Monotone comparative statics
Finite Data and GARP

Econon 2100 Fall 2017

Lecture 7, September 19

- Problem Set 3 is due in Kelly's mailbox by 5pm today

Outline

1. Comparative Statics Without Calculus
2. Supermodularity and Single Crossing
3. Topkis and Milgrom&Shannon's Theorems
4. Finite Data

Comparative Statics Without Calculus

Remark

- Let $$x^*(q) = \arg\max f(x,q), \text{ subject to } q \in \Theta,\ x \in S(q)$$ Using the implicit function theorem, one can show that if there are complementarities between the choice variable x and the parameter q, the optimum increases in q.

First Order Condition: $f_x(x;q) = 0$. **Second Order Condition**: $f_{xx}(x;q) < 0$.
IFT:
$$x_q^*(q) = -\frac{f_{xq}(x;q)}{f_{xx}(x;q)}.$$
Then
$$x_q^*(q) \geq 0 \textit{ if and only if } f_{xq}(x;q) \geq 0$$

Here $g_\cdot$ denotes the derivative of g with respect to $\cdot$.

Issues with implicit function theorem:

1. IFT needs calculus.
2. Conclusions hold only in a neighborhood of the optimum.
3. Results are dependent on the functional form used for the objective function.
   1. In particular, IFT gives cardinal results that depend on the assumptions on f.

Monotone Comparative Statics

Objectives

- With monotone comparative statics, we have results about "changes" that:
  - do not need calculus;
  - are not necessarily only local (around the optimum);
  - are ordinal in the sense of being robust to monotonic transformations.

- One can get conclusions similar to IFT without calculus.
- The downside is that the results are not as strong.

Main Idea: Complementarities

- The central idea generalizes the notion of complementarities between endogenous variable and parameters.
  - With calculus, this is the assumption $f_{xq}(x; q) \geq 0$.

- We would also like to account for the possibility that the optimum is not unique, so that $x^*(q)$ is not a function.
- First, what does it mean for a correspondence to be increasing?

Strong Set Order

- Ranking real numbers is easy, but how can we express the idea that one set is bigger than another set?

Definition

For two sets of real numbers A and B, define the binary relation $\geq_s$ as follows:

$$\boxed{\mathrm{A} \geq_s \mathrm{B} \quad \text{if} \quad \begin{array}{c} \text{for any } a \in A \text{ and } b \in B \\ \min\{a, b\} \in B \quad \text{and} \quad \max\{a, b\} \in A \end{array}}$$

$A \geq_s B$ reads "A is greater than or equal to B in the strong set order".

- Generalizes the notion of greater than from numbers to sets of numbers.
- This definition reduces to the standard definition when sets are singletons.

Example

Suppose $A = \{1, 3\}$ and $B = \{0, 2\}$. Then, A is not greater than or equal to B in the strong set order.

Non-Decreasing Correspondences

Definition

We say a correspondence $g : \mathbf{R}^n \to 2^{\mathbf{R}}$ is non-decreasing in q if and only if

$$x' > x \qquad \text{implies} \qquad g(x') \geq_s g(x)$$

- Thus, $x' > x$ implies that for any $y' \in g(x')$ and $y \in g(x)$: $\min\{y', y\} \in g(x)$ and $\max\{y', y\} \in g(x')$.
  - Larger points in the domain correspond to larger sets in the codomain.
- Generalizes the notion of increasing function to correspondences.

Exercise

Prove that if $g(\cdot)$ is non-decreasing and $\min g(x)$ exists for all x, then $\min g(x)$ is non-decreasing.

Exercise

Prove that if $g(\cdot)$ is non-decreasing and $\max g(x)$ exists for all x, then $\max g(x)$ is non-decreasing.

Monotone Comparative Statics: Simplest Case

Set up

- Suppose the function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ is the objective function; this is not necessarily concave or differentiable, and the optimizer could be set valued.

- Let
$$x^*(q) = \arg\max f(x, q), \text{ subject to } q \in \Theta; x \in S(q)$$
  - Note: for any strictly increasing h, this problem is equivalent to
$$x^*(q) = \arg\max h(f(x, q)), \text{ subject to } q \in \Theta; x \in S(q)$$
  - $h(\cdot)$ may destroy smoothness or concavity properties of the objective function so IFT may not work.

- For now, assume $S(\cdot)$ is independent of q (ignore the constraints), and that both x and q are real variables.
- Assume existence of a solution, but not uniqueness.

Supermodularity

Definition

The function $f : \mathbf{R}^2 \to \mathbf{R}$ is supermodular in $(x; q)$ if

$$\text{for all } x' > x \qquad f(x'; q) - f(x; q) \text{ is non-decreasing in } q.$$

- If f is supermodular in $(x; q)$, then the incremental gain to a higher x is greater when q is higher.
- This is the idea that x and q are "complements".

