# Monotone Comparative Statics
# Finite Data and GARP

Econ 2100 Fall 2018

Lecture 7, September 19

- Problem Set 3 was due by 10:30am today

Outline

1. Comparative Statics Without Calculus
2. Supermodularity and Single Crossing
3. Topkis and Milgrom&Shannon's Theorems
4. Finite Data

# Comparative Statics Without Calculus

**Remark**

- Let $$x^*(q) = \arg\max f(x,q), \text{ subject to } q \in \Theta,\ x \in S(q)$$
  Using the implicit function theorem, one can show that if there are complementarities between the choice variable $x$ and the parameter $q$, the optimum increases in $q$.

**First Order Condition**: $f_x(x;q) = 0$. **Second Order Condition**: $f_{xx}(x;q) < 0$.
**IFT**:
$$x_q^*(q) = -\frac{f_{xq}(x;q)}{f_{xx}(x;q)}.$$
*Then*
$$x_q^*(q) \geq 0 \textit{ if and only if } f_{xq}(x;q) \geq 0$$

Here $g.$ denotes the derivative of $g$ with respect to $\cdot$.

**Issues with implicit function theorem:**

1. IFT needs calculus.
2. Conclusions hold only in a neighborhood of the optimum.
3. Results are dependent on the functional form used for the objective function.
    1. In particular, IFT gives cardinal results that depend on the assumptions on $f$.

# Monotone Comparative Statics

## Objectives

- With monotone comparative statics, we have results about "changes" that:
  - do not need calculus;
  - are not necessarily only local (around the optimum);
  - are ordinal in the sense of being robust to monotonic transformations.

- One can get conclusions similar to IFT without calculus.
- The downside is that the results are not as strong.

## Main Idea: Complementarities

- The central idea generalizes the notion of complementarities between endogenous variable and parameters.
  - With calculus, this is the assumption $f_{xq}(x; q) \geq 0$.

- We would also like to account for the possibility that the optimum is not unique, so that $x^*(q)$ is not a function.
- First, what does it mean for a correspondence to be increasing?

# Strong Set Order

- Ranking real numbers is easy, but how can we express the idea that one set is bigger than another set?

**Definition**

For two sets of real numbers $A$ and $B$, define the binary relation $\geq_s$ as follows:

$$\boxed{\text{A}\geq_s\text{B} \quad \text{if} \quad \begin{array}{c} \text{for any } a \in A \text{ and } b \in B \\ \min\{a,b\} \in B \quad \text{and} \quad \max\{a,b\} \in A \end{array}}$$

$A \geq_s B$ reads "$A$ is greater than or equal to $B$ in the strong set order".

- Generalizes the notion of greater than from numbers to sets of numbers.
- This definition reduces to the standard definition when sets are singletons.

**Example**

Suppose $A = \{1, 3\}$ and $B = \{0, 2\}$. Then, $A$ is not greater than or equal to $B$ in the strong set order.

# Non-Decreasing Correspondences

**Definition**

We say a correspondence $g : \mathbf{R}^n \to 2^{\mathbf{R}}$ is non-decreasing in $q$ if and only if

$$x' > x \qquad \text{implies} \qquad g(x') \geq_s g(x)$$

- Thus, $x' > x$ implies that for any $y' \in g(x')$ and $y \in g(x)$: $\min\{y', y\} \in g(x)$ and $\max\{y', y\} \in g(x')$.
  - Larger points in the domain correspond to larger sets in the codomain.
- Generalizes the notion of increasing function to correspondences.

**Exercise**

Prove that if $g(\cdot)$ is non-decreasing and $\min g(x)$ exists for all $x$, then $\min g(x)$ is non-decreasing.

**Exercise**

Prove that if $g(\cdot)$ is non-decreasing and $\max g(x)$ exists for all $x$, then $\max g(x)$ is non-decreasing.

