Afriat’s Theorem and Generalized Axiom of Revealed Preference; Production

Econ 2100  Fall 2015

Lecture 8, September 28

Outline

1. Finite Data
2. Generalized Axiom of Revealed Preferences
3. Afriat’s Theorem
4. Production Sets and Production Functions
5. Profits Maximization, Supply Correspondence, and Profit Function
Data and Rationality: Motivation

Main Idea
- We observe data and want to know if it could have been the result of maximizing a preference relation or a utility function.

Demand data observations
- We observe \( N \) choices made by one consumer.
- These are:
  \[
  x^1, p^1, w^1 \quad x^2, p^2, w^2 \quad x^3, p^3, w^3 \quad \ldots \quad x^N, p^N, w^N
  \]
  with the properties that:
  - \((x^j, p^j, w^j) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^n \times \mathbb{R}_+\) for all \( j = 1, \ldots, N \); and
  - \( p^j \cdot x^j \leq w^j \) for all \( j = 1, \ldots, N \).
- These are finitely many observations.

- What conditions must these observations satisfy for us to conclude they are the result of the maximizing a preference relation or a utility function?
- Something similar to, but stronger than, revealed preference.
An Example (from Kreps)

- Suppose we observe the following choices among 3 goods

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<tr>
<td>1</td>
<td>300</td>
<td>(10, 10, 10)</td>
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<tr>
<td>2</td>
<td>130</td>
<td>(10, 1, 2)</td>
<td>(9, 25, 7.5)</td>
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<td>3</td>
<td>110</td>
<td>(1, 1, 10)</td>
<td>(15, 5, 9)</td>
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- Are these choices consistent with our theory? Is there a preference/utility function that generates these choices?
  - Sure: suppose the consumer strictly prefers \((500, 500, 500)\) to everything, and is indifferent among all other bundles.
  - Since \((500, 500, 500)\) is never affordable, any other choice is rationalizable.
    - This seems silly, and not something that should worry us since we can never rule it out. We want at least local non-satiation.

- Observations:
  - The consumer spends all her income in all cases.
  - At prices \((10, 10, 10)\) the bundle \((15, 5, 9)\) could have been chosen.
  - At prices \((10, 1, 2)\) the bundle \((10, 10, 10)\) could have been chosen.
  - At prices \((1, 1, 10)\) the bundle \((9, 25, 7.5)\) could have been chosen.
Consequences of Local Non Satiation

The following is slightly different from our previous result.

Lemma

Suppose $\succsim$ is locally non-satiated, and let $x^*$ be an element of Walrasian demand so that

$$x^* \succsim x \quad \text{for all } x \in \{x \in X : p \cdot x \leq w\}.$$ 

Then

$$x^* \succsim x \quad \text{when } p \cdot x = w$$

and

$$x^* \succ x \quad \text{when } p \cdot x < w$$

The maximal bundle is weakly preferred to any bundle that costs the same.
The maximal bundle is strictly preferred to any bundle that costs less.

Proof.

The first part is immediate.
For the second part note that if $p \cdot x < w$ by local non satiation (and continuity of $p \cdot x$) there exists some $x'$ such that $x' \succ x$ and $p \cdot x' \leq w$.
Thus $x'$ is affordable and $x^* \succsim x' \succ x$ as desired.
Example Continued

Consumer’s choices among 3 goods

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<tr>
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<th>w</th>
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- Using local non-satiation we can conclude the following.

- At prices (10, 10, 10), (15, 5, 9) costs less than (10, 10, 10).
  - Hence (10, 10, 10) ∼ (15, 5, 9).

- At prices (10, 1, 2), (9, 25, 7.5) costs as much as (10, 10, 10).
  - Hence (9, 25, 7.5) ∼ (10, 10, 10).

- At prices (1, 1, 10), (9, 25, 7.5) costs less than (15, 5, 9).
  - Hence (15, 5, 9) ∼ (9, 25, 7.5).

- Putting these together:
  
  (10, 10, 10) ∼ (15, 5, 9) ∼ (9, 25, 7.5) ∼ (10, 10, 10)

- These data are not consistent with our utility/preference maximization model: they violate transitivity.
Directly Revealed Preference

Definitions
Suppose \( \{x^j, p^j, w^j\}_{j=1}^N \) is a finite set of demand data that satisfy \( p^j \cdot x^j \leq w^j \).