Question 1, Problem Set 4.

Show that supermodularity is equivalent to the property that

$$q' > q \qquad \text{implies} \qquad f(x; q') - f(x; q) \text{ is non-decreasing in } x.$$

Differentiable Version of Supermodularity

- When f is smooth, supermodularity has a characterization in terms of derivatives.

Lemma

A twice continuously differentiable function $f : \mathbf{R}^2 \to \mathbf{R}$ *is supermodular in* $(x; q)$ *if and only if* $D_{xq} f(x; q) \geq 0$ *for all* $(x; q)$.

- The inequality in the definition of supermodularity is just the discrete version of the mixed-partial condition in the lemma.
  - $q' > q$ implies $f(x; q') - f(x; q)$ is non-decreasing in x

Topkis' Monotonicity Theorem

Theorem (Easy Topkis' Monotonicity Theorem)

If f is supermodular in $(x;q)$, then $x^*(q) = \arg\max f(x,q)$ is non-decreasing.

Proof.

Let $q' > q$ and take $x \in x^*(q)$ and $x' \in x^*(q')$. Show that $x^*(q') \geq_s x^*(q)$.

- First show that $\max\{x, x'\} \in x^*(q')$
  - $x \in x^*(q)$ implies $f(x;q) - f(\min\{x,x'\};q) \geq 0$.
  - $x \in x^*(q)$ also implies that $f(\max\{x,x'\};q) - f(x';q) \geq 0$
    - verify these by checking two cases, $x > x'$ and $x' > x$.
  - By supermodularity, $f(\max\{x,x'\};q') - f(x';q') \geq 0$,
  - Hence $\max\{x,x'\} \in x^*(q')$.
- Now show that $\min\{x,x'\} \in x^*(q)$
  - $\max\{x,x'\} \in x^*(q')$ implies that $f(x';q') - f(\max\{x,x'\},q) \geq 0$,
    - or equivalently $f(\max\{x,x'\},q) - f(x';q') \leq 0$.
  - $\max\{x,x'\} \in x^*(q')$ also implies that $f(\max\{x,x'\};q') - f(x';q') \geq 0$,
  - which by supermodularity implies $f(x;q) - f(\min\{x,x'\};q) \leq 0$
  - Hence $\min\{x,x'\} \in x^*(q)$.

□

Topkis' Monotonicity Theorem

Theorem (Easy Topkis' Monotonicity Theorem)

If f is supermodular in $(x; q)$, then $x^*(q) = \arg\max f(x, q)$ is non-decreasing.

- Supermodularity is sufficient to draw comparative statics conclusions in optimization problems without other assumptons.
- Topkis' Theorem follows from the IFT whenever the standard full-rank condition in the IFT holds.
  - At a maximum, if $D_{xx} f(x^*, q) \neq 0$, it must be negative (by the second-order condition), hence the IFT tells you that $x^*(q)$ is strictly increasing.

Example

Profit Maximization Without Calculus

- A monopolist chooses output q to solve $\max p(q)q - c(q, \mu)$.
  - $p(\cdot)$ is the demand (price) function
  - $c(\cdot)$ is the cost function
    - costs depend on the existing technology, described by some parameter μ.
- Let $q^*(\mu)$ be the monopolist's optimal quantity.
- Suppose $-c(q, \mu)$ is supermodular in (q, μ); then the entire objective function is also supermodular in (q, μ).
  - this follows because the first term of the objective does not depend on μ.
- Notice that supermodularity says that for all $q' > q$, $-c(q'; \mu) + c(q; \mu)$ is nondecreasing in μ.
  - in other words, the marginal cost is decreasing in μ.
- Conclusion: by Topkis' theorem q^* is nondecreasing as long as the marginal cost of production decreases in the technological progress parameter μ.

Single-Crossing

- In constrained maximization problems, $x \in S(q)$, supermodularity is not enough for Topkis' theorem.

Definition

The function $f : \mathbf{R}^2 \to \mathbf{R}$ satisfies the single-crossing condition in $(x; q)$ if for all $x' > x$, $q' > q$

$$f(x'; q) - f(x; q) \geq 0 \qquad \text{implies} \qquad f(x'; q') - f(x; q') \geq 0$$

and

$$f(x'; q) - f(x; q) > 0 \qquad \text{implies} \qquad f(x'; q') - f(x; q') > 0.$$

- As a function of the second argument, the marginal return can cross 0 at most once; whenever it crosses 0, as the second argument continues to increase, the marginal return is going to remain positive.