# Monotone Comparative Statics: Simplest Case

## Set up

- Suppose some function $f : \mathbf{R}^2 \to \mathbf{R}$ is the objective function; this is not necessarily concave or differentiable, and the optimizer could be set valued.
- Let
$$x^*(q) = \arg\max f(x,q), \text{ subject to } q \in \Theta; x \in S(q)$$
  - Note: for any strictly increasing $h$, this problem is equivalent to
$$x^*(q) = \arg\max h(f(x,q)), \text{ subject to } q \in \Theta; x \in S(q)$$
  - $h(\cdot)$ may destroy smoothness or concavity properties of the objective function so IFT may not work.
- For now, assume $S(\cdot)$ is independent of $q$ (ignore the constraints), and that both $x$ and $q$ are real variables.
- Assume existence of a solution, but not uniqueness.

# Supermodularity

**Definition**

The function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ is supermodular in $(x; q)$ if

$$\text{for all } x' > x \qquad f(x'; q) - f(x; q) \text{ is non-decreasing in } q.$$

- If $f$ is supermodular in $(x; q)$, then the incremental gain to a higher $x$ is greater when $q$ is higher.
- This is the idea that $x$ and $q$ are "complements".

**Question 1, Problem Set 4.**

Show that supermodularity is equivalent to the property that

$$q' > q \qquad \text{implies} \qquad f(x; q') - f(x; q) \text{ is non-decreasing in } x.$$

# Differentiable Version of Supermodularity

- When $f$ is smooth, supermodularity has a characterization in terms of derivatives.

**Lemma**

*A twice continuously differentiable function* $f : \mathbf{R}^2 \to \mathbf{R}$ *is supermodular in* $(x; q)$ *if and only if* $D_{xq} f(x; q) \geq 0$ *for all* $(x; q)$.

- The inequality in the definition of supermodularity is just the discrete version of the mixed-partial condition in the lemma.
  - $q' > q$ implies $f(x; q') - f(x; q)$ is non-decreasing in $x$

# Topkis' Monotonicity Theorem

**Theorem (Easy Topkis' Monotonicity Theorem)**

*If $f$ is supermodular in $(x;q)$, then $x^*(q) = \arg\max f(x,q)$ is non-decreasing.*

**Proof.**

Let $q' > q$ and take $x \in x^*(q)$ and $x' \in x^*(q')$. Show that $x^*(q') \geq_s x^*(q)$.

- First show that $\max\{x, x'\} \in x^*(q')$
  - $x \in x^*(q)$ implies $f(x;q) - f(\min\{x,x'\};q) \geq 0$.
  - $x \in x^*(q)$ also implies that $f(\max\{x,x'\};q) - f(x';q) \geq 0$
    - verify these by checking two cases, $x > x'$ and $x' > x$.
  - By supermodularity, $f(\max\{x,x'\};q') - f(x';q') \geq 0$,
  - Hence $\max\{x,x'\} \in x^*(q')$.
- Now show that $\min\{x,x'\} \in x^*(q)$
  - $\max\{x,x'\} \in x^*(q')$ implies that $f(x';q') - f(\max\{x,x'\},q) \geq 0$,
    - or equivalently $f(\max\{x,x'\},q) - f(x';q') \leq 0$.
  - $\max\{x,x'\} \in x^*(q')$ also implies that $f(\max\{x,x'\};q') - f(x';q') \geq 0$,
  - which by supermodularity implies $f(x;q) - f(\min\{x,x'\};q) \leq 0$
  - Hence $\min\{x,x'\} \in x^*(q)$.

□

# Topkis' Monotonicity Theorem

**Theorem (Easy Topkis' Monotonicity Theorem)**

*If $f$ is supermodular in $(x; q)$, then $x^*(q) = \arg\max f(x, q)$ is non-decreasing.*

- Supermodularity is sufficient to draw comparative statics conclusions in optimization problems without other assumptons.
- Topkis' Theorem follows from the IFT whenever the standard full-rank condition in the IFT holds.
  - At a maximum, if $D_{xx} f(x^*, q) \neq 0$, it must be negative (by the second-order condition), hence the IFT tells you that $x^*(q)$ is strictly increasing.