- The directly revealed weak preference is defined by
  \[
  x^j \succeq^R x^k \quad \text{if} \quad p^j \cdot x^k \leq w^j
  \]

- The directly revealed strict preference is defined by
  \[
  x^j \succ^R x^k \quad \text{if} \quad p^j \cdot x^k < w^j
  \]

These are similar to the revealed preference given by a choice rule:

- if \( x^k \) was affordable, but the consumer chose \( x^j \) instead, she revealed that \( x^j \) is at least as good a choice as \( x^k \).
- if \( x^k \) was strictly affordable, but the consumer chose \( x^j \) instead, she revealed that \( x^j \) is a strictly better choice than \( x^k \).

Remark: The strict relation is derived from observation

- \( \succ^R \) is defined directly; it does not necessarily correspond to the asymmetric component of \( \succeq^R \).
Indirectly Revealed Preference

Definitions

Suppose $\{x^j, p^j, w^j\}_{j=1}^N$ is a finite set of demand data that satisfy $p^j \cdot x^j \leq w^j$.

- The **indirectly revealed weak preference** is defined by

$$x^j \preceq R x^i_1, \quad x^i_1 \preceq R x^i_2, \quad \ldots \quad x^i_m \preceq R x^k$$

if there exists

$$x^j \preceq I x^i_1, x^i_2, \ldots, x^i_m \quad \ldots \quad x^i_m \preceq R x^k$$

such that

- The **indirectly revealed strict preference** is defined by

$$x^j \succeq R x^i_1, \quad x^i_1 \succeq R x^i_2, \quad \ldots$$

if there exists

$$x^j \succeq I x^i_1, x^i_2, \ldots, x^i_m \quad \ldots$$

such that

$$x^i_m \preceq R x^k$$

and one of the relations in the chain is strict

- $\preceq I$ is also defined directly from observation, and maybe different than the asymmetric component of $\preceq I$.

Exercise

Verify that $x^j \succeq R x^k$ implies $x^j \succeq I x^k$.
Verify that if $x^j \succeq R x^k$ and $x^k \succeq I x^i$, then $x^j \succeq I x^i$. 
Generalized Axiom of Revealed Preference

Axiom (Generalized Axiom of Revealed Preference - GARP)

If $x^j \preceq^R x^k$, then $\not\exists [x^k \succ^I x^j]$.

- If a choice is directly revealed weakly preferred to another, then this other possibility cannot be indirectly revealed strictly preferred to the original choice.
- Note that if $p^j \cdot x^j < w^j$ then $x^j \preceq^R x^j$ and therefore $x^j \succ^I x^j$ and therefore GARP does not hold.
  - GARP implies the consumer spends all her money.
- Now suppose $x^j \preceq^R x^k$ and $x^k \succ^I x^j$. Then there is a chain of weak preference from $j$ to $k$, and a chain back (from $k$ to $j$) that has at least one strict preference.
  - This is a cycle with a strict preference inside; if such a cycle exists one can construct a “money pump” to make money off the consumer and leave her exactly at the bundle she started from.
  - GARP rules these cycles out.

Problem 1, Problem Set 5.

Show that GARP is equivalent to the following: $\text{If } x^j \preceq^I x^k \text{ then } \not\exists [x^k \succ^I x^j]$.
Utility and Demand Data

**Definition**

A utility function $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ rationalizes a finite set of demand data $\{x^j, p^j, w^j\}_{j=1}^N$ if

$$u(x^j) \geq u(x) \quad \text{whenever} \quad p^j \cdot x \leq w^j$$

- This definition can alternatively be stated by the condition

  $$x^j \in \arg \max_{x \in \mathbb{R}^n_+} u(x) \text{ subject to } p^j \cdot x \leq w^j$$

- If the observations satisfy the definition, the data is consistent with the behavior of an individual who, in each observation, chooses a utility-maximizing bundle subject to the corresponding budget constraint.
Afriat’s Theorem

**Theorem (Afriat)**

Given a finite set of demand data \(\{(x^j, p^j, w^j)\}_{k=1}^N\), the following are equivalent:

1. There exists a locally nonsatiated utility function which rationalizes the data.
2. The data satisfy the Generalized Axiom of Revealed Preference.
3. There exist numbers \(\nu^j, \lambda^j > 0\) such that
   \[
   \nu^k \leq \nu^j + \lambda^j [p^j \cdot x^k - w^j], \quad \text{for all } j, k = 1, \ldots, N \quad \text{(G)}
   \]
4. There exists a continuous, strictly increasing, and concave utility function which rationalizes the data.

