in \mathbb{R}^L$

- $Rz_i$ represents the ‘financial income’ attainable by a consumer who chooses portfolio $z_i$.

- For a fixed $R$, the set of all possible values for $Rz_i$ determines the possible incomes available to the consumer. As $R$ changes so does this set.
A Radner equilibrium with asset markets is given by assets prices $q^* \in \mathbb{R}^K$, spot prices $p_s^* \in \mathbb{R}^L$ in each state $s$, portfolios $z_i^* \in \mathbb{R}^K$, and spot market plans $x_{si}^* \in \mathbb{R}^L$ for each $s$ such that:

1. For each $i$, $z_i^*$ and $x_i^*$ solve

$$\max_{z_i \in \mathbb{R}^K, x_i \in \mathbb{R}^L_+} U_i(x_i)$$

subject to

$$\sum_{k=1}^{K} q_k^* z_{ki} \leq 0 \quad \text{and} \quad p_s^* \cdot x_{si} \leq p_s^* \cdot \omega_{si} + \sum_{k=1}^{K} p_{1s}^* z_{ki} r_{sk}$$

2. All markets clear; that is:

$$\sum_{i=1}^{l} z_{ki}^* \leq 0 \quad \text{and} \quad \sum_{i=1}^{l} x_{si}^* \leq \sum_{i=1}^{l} \omega_{si} \quad \text{for all } s \text{ and } k$$

- The definition is familiar, but there are two new objects: the optimal portfolio and the equilibrium asset prices.
- What can one say about them? What do we know about $q^*$?
Definition
An asset structure $R$ is complete if $\text{rank } R = S$.

Example
There are $S$ different Arrow securities; then

$$R = \begin{bmatrix}
1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & 1
\end{bmatrix}$$

which is the identity matrix and therefore has rank $S$.

Example
There are three states, so $S = 3$, and $R = \begin{bmatrix} 1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \end{bmatrix}$ which has full rank.
Completing Markets with Options

Spanning and Options

Consider the European Call Option on asset \( r = (r_1, r_2, r_3, r_4) \) at strike price \( c \).

- The return vector of this call option is
  \[
  r(c) = (\max\{0, r_1 - c\}, \ldots, \max\{0, r_4 - c\})
  \]
- Let \( r = (4, 3, 2, 1) \), consider three options given by \( r(3.5), r(2.5) \), and \( r(1.5) \).
- The return matrix given by these four assets
  \[
  R = \begin{bmatrix}
  4 & 0.5 & 1.5 & 2.5 \\
  3 & 0 & 0.5 & 1.5 \\
  2 & 0 & 0 & 0.5 \\
  1 & 0 & 0 & 0
  \end{bmatrix}
  \]
  which has full rank.
- So even if there is only one asset and four states (so that \( K < S \) which obviously would imply \( \text{rank } R < S \)), one can have complete markets by introducing three call options: derivatives can complete markets.
Remark

The crucial feature of an asset structure is the wealth it can generate.

- This is the set of all incomes that can be achieved constructing some portfolio: 
  \[ \text{Range } R = \{ w \in \mathbb{R}^S : w = Rz \text{ for some } z \in \mathbb{R}^K \} \]
- When two return matrices have the same range (i.e. they ‘span’ the same wealth levels) they yield the same equilibrium allocation.

Proposition

Suppose prices \( q^* \in \mathbb{R}^K \) and \( p^* \in \mathbb{R}^{LS} \) and plans \( x^* \in \mathbb{R}^{LSI} \) and \( z^* \in \mathbb{R}^{KI} \) constitute a Radner equilibrium for an asset structure with \( S \times K \) return matrix \( R \). Let \( \hat{R} \) be an \( S \times \hat{K} \) second asset returns structure. If Range \( R = \text{Range } \hat{R} \), then \( x^* \) is also an equilibrium allocation for the economy that has \( \hat{R} \) as its asset matrix.