# Example

## Profit Maximization Without Calculus

- A monopolist chooses output $q$ to solve $\max p(q)q - c(q,\mu)$.
  - $p(\cdot)$ is the demand (price) function
  - $c(\cdot)$ is the cost function
    - costs depend on the existing technology, described by some parameter $\mu$.
- Let $q^*(\mu)$ be the monopolist's optimal quantity.
- Suppose $-c(q,\mu)$ is supermodular in $(q,\mu)$; then the entire objective function is also supermodular in $(q,\mu)$.
  - this follows because the first term of the objective does not depend on $\mu$.
- Notice that supermodularity says that for all $q' > q$, $-c(q';\mu) + c(q;\mu)$ is nondecreasing in $\mu$.
  - in other words, the marginal cost is decreasing in $\mu$.
- Conclusion: by Topkis' theorem $q^*$ is nondecreasing as long as the marginal cost of production decreases in the technological progress parameter $\mu$.

# Single-Crossing

- In constrained maximization problems, $x \in S(q)$, supermodularity is not enough for Topkis' theorem.

**Definition**

The function $f : \mathbf{R}^2 \to \mathbf{R}$ satisfies the single-crossing condition in $(x; q)$ if for all $x' > x$, $q' > q$

$$f(x'; q) - f(x; q) \geq 0 \qquad \text{implies} \qquad f(x'; q') - f(x; q') \geq 0$$

and

$$f(x'; q) - f(x; q) > 0 \qquad \text{implies} \qquad f(x'; q') - f(x; q') > 0.$$

- As a function of the second argument, the 'marginal return' can cross 0 at most once; whenever it crosses 0, as the second argument continues to increase, the marginal return will remain positive.

**Theorem**

*If $f$ satisfies single crossing in $(x; q)$, then $x^*(q) = \arg\max_{x \in S(q)} f(x; q)$ is nondecreasing. Moreover, if $x^*(q)$ is nondecreasing in $q$ for all constraint choice sets $S$, then $f$ satisfies single-crossing in $(x; q)$.*

# Monotone Comparative Statics

## $n$-dimensional choice variable and $m$-dimensional parameter vector

- Next, we generalize to higher dimensions.

**Definitions**

Suppose $x, y \in \mathbf{R}^n$.

- The join of $x$ and $y$ is defined by
$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}).$$
- The meet of $x$ and $y$ is defined by
$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

- Draw a picture.

# Strong Set Order

- We generalize the strong set order definition to $\mathbf{R}^n$.

**Definition (Strong set order in $\mathbf{R}^n$)**

The binary relation $\geq_s$ is defined as follows: for $A, B \subset \mathbf{R}^n$,

$$\boxed{A \geq_s B \qquad \text{if} \qquad \begin{array}{c} \text{for any } a \in A \text{ and } b \in B \\ \\ a \wedge b \in B \quad \text{and} \quad a \vee b \in A \end{array}}$$

- The meet is in the smaller set, while the join is in the larger set.
- One uses this to talk about non-decreasing $\mathbf{R}^n$-valued correspondences.

- We look at functions $f(\mathbf{x}; \mathbf{q})$ where the first argument represents the endogenous variables and the second represents the parameters.

# Quasi-Supermodularity

## Definition

The function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is quasi-supermodular in its first argument if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\mathbf{q} \in \mathbf{R}^m$:

1. $$f(\mathbf{x}; \mathbf{q}) \geq f(\mathbf{x} \wedge \mathbf{y}; \mathbf{q}) \quad \Rightarrow \quad f(\mathbf{x} \vee \mathbf{y}; \mathbf{q}) \geq f(\mathbf{y}; \mathbf{q});$$
2. $$f(\mathbf{x}; \mathbf{q}) > f(\mathbf{x} \wedge \mathbf{y}; \mathbf{q}) \quad \Rightarrow \quad f(\mathbf{x} \vee \mathbf{y}; \mathbf{q}) > f(\mathbf{y}; \mathbf{q}).$$

- This generalizes the 'mixed' second partial derivatives typically used to make statements about complementarities.
- Quasi-supermodularity is an ordinal property; for differentiable functions there is a sufficient condition for quasi-supermodularity.