For finite data, the only implication of preference maximization is GARP; conversely, if the data violate GARP, we conclude the consumer is not maximizing any continuous, strictly monotone, and convex preference relation.

Using a finite number of observations, one cannot distinguish a continuous, strictly monotone, and convex preference from a locally nonsatiated one.

One way to test if demand data falsify utility maximization is to see if one can find the numbers in (G).

These numbers help us construct a function that rationalizes data: each observation has “utils” \(\nu^j\) and a “budget violating penalty/prize” \(\lambda^j\); every choice is rational given these.
Afriat’s Theorem

**Theorem (Afriat)**

Given a finite set of demand data \( \{ (x_j^i, p_j^i, w_j^i) \}_{k=1}^N \), the following are equivalent:

1. There exists a locally nonsatiated utility function which rationalizes the data.
2. The data satisfy the Generalized Axiom of Revealed Preference.
3. There exist numbers \( v_j^i, \lambda_j^i > 0 \) such that
   \[
   v_k^i \leq v_j^i + \lambda_j^i [p_j^i \cdot x_k^i - w_j^i], \quad \text{for all } j, k = 1, \ldots, N \tag{G}
   \]
4. There exists a continuous, strictly increasing, and concave utility function which rationalizes the data.

**Proof Strategy**

- The fact (4) implies (1) is immediate (strictly increasing implies locally non-satiated).
- (1) implies (2) is the second exercise of Problem Set 5.
- We will skip the remainder of the proof, but see Kreps or Varian for it.
The Afriat Numbers

\[ \exists v^j, \lambda^j > 0 \text{ s.t. } v^k \leq v^j + \lambda^j [p^j \cdot x^k - w^j] \text{ for all } j, k = 1, \ldots, N \quad (G) \]

- One uses the numbers \( v^k \) and \( \lambda^k \) to construct a function that rationalizes the data;
  - the theorem then says that every choice is rational.

Where are the Afriat’s numbers coming from?

- Consider utility maximization subject to a budget constraint:
  \[ x^* \in \arg \max_{x \in \mathbb{R}^n_+} u(x) \text{ s.t. } p \cdot x \leq w. \]
  - The solution maximizes the Lagrangean:
    \[ L(x, \lambda) = u(x) + \lambda (w - p \cdot x). \]
    - Let \( x^* \) and \( \lambda^* \) be the optimal consumption and multiplier values, respectively.
    - Then we have \( L(x^*, \lambda^*) \geq L(x, \lambda^*) \) for all \( x \in \mathbb{R}^n_+ \). That is,
      \[ u(x^*) + \lambda^* \left( w - p \cdot x^* \right) \geq u(x) + \lambda^* (w - p \cdot x). \]
      - \( = 0 \) since budget constraint binds
    - Thus for all \( x \in \mathbb{R}^n_+ \)
      \[ u(x) \leq u(x^*) + \lambda^* (p \cdot x - w) \]
Intuition Behind the Afriat Numbers

\[ \exists v^i, \lambda^i > 0 \text{ s.t. } v^k \leq v^j + \lambda^j[p^j \cdot x^k - w^j] \text{ for all } j, k = 1, \ldots, N \quad (G) \]

Another intuition behind Afriat’s numbers

- Rewrite (G) as
  \[ v^k - v^j \leq \lambda^j[p^j \cdot x^k - w^j]. \]
- Suppose that \( p^j \cdot x^k - w^j > 0 \). Then \( x^k \) was unaffordable at \( p^j \) and \( w^j \).
  - Even if \( x^k \) yields more “utils” than \( x^j \), at \( p^j \) and \( w^j \) the penalty for violating the budget constraint (given by \( \lambda^j[p^j \cdot x^k - w^j] \)) is too large for the change in the value of the choice (given by \( v^k - v^j \)) to compensate for it.
  - Thus, the decision maker chose \( x^j \) over all other alternatives at \( p^j \) and \( w^j \).
- Suppose that \( p^j \cdot x^k - w^j \leq 0 \). Then \( x^k \) was affordable at \( p^j \) and \( w^j \).
  - But then (G) implies that
    \[ v^j - v^k \geq \lambda^j[w^j - p^j \cdot x^k]. \]
  - So the decision maker chose \( x^j \) over \( x^k \) because the difference in “utils” is higher than the bonus from having some extra money left over by choosing \( x^k \) instead of \( x^j \).
Afriat’s Theorem