- The proof has two steps.
  - First, show that the two budget sets corresponding to \( R \) and \( \hat{R} \) are the same.
  - Second, find portfolios \( z \) such that
    \[ \sum_i z_i = 0 \text{ and } m_i = \left[ p_{1i}^* \cdot (x_{1i}^* - \omega_{1i}), \ldots, p_{Si}^* \cdot (x_{Si}^* - \omega_{Si}) \right]^T = \hat{R}z_i \]

- This is question 1 in Problem Set 7
Remember, $i$’s budget set is:

$$B_i(p, q, R) = \left\{ x_i \in \mathbb{R}^{LS}_+ : \exists z_i \in \mathbb{R}^K \text{ such that } q \cdot z_i \leq 0 \right\}$$

where we normalized $p_1 = 1$.

The set of income levels that can be achieved using some portfolio is:

$$\text{Range } R = \{ w \in \mathbb{R}^S : w = Rz \text{ for some } z \in \mathbb{R}^K \}$$

This is a linear space.

By the previous proposition, if two return matrices have the same range they ‘span’ the same wealth levels and yield the same equilibrium allocation.
Proposition

Assume the asset markets are complete. Then

1. Suppose consumption plans $x^* \in \mathbb{R}^{LSI}$ and prices $p^* \in \mathbb{R}^{LS}_{++}$ constitute an Arrow-Debreu equilibrium. Then, there are asset prices $q^* \in \mathbb{R}^{K}_{++}$ and portfolio choices $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ such that: $z^*$, $q^*$, $x^*$, and spot prices $p_s^* \in \mathbb{R}^{S}_{++}$ for each $s$ form a Radner equilibrium.

2. Suppose consumption plans $x^* \in \mathbb{R}^{LSI}$, portfolio plans $z^* \in \mathbb{R}^{KI}$ and prices $q^* \in \mathbb{R}^{S}_{++}$ and spot prices $p_s^* \in \mathbb{R}^{S}_{++}$ for each $s$ constitute a Radner equilibrium. Then, there are $S$ strictly positive numbers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}^{S}_{++}$ such that the allocation $x^*$ and the state-contingent commodities price vector $(\mu_1 p_1^*, \ldots, \mu_S p_S^*) \in \mathbb{R}^{LS}$ form an Arrow Debreu Equilibrium.

- When asset markets are complete, agents are effectively unrestricted in the amounts of trades they can make across states. Then, the Radner equilibrium with assets is equivalent to an Arrow-Debreu equilibrium.
- Because we can always set $p_{1s} = 1$, each multiplier is interpreted as the value at time 0 of a dollar at time 1 in state $s$.
- Prove this as an exercise.
An arbitrage opportunity is the possibility to make strictly positive profits at no risk.

- Under what conditions on asset prices such opportunities do not exist?

**Definition**

Asset prices $\hat{q} \in \mathbb{R}^K$ satisfy absence of arbitrage if there does not exists a $z \in \mathbb{R}^K$ such that

$$\hat{q} \cdot z \leq 0, \quad Rz \geq 0 \quad \text{and} \quad Rz \neq 0.$$

- $\hat{q}$ are sometimes called no-arbitrage prices.

This can be interpreted as

- affordable portfolio
- strictly positive profits for sure

- No arbitrage means there does not exists an affordable portfolio that yields non-negative returns in all states and strictly positive returns in some state.

- Preferences are nowhere to be seen in this definition.
No arbitrage imposes restrictions on prices.

Let
\[ R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]
so that
\[ Rz = \begin{bmatrix} z_1 + z_2 + z_3 \\ z_1 + z_2 \\ z_1 \end{bmatrix} \]

Here, no arbitrage implies that if \( z_1 > 0 \) then \( q_1 > 0 \). Why?

- Can always pick \( z_2 \) and \( z_3 \) equal to zero.

This result generalizes beyond the example.
If $r_k \geq 0$ and $r_k \neq 0$ for all $k$ (all assets have non-negative, non-zero, returns), then no arbitrage implies $q_k > 0$ for all $k$.

Think of the portfolio $z$ defined as

$$z_k > 0 \text{ and } z_{k'} = 0 \text{ for all } k' \neq k.$$ 

The portfolio returns are

$$Rz = r_{sk} z_k \text{ for } s = 1, \ldots, S$$

this must be strictly positive in some state.

Therefore, the only way to avoid arbitrage is for the price of asset $k$ to be strictly positive.