## Exercise

1. Prove that if $f$ is quasi-supermodular in $x$, then $h \circ f$ is quasi-supermodular in $x$ for any strictly increasing $h : f(\mathbf{R}^n \times \mathbf{R}^m) \to \mathbf{R}$.
2. Suppose $f(\mathbf{x}; \mathbf{q})$ is twice differentiable in $x$ and $\frac{\partial^2 f}{\partial x_i x_j} > 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Then $f$ is quasi-supermodular in $\mathbf{x}$.

# Single-Crossing Property

**Definition**

The function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ satisfies the single-crossing property if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\mathbf{q}, \mathbf{r} \in \mathbf{R}^m$ such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{q} \geq \mathbf{r}$:

1. $$f(\mathbf{x}; \mathbf{r}) \geq f(\mathbf{y}; \mathbf{r}) \quad \Rightarrow \quad f(\mathbf{x}; \mathbf{q}) \geq f(\mathbf{y}; \mathbf{q});$$
2. $$f(\mathbf{x}; \mathbf{r}) > f(\mathbf{y}; \mathbf{r}) \quad \Rightarrow \quad f(\mathbf{x}; \mathbf{q}) > f(\mathbf{y}; \mathbf{q}).$$

- The "marginal return" $f(\mathbf{x}; \cdot) - f(\mathbf{y}; \cdot)$ as a function of the second argument can cross 0 at most once.
- The single-crossing property is an ordinal property; for differentiable functions there is a sufficient condition for single-crossing.

**Exercise**

1. Prove that if $f$ satisfies the single-crossing property, then $h \circ f$ satisfies the single-crossing property for any strictly increasing $h : f(\mathbf{R}^n \times \mathbf{R}^m) \to \mathbf{R}$.
2. Suppose $f(\mathbf{x}; \mathbf{q})$ is twice differentiable and $\frac{\partial^2 f}{\partial x_i x_j} > 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then $f$ satisfies the single-crossing property.

# Monotone Comparative Statics

**Theorem (easy Milgrom and Shannon)**

*Let* $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$. *Define* $x^*(\mathbf{q}) = \arg\max_{x \in R^n} f(\mathbf{x}; \mathbf{q})$. *Suppose* $|x^*(\mathbf{q})| = 1$ *for all* $\mathbf{q}$ *and* $f(\mathbf{x}; \mathbf{q})$ *is quasi-supermodular in its first argument and satisfies the single-crossing property. Then*

$$\mathbf{q} \geq \mathbf{r} \Rightarrow x^*(\mathbf{q}) \geq x^*(\mathbf{r}).$$

- 'Easy' because it assumes the optimum is unique (thus, the proof does not use 'strict' quasi-supermodularity and single-crossing).

**Proof.**

Suppose $\mathbf{q} \geq \mathbf{r}$. Then:

$$\begin{array}{rll}
& f\left(x^*(\mathbf{r}); \mathbf{r}\right) \geq f\left(x^*(\mathbf{q}) \wedge x^*(\mathbf{r}); \mathbf{r}\right) & \text{by definition of } x^*(\mathbf{q}) \\
\Rightarrow & f\left(x^*(\mathbf{q}) \vee x^*(\mathbf{r}); \mathbf{r}\right) \geq f(x^*(\mathbf{q}); \mathbf{r}) & \text{by quasi-supermodularity in } \mathbf{x} \\
\Rightarrow & f\left(x^*(\mathbf{q}) \vee x^*(\mathbf{r}); \mathbf{q}\right) \geq f(x^*(\mathbf{q}); \mathbf{q}) & \text{by Single Crossing} \\
\Rightarrow & x^*(\mathbf{q}) \vee x^*(\mathbf{r}) = x^*(\mathbf{q}) & \text{since } |x^*(\mathbf{q})| \text{ equals } 1 \\
\Rightarrow & x^*(\mathbf{q}) \geq x^*(\mathbf{r}) & \text{by Question 2, PS4}
\end{array}$$

□

- This result can be extended to constrained optimization problems (Question 3, Problem Set 4), and to multi-valued optimizers (see Milgrom & Shannon).

# Data and Rationality: Motivation

## Main Idea

- We observe data and want to know if it could have been the result of maximizing a preference relation or a utility function.