Theorem

If f satisfies single crossing in $(x; q)$, then $x^*(q) = \arg\max_{x \in S(q)} f(x; q)$ is nondecreasing. Moreover, if $x^*(q)$ is nondecreasing in q for all constraint choice sets S, then f satisfies single-crossing in $(x; q)$.

Monotone Comparative Statics

n-dimensional choice variable and m-dimensional parameter vector

- Next, we generalize to higher dimensions.

Definitions

Suppose $x, y \in \mathbf{R}^n$.

- The join of x and y is defined by
$$x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, \ldots, \max\{x_n, y_n\}).$$
- The meet of x and y is defined by
$$x \wedge y = (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\}).$$

- Draw a picture.

Strong Set Order

- We generalize the strong set order definition to $\mathbf{R}^n$.

Definition (Strong set order in $\mathbf{R}^n$)

The binary relation $\geq_s$ is defined as follows: for $A, B \subset \mathbf{R}^n$,

$$\boxed{A \geq_s B \qquad \text{if} \qquad \begin{array}{c} \text{for any } a \in A \text{ and } b \in B \\ \\ a \wedge b \in B \quad \text{and} \quad a \vee b \in A \end{array}}$$

- The meet is in the smaller set, while the join is in the larger set.
- One uses this to talk about non-decreasing $\mathbf{R}^n$-valued correspondences.

- We look at functions $f(\mathbf{x}; \mathbf{q})$ where the first argument represents the endogenous variables and the second represents the parameters.

Quasi-Supermodularity

Definition

The function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is quasi-supermodular in its first argument if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\mathbf{q} \in \mathbf{R}^m$:

1. $$f(\mathbf{x}; \mathbf{q}) \geq f(\mathbf{x} \wedge \mathbf{y}; \mathbf{q}) \quad \Rightarrow \quad f(\mathbf{x} \vee \mathbf{y}; \mathbf{q}) \geq f(\mathbf{y}; \mathbf{q});$$
2. $$f(\mathbf{x}; \mathbf{q}) > f(\mathbf{x} \wedge \mathbf{y}; \mathbf{q}) \quad \Rightarrow \quad f(\mathbf{x} \vee \mathbf{y}; \mathbf{q}) > f(\mathbf{y}; \mathbf{q}).$$

- This generalizes the 'mixed' second partial derivatives typically used to make statements about complementarities.
- Quasi-supermodularity is an ordinal property; for differentiable functions there is a sufficient condition for quasi-supermodularity.

Exercise

1. Prove that if f is quasi-supermodular in x, then $h \circ f$ is quasi-supermodular in x for any strictly increasing $h : f(\mathbf{R}^n \times \mathbf{R}^m) \to \mathbf{R}$.
2. Suppose $f(x; q)$ is twice differentiable in x and $\frac{\partial^2 f}{\partial x_i x_j} > 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Then f is quasi-supermodular in x.

Single-Crossing Property

Definition

The function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ satisfies the single-crossing property if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ and $\mathbf{q}, \mathbf{r} \in \mathbf{R}^m$ such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{q} \geq \mathbf{r}$:

1. $f(\mathbf{x}; \mathbf{r}) \geq f(\mathbf{y}; \mathbf{r}) \quad \Rightarrow \quad f(\mathbf{x}; \mathbf{q}) \geq f(\mathbf{y}; \mathbf{q})$;
2. $f(\mathbf{x}; \mathbf{r}) > f(\mathbf{y}; \mathbf{r}) \quad \Rightarrow \quad f(\mathbf{x}; \mathbf{q}) > f(\mathbf{y}; \mathbf{q})$.

- The "marginal return" $f(\mathbf{x}; \cdot) - f(\mathbf{y}; \cdot)$ as a function of the second argument can cross 0 at most once.
- The single-crossing property is an ordinal property; for differentiable functions there is a sufficient condition for single-crossing.

Exercise

1. Prove that if f satisfies the single-crossing property, then $h \circ f$ satisfies the single-crossing property for any strictly increasing $h : f(\mathbf{R}^n \times \mathbf{R}^m) \to \mathbf{R}$.
2. Suppose $f(x; q)$ is twice differentiable and $\frac{\partial^2 f}{\partial x_i x_j} > 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then f satisfies the single-crossing property.

Monotone Comparative Statics

Theorem (easy Milgrom and Shannon)

Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$*. Define* $x^*(q) = \arg\max_{x \in R^n} f(x; q)$*. Suppose* $|x^*(q)| = 1$ *for all* q *and* $f(x; q)$ *is quasi-supermodular in its first argument and satisfies the single-crossing property. Then*

$$q \geq r \Rightarrow x^*(q) \geq x^*(r).$$

- 'Easy' because it assumes the optimum is unique (thus, the proof does not use 'strict' quasi-supermodularity and single-crossing).

Proof.