Thus no arbitrage imposes

$$q \cdot z = q_k z_k > 0$$

since we already have $Rz \geq 0$ and $Rz \neq 0$. 
Theorem

Assume $r_k \geq 0$ and $r_k \neq 0$ for all $k$ (return vectors are nonnegative and non zero), and suppose preferences are strictly monotone. Then, any $q^* \in \mathbb{R}^K$ that is part of a Radner equilibrium satisfies no-arbitrage.

- Under simple restrictions on assets and preferences, equilibrium implies absence of arbitrage.
- Intuitively, arbitrage implies some money is left on the table.
- Given the efficiency properties of equilibrium, this should not happen.
- This is similar in spirit to the First Welfare Theorem.
We want to show that there cannot be arbitrage in equilibrium.

**Proof.**

Suppose not; then, $q^*$ is an equilibrium and there exists a portfolio $z$ such that

$q^* \cdot z \leq 0,$ \hspace{1cm} $Rz \geq 0,$ \hspace{1cm} and \hspace{1cm} $Rz \neq 0$

- Suppose an individual buys some amount of this portfolio:
  - it doesn’t cost her anything, since $q^* \cdot z \leq 0$;
  - given $Rz \geq 0$ and $Rz \neq 0$, this action strictly increases her ‘financial’ income (all columns of $R$ are positive and no entry is negative).

  Hence, she can relax her budget constraint.

  Because preferences are strictly monotone, she will increase her utility by using this additional income to buy more of some goods on the spot markets.

- This contradicts the assumption that $q^*$ is an equilibrium.
Theorem (Martingale Pricing)

Assume \( r_k \geq 0 \) and \( r_k \neq 0 \) for all \( k \) (return vectors are nonnegative and non zero), and suppose preferences are strictly monotone. Then, for any \( q^* \in \mathbb{R}^K \) that is part of a Radner equilibrium, we can find non-negative numbers \( \mu_s \geq 0 \) which satisfy

\[
[q^*]^T = \mu \cdot R
\]

or

\[
q_k^* = \sum_{s=1}^{S} \mu_s r_{sk} \quad \text{for all } k.
\]

What does this equality imply?

- Prices are a linear combination of dividends.
- Suppose there is an asset which pays a constant amount in all states: that is \( r_{sk} = c \) for all \( s \).
- Normalize the price of this asset to be \( c \), so that

\[
c = \sum_{s=1}^{S} \mu_s c \quad \Rightarrow \quad 1 = \sum_{s=1}^{S} \mu_s
\]

- Thus, equilibrium asset prices must equal an expected value of their returns.
- The ‘probability’ used to compute the expected value is the same for all assets.
Since we have already shown that equilibrium implies no-arbitrage, to prove the Martingale Pricing Theorem we only need to prove the following lemma.

**Lemma**

Suppose $r_k \geq 0$ and $r_k \neq 0$ for all $k$ (return vectors are nonnegative and non-zero). If asset prices $q \in \mathbb{R}^K$ satisfy the no-arbitrage condition, then there exists a vector $\mu \in \mathbb{R}^S_+$ such that

$$q^T = \mu \cdot R$$

The proof follows from convexity of the set of no arbitrage portfolios.

The main step uses the separating hyperplane theorem, and should remind you of the proof of the welfare theorems.

Separate the positive orthant from the set of incomes that can be obtained given a no-arbitrage (equilibrium) price vector.
Proof of Lemma: Convex Sets

- Assume no row of $R$ is zero (there is no state where all assets yield zero).
  - This is without loss of generality because if such a row existed, one can pick an arbitrary $\mu_s$ for that row.
- Pick a no arbitrage price vector $\hat{q} \in \mathbb{R}^K$
- Given $\hat{q}$, consider the following set
  \[ V = \{ v \in \mathbb{R}^S : v = Rz \text{ for some } z \in \mathbb{R}^K \text{ with } \hat{q} \cdot z = 0 \} \]
  - This is the set of ‘financial’ incomes attainable with ‘zero cost’ portfolios.
  - $V$ is a convex set (prove it as an exercise).
  - Note that if $v \in V$, then $-v \in V$
    - one can always ‘reverse’ all trades in $z$ and $\hat{q} \cdot (-z) = -(\hat{q} \cdot z) = 0$.
    - Furthermore, $0 \in V$.
- Since $\hat{q} \cdot z = 0$, no-arbitrage implies the financial income from $z$ cannot be strictly positive ($Rz \leq 0$).
- Thus, $V$ can intersect the $S$-dimensional non-negative orthant only at 0:
  \[ V \cap \{ \mathbb{R}_+^S \setminus \{0\} \} = \emptyset. \]
- Graph these sets.
Proof of Lemma: Separation

Since $V$ and $\{\mathbb{R}^S_+ \setminus \{0\}\}$ are convex, and the origin ($\{0\}$) belongs to $V$, we can apply the separating hyperplane theorem at $0$.