## Demand data observations

- We observe $N$ consumption choices made by an individual, given her income and prices (also observable):

$$\mathbf{x}^1, \mathbf{p}^1, w^1 \qquad \mathbf{x}^2, \mathbf{p}^2, w^2 \qquad \mathbf{x}^3, \mathbf{p}^3, w^3 \qquad \ldots \qquad \mathbf{x}^N, \mathbf{p}^N, w^N$$

  These satisfy:

  - $(\mathbf{x}^j, \mathbf{p}^j, w^j) \in \mathbf{R}^n_+ \times \mathbf{R}^n_{++} \times \mathbf{R}_+$ for all $j = 1, .., N$; and
  - $\mathbf{p}^j \cdot \mathbf{x}^j \leq w^j$ for all $j = 1, .., N$.

- In other words, we have finitely many observations on behavior.

- What conditions must these observations satisfy for us to conclude they are the result of the maximizing of a preference relation or a utility function?
- Answer: Something similar to, but stronger than, revealed preference.

# An Example (from Kreps)

- Suppose we observe the following choices among 3 goods

| | $w$ | $p$ | $x$ |
|---|---|---|---|
| 1 | 300 | $(10, 10, 10)$ | $(10, 10, 10)$ |
| 2 | 130 | $(10, 1, 2)$ | $(9, 25, 7.5)$ |
| 3 | 110 | $(1, 1, 10)$ | $(15, 5, 9)$ |

- Are these choices consistent with our theory? Is there a preference/utility function that generates these choices?
    - Sure: suppose the consumer strictly prefers $(500, 500, 500)$ to anything else, and is indifferent among all other bundles.
    - Since $(500, 500, 500)$ is never affordable, any other choice is rationalizable.
        - This seems silly, and not something that should worry us since we can never rule it out.
- We may want at least local non satiation.
- Observations:
    - The consumer spends all her income in all cases. Furthermore:
    - At prices $(10, 10, 10)$ the bundle $(15, 5, 9)$ could have been chosen.
    - At prices $(10, 1, 2)$ the bundle $(10, 10, 10)$ could have been chosen.
    - At prices $(1, 1, 10)$ the bundle $(9, 25, 7.5)$ could have been chosen.

# Consequences of Local Non Satiation

- The following is slightly different from Full Expenditure.

**Lemma**

*Suppose $\succsim$ is locally non-satiated, and let $\mathbf{x}^*$ be an element of Walrasian demand (therefore $\mathbf{x}^* \succsim \mathbf{x}$ for all $\mathbf{x} \in \{\mathbf{x} \in X : p \cdot \mathbf{x} \leq w\}$). Then*

$$\mathbf{x}^* \succsim \mathbf{x} \qquad \textit{when} \qquad p \cdot \mathbf{x} = w$$

*and*

$$\mathbf{x}^* \succ \mathbf{x} \qquad \textit{when} \qquad p \cdot \mathbf{x} < w$$

- The maximal bundle is weakly preferred to any bundle that costs the same.
- The maximal bundle is strictly preferred to any bundle that costs less.

**Proof.**

The first part is immediate.
For the second part note that if $p \cdot \mathbf{x} < w$ by local non satiation (and continuity of $p \cdot \mathbf{x}$) there exists some $\mathbf{x}'$ such that $\mathbf{x}' \succ \mathbf{x}$ and $p \cdot \mathbf{x}' \leq w$.
Thus $\mathbf{x}'$ is affordable and $\mathbf{x}^* \succsim \mathbf{x}' \succ \mathbf{x}$ as desired. $\square$

# Example Continued

Consumer's choices among 3 goods

| | $w$ | $p$ | $x$ |
|---|---|---|---|
| 1 | 300 | $(10, 10, 10)$ | $(10, 10, 10)$ |
| 2 | 130 | $(10, 1, 2)$ | $(9, 25, 7.5)$ |
| 3 | 110 | $(1, 1, 10)$ | $(15, 5, 9)$ |