Summary

- For finite data sets, preference maximization is equivalent to GARP.
- With GARP, we can find utility values for every data point (Afriat numbers).
- The decision maker behaved as if each chosen data point was optimal (i.e. utility-maximizing).
- The theorem also gives us a constructive proof of the convexity of the upper contour sets of any element in the data set.
  - These are convex since they are the intersections of half spaces (this comes from the proof that (3) implies (4)); however, the utility representation is concave.
    - In general convex upper contour sets are associated with convex preferences and quasiconcave utility.
    - Therefore using finite demand data one cannot test convexity of preferences.
- Using a finite number of observations we cannot distinguish a consumer with continuous, strictly monotone, and convex preferences from a consumer with locally nonsatiated preference (the extra assumptions have absolutely no bite for finite data sets).
Producers and Production Sets

Producers are profit maximizing firms that buy inputs and use them to produce and then sell outputs.

The plural is important because most firms produce more than one good.

The standard textbook description in intermediate microeconomics focuses on one output and a few inputs (two in most cases).

Usually, you may have seen production described by a function that takes inputs as the domain and output(s) as the range.

Here we focus on a more general and abstract version of this same idea.

**Definition**

A production set is a subset \( Y \subseteq \mathbb{R}^n \).

- Negative numbers denote inputs.
- What is \( p \cdot y \)?
Production Set Properties

**Definition**

\( Y \subseteq \mathbb{R}^n \) satisfies:

- **no free lunch** if \( Y \cap \mathbb{R}_+^n \subseteq \{0_n\} \);
- **possibility of inaction** if \( 0_n \in Y \);
- **free disposal** if \( y \in Y \) implies \( y' \in Y \) for all \( y' \leq y \);
- **irreversibility** if \( y \in Y \) and \( y \neq 0_n \) imply \(-y \notin Y \);
- **nonincreasing returns to scale** if \( y \in Y \) implies \( \alpha y \in Y \) for all \( \alpha \in [0, 1] \);
- **nondecreasing returns to scale** if \( y \in Y \) implies \( \alpha y \in Y \) for all \( \alpha \geq 1 \);
- **constant returns to scale** if \( y \in Y \) implies \( \alpha y \in Y \) for all \( \alpha \geq 0 \);
- **additivity** if \( y, y' \in Y \) imply \( y + y' \in Y \);
- **convexity** if \( Y \) is convex;
- \( Y \) is a **convex cone** if for any \( y, y' \in Y \) and \( \alpha, \beta \geq 0 \), \( \alpha y + \beta y' \in Y \).

Draw Pictures.
Some of these properties are related to each other.

**Exercise**

$Y$ satisfies additivity and nonincreasing returns if and only if it is a convex cone.

**Exercise**

For any convex $Y \subset \mathbb{R}^n$ such that $0_n \in Y$, there is a convex $Y' \subset \mathbb{R}^{n+1}$ that satisfies constant returns such that $Y = \{y \in \mathbb{R}^n : (y, -1) \in \mathbb{R}^{n+1}\}$. 
Production Functions

Let $y \in \mathbb{R}^m_+$ denote outputs while $x \in \mathbb{R}^n_+$ represent inputs; if the two are related by a function $f : \mathbb{R}^n_+ \to \mathbb{R}^m_+$, we write $y = f(x)$ to say that $y$ units of outputs are produced using $x$ amount of the inputs.

- When $m = 1$, this is the familiar one output many inputs production function.
- Production sets and the familiar production function are related.

Exercise

Suppose the firm’s production set is generated by a production function $f : \mathbb{R}^n_+ \to \mathbb{R}^m_+$, where $\mathbb{R}^n_+$ represents its $n$ inputs and $\mathbb{R}^m_+$ represents its $m$ outputs. Let

$$Y = \{(-x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ : y \leq f(x)\}.$$ 

Prove the following:

1. $Y$ satisfies no free lunch, possibility of inaction, free disposal, and irreversibility.
2. Suppose $m = 1$. $Y$ satisfies constant returns to scale if and only if $f$ is homogeneous of degree one, i.e. $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$.
3. Suppose $m = 1$. $Y$ satisfies convexity if and only if $f$ is concave.
Transformation Function

- We can describe production sets using a function.