- By separating hyperplane theorem, there exists a nonzero $\mu' \in \mathbb{R}^S$ such that
  \[ \mu' \cdot v \leq 0 \quad \text{for each } v \in V \]
  and
  \[ \mu' \cdot v \geq 0 \quad \text{for each } v \in \mathbb{R}^S_+. \]

- Notice that $\mu' \geq 0$;
  - if not, $\mu' < 0$ and for some $v \in \mathbb{R}^S_+$ one would have $\mu' \cdot v < 0$ contradicting separation.

- Because $v$ and $-v$ are in $V$ we have
  \[ \mu' \cdot v \leq 0 \quad \text{and} \quad \mu' \cdot (-v) = -(\mu' \cdot v) \leq 0 \]

- Since $\mu' \geq 0$, these two inequalities imply $\mu' \cdot v = 0$. 
Proof of Lemma: Linear Pricing

- Consider the row vector $\mu' \cdot R \in \mathbb{R}^K$.

We need to show that $\hat{q}^T$ is proportional to $\mu' \cdot R$.

- Because $R$ has all nonnegative entries (and no zero row) and $\mu' \geq 0$, we have $\mu' \cdot R \geq 0^T$ and $\mu' \cdot R \neq 0^T$.

- Suppose $\hat{q}^T$ is not proportional to $\mu' \cdot R$ (there exists no real number $\alpha > 0$ such that $\hat{q}^T = \alpha(\mu' \cdot R)$) and find a contradiction.
  - If this is the case, there exists a $\bar{z}$ such that $\hat{q} \cdot \bar{z} = 0$ ($\bar{z}$ is orthogonal to $\hat{q}$) and $\mu' \cdot R\bar{z} > 0$.
  - Let $\bar{v} = R\bar{z}$; then $\bar{v} \in V$ and $\mu' \cdot \bar{v} > 0$ contradicting separation ($\mu' \cdot v = 0$ for each $v \in V$).

- Thus, $\hat{q}^T$ is proportional to $\mu' \cdot R$ and $\hat{q}^T = \alpha \mu' \cdot R$ for some real number $\alpha > 0$.

- If we let $\mu = \alpha \mu'$ we get the result $\hat{q}^T = \mu \cdot R$ concluding the proof of the Lemma.
We have proved the following

**Lemma**

Suppose $r_k \geq 0$ and $r_k \neq 0$ for all $k$ (return vectors are nonnegative and non zero). If asset prices $q \in \mathbb{R}^K$ satisfy the no-arbitrage condition, then there exists a vector $\mu \in \mathbb{R}_+^S$ such that

$$q^T = \mu \cdot R$$

Notice that all we need for linear pricing is no-arbitrage.

There are no restrictions on preferences for the Lemma!

The connection with equilibrium is made by adding strictly monotone preferences.
Implications of Linear Pricing

Suppose there is an asset that delivers one unit of good 1 in all states:

\[ r_1 = (1, \ldots, 1) \]

- Let \( q \) be a no-arbitrage price vector and normalize \( q_1 = 1 \).
- If \( \mu \) satisfies \( q^T = \mu \cdot R \), we have

\[
\mu = (\mu_1, \ldots, \mu_S) \geq 0 \quad \text{and} \quad \sum_{s=1}^{S} \mu_s = \mu \cdot r_1 = q_1 = 1
\]

- Hence, for all assets different from asset 1,

\[
q_k = \sum_{s=1}^{S} \mu_s r_{sk} \geq \min_s r_{sk} \quad \text{and} \quad q_k = \sum_{s=1}^{S} \mu_s r_{sk} \leq \max_s r_{sk}
\]

- Intuitively, prices must be between the lowest and highest dividend.