- Using local non satiation we can conclude the following.
- Since at prices $(10, 10, 10)$, the bundle $(15, 5, 9)$ costs less than $(10, 10, 10)$:
  - $(10, 10, 10) \succ (15, 5, 9)$.
- Since at prices $(10, 1, 2)$, the bundle $(9, 25, 7.5)$ costs as much as $(10, 10, 10)$:
  - $(9, 25, 7.5) \succsim (10, 10, 10)$.
- Since at prices $(1, 1, 10)$, the bundle $(9, 25, 7.5)$ costs less than $(15, 5, 9)$.
  - $(15, 5, 9) \succ (9, 25, 7.5)$.
- Putting these together:

$$(10, 10, 10) \succ (15, 5, 9) \succ (9, 25, 7.5) \succsim (10, 10, 10)$$

- These observations are not consistent with utility/preference maximization theory: they violate transitivity.

# Directly Revealed Preference

## Definitions

Suppose $\{x^j, p^j, w^j\}_{j=1}^N$ is a finite set of demand data that satisfy $p^j \cdot x^j \leq w^j$.

- The directly revealed weak preference is defined by
$$x^j \succsim^R x^k \quad \text{if} \quad p^j \cdot x^k \leq w^j$$
- The directly revealed strict preference is defined by
$$x^j \succ^R x^k \quad \text{if} \quad p^j \cdot x^k < w^j$$

- These are similar to the revealed preference given by a choice rule:
  - if $x^k$ was affordable, but the consumer chose $x^j$ instead, she revealed that $x^j$ is at least as good a choice as $x^k$.
  - if $x^k$ was strictly affordable, but the consumer chose $x^j$ instead, she revealed that $x^j$ is a strictly better choice than $x^k$.

## Remark: The strict relation is derived from observation

- $\succ^R$ is defined directly; it does not necessarily correspond to the asymmetric component of $\succsim^R$.

# Indirectly Revealed Preference

## Definitions

Suppose $\{\mathbf{x}^j, \mathbf{p}^j, w^j\}_{j=1}^N$ is a finite set of demand data that satisfy $\mathbf{p}^j \cdot \mathbf{x}^j \le w^j$.

- The indirectly revealed weak preference is defined by

$$\mathbf{x}^j \succsim^I \mathbf{x}^k \quad \begin{array}{c}\text{if there exists}\\ \mathbf{x}^{i_1}, \mathbf{x}^{i_2}, \dots, \mathbf{x}^{i_m}\\ \text{such that}\end{array} \quad \begin{array}{c}\mathbf{x}^j \succsim^R \mathbf{x}^{i_1},\\ \mathbf{x}^{i_1} \succsim^R \mathbf{x}^{i_2},\\ \dots\\ \mathbf{x}^{i_m} \succsim^R \mathbf{x}^k\end{array}$$

- The indirectly revealed strict preference is defined by

$$\mathbf{x}^j \succ^I \mathbf{x}^k \quad \begin{array}{c}\text{if there exists}\\ \mathbf{x}^{i_1}, \mathbf{x}^{i_2}, \dots, \mathbf{x}^{i_m}\\ \text{such that}\end{array} \quad \begin{array}{c}\mathbf{x}^j \succsim^R \mathbf{x}^{i_1},\\ \mathbf{x}^{i_1} \succsim^R \mathbf{x}^{i_2},\\ \dots\\ \mathbf{x}^{i_m} \succsim^R \mathbf{x}^k\end{array} \quad \begin{array}{c}\text{and one of the relations}\\ \\ \text{in the chain is strict}\end{array}$$

- $\succ^I$ is also defined directly from observation, and maybe different than the asymmetric component of $\succsim^I$.

## Exercise

Verify that $\mathbf{x}^j \succsim^R \mathbf{x}^k$ implies $\mathbf{x}^j \succsim^I \mathbf{x}^k$.
Verify that if $\mathbf{x}^j \succsim^R \mathbf{x}^k$ and $\mathbf{x}^k \succ^I \mathbf{x}^i$, then $\mathbf{x}^j \succ^I \mathbf{x}^i$.

# Next Class

- Generalized Axiom or Revealed Preferences and Afriat's Theorem
- Production
- Firms' Choices