**Definition**

Given a production set \( Y \subseteq \mathbb{R}^n \), the **transformation function** \( F : Y \to \mathbb{R} \) is defined by

\[
Y = \{ y \in Y : F(y) \leq 0 \text{ and } F(y) = 0 \text{ if and only if } y \text{ is on the boundary of } Y \};
\]

the **transformation frontier** is

\[
\{ y \in \mathbb{R}^n : F(y) = 0 \}
\]

**Definition**

Given a differentiable transformation function \( F \) and a point on its transformation frontier \( y \), the **marginal rate of transformation** for goods \( i \) and \( j \) is given by

\[
MRT_{i,j} = \frac{\frac{\partial F(y)}{\partial y_i}}{\frac{\partial F(y)}{\partial y_j}}
\]

- Since \( F(y) = 0 \) we have

\[
\frac{\partial F(y)}{\partial y_i} dy_i + \frac{\partial F(y)}{\partial y_j} dy_j = 0
\]

- MRT is the slope of the transformation frontier at \( y \).
Profit Maximization

Profit Maximizing Assumption

The firm’s objective is to choose a production vector on the transformation frontier as to maximize profits given prices \( p \in \mathbb{R}^n_+ \):

\[
\max_{y \in Y} \quad p \cdot y
\]

or equivalently

\[
\max_{y} \quad p \cdot y \quad \text{subject to} \quad F (y) \leq 0
\]

- Does this distinguish between revenues and costs? How?
- Using the single output production function:

\[
\max_{x \geq 0} \quad pf (x) - w \cdot x
\]

- here \( p \in \mathbb{R}^l_+ \) is the price of output and \( w \in \mathbb{R}^l_+ \) is the price of inputs.
First Order Conditions For Profit Maximization

\[
\max p \cdot y \quad \text{subject to} \quad F(y) = 0
\]

Profit Maximizing

- The FOC are
  \[ p_i = \lambda \frac{\partial F(y)}{\partial y_i} \quad \text{for each} \quad i \]
  or
  \[ p = \lambda \nabla F(y) \] in matrix form

- Therefore
  \[
  \frac{1}{\lambda} = \frac{\frac{\partial F(y)}{\partial y_i}}{p_i} \quad \text{for each} \quad i
  \]
  the marginal product per dollar spent or received is equal across all goods.

- Using this formula for \( i \) and \( j \):
  \[
  \frac{\frac{\partial F(y)}{\partial y_i}}{\frac{\partial F(y)}{\partial y_j}} = MRT_{i,j} = \frac{p_i}{p_j} \quad \text{for each} \quad i, j
  \]
  the Marginal Rate of Transformation equals the price ratio.
Supply Correspondence and Profit Functions

**Definition**

Given a production set \( Y \subseteq \mathbb{R}^n \), the supply correspondence \( y^* : \mathbb{R}_{++}^n \rightarrow \mathbb{R}^n \) is:

\[
y^*(p) = \arg \max_{y \in Y} p \cdot y.
\]

- Tracks the optimal choice as prices change (similar to Walrasian demand).

**Definition**

Given a production set \( Y \subseteq \mathbb{R}^n \), the profit function \( \pi : \mathbb{R}_{++}^n \rightarrow \mathbb{R} \) is:

\[
\pi(p) = \max_{y \in Y} p \cdot y.
\]

- Tracks maximized profits as prices change (similar to indirect utility function).

**Proposition**

If \( Y \) satisfies non-decreasing returns to scale either \( \pi(p) \leq 0 \) or \( \pi(p) = +\infty \).

**Proof.**

Question 4, Problem Set 5.
1. Show that GARP is equivalent to the following: If $x^j \succeq^I x^k$ then not $[x^k \succ^I x^j]$.

2. Suppose $\{(x^j, p^j, w^j)\}^{N}_{k=1}$ is a finite set of demand data. Prove that if there exists a locally nonsatiated utility function which rationalizes the data, then the data satisfy the Generalized Axiom of Revealed Preference.

3. Consider the following data set of four demand observations for two commodities.

```
x   p   w
1   (3, 9) (3, 3) 36
2   (12, 1) (1, 8) 20
3   (4, 2) (2, 3) 14
4   (1, 1) (4, 4)  8
```

Find $\succ^R$, $\succeq^R$, $\succ^I$, and $\succeq^I$ for these observations. Check that the data satisfy GARP.

4. Prove that if $Y$ satisfies non decreasing returns to scale either $\pi(p) \leq 0$ or $\pi(p) = +\infty$. 