Suppose $q \geq r$. Then:

$$\begin{array}{rll} & f(x^*(r); r) \geq f(x^*(q) \wedge x^*(r); r) & \text{by definition of } x^*(q) \\ \Rightarrow & f(x^*(q) \vee x^*(r); r) \geq f(x^*(q); r) & \text{by quasi-supermodularity in } x \\ \Rightarrow & f(x^*(q) \vee x^*(r); q) \geq f(x^*(q); q) & \text{by Single Crossing} \\ \Rightarrow & x^*(q) \vee x^*(r) = x^*(q) & \text{since } |x^*(q)| \text{ equals } 1 \\ \Rightarrow & x^*(q) \geq x^*(r) & \text{by Question 2, PS4} \end{array}$$

□

- This result can be extended to constrained optimization problems (Question 3, Problem Set 4), and to multi-valued optimizers (see Milgrom & Shannon).

Data and Rationality: Motivation

Main Idea

- We observe data and want to know if it could have been the result of maximizing a preference relation or a utility function.

Demand data observations

- We observe N consumption choices made by an individual, given her income and prices (also observable):

$$x^1, p^1, w^1 \qquad x^2, p^2, w^2 \qquad x^3, p^3, w^3 \qquad \dots \qquad x^N, p^N, w^N$$

  These satisfy:
  - $(x^j, p^j, w^j) \in \mathbf{R}^n_+ \times \mathbf{R}^n_{++} \times \mathbf{R}_+$ for all $j = 1, .., N$; and
  - $p^j \cdot x^j \leq w^j$ for all $j = 1, .., N$.
- In other words, we have finitely many observations on behavior.

- What conditions must these observations satisfy for us to conclude they are the result of the maximizing of a preference relation or a utility function?
- Answer: Something similar to, but stronger than, revealed preference.

An Example (from Kreps)

- Suppose we observe the following choices among 3 goods

	w	p	x
1	300	$(10, 10, 10)$	$(10, 10, 10)$
2	130	$(10, 1, 2)$	$(9, 25, 7.5)$
3	110	$(1, 1, 10)$	$(15, 5, 9)$

- Are these choices consistent with out theory? Is there a preference/utility function that generates these choices?
  - Sure: suppose the consumer strictly prefers $(500, 500, 500)$ to anything else, and is indifferent among all other bundles.
  - Since $(500, 500, 500)$ is never affordable, any other choice is rationalizable.
    - This seems silly, and not something that should worry us since we can never rule it out.
- We may want at least local non satiation.
- Observations:
  - The consumer spends all her income in all cases. Furthermore:
  - At prices $(10, 10, 10)$ the bundle $(15, 5, 9)$ could have been chosen.
  - At prices $(10, 1, 2)$ the bundle $(10, 10, 10)$ could have been chosen.
  - At prices $(1, 1, 10)$ the bundle $(9, 25, 7.5)$ could have been chosen.

Consequences of Local Non Satiation

- The following is slightly different from Full Expenditure.

Lemma

Suppose $\succsim$ is locally non-satiated, and let x^* be an element of Walrasian demand (therefore $x^* \succsim x$ for all $x \in \{x \in X : p \cdot x \leq w\}$). Then

$$x^* \succsim x \qquad \text{when} \qquad p \cdot x = w$$

and

$$x^* \succ x \qquad \text{when} \qquad p \cdot x < w$$

- The maximal bundle is weakly preferred to any bundle that costs the same.
- The maximal bundle is strictly preferred to any bundle that costs less.

Proof.

The first part is immediate.
For the second part note that if $p \cdot x < w$ by local non satiation (and continuity of $p \cdot x$) there exists some x' such that $x' \succ x$ and $p \cdot x' \leq w$.
Thus x' is affordable and $x^* \succsim x' \succ x$ as desired. $\square$

Example Continued

Consumer's choices among 3 goods

	w	p	x
1	300	$(10, 10, 10)$	$(10, 10, 10)$
2	130	$(10, 1, 2)$	$(9, 25, 7.5)$
3	110	$(1, 1, 10)$	$(15, 5, 9)$

- Using local non satiation we can conclude the following.
- Since at prices $(10, 10, 10)$, the bundle $(15, 5, 9)$ costs less than $(10, 10, 10)$:
  - $(10, 10, 10) \succ (15, 5, 9)$.
- Since at prices $(10, 1, 2)$, the bundle $(9, 25, 7.5)$ costs as much as $(10, 10, 10)$:
  - $(9, 25, 7.5) \succsim (10, 10, 10)$.
- Since at prices $(1, 1, 10)$, the bundle $(9, 25, 7.5)$ costs less than $(15, 5, 9)$.
  - $(15, 5, 9) \succ (9, 25, 7.5)$.
- Putting these together:

$$(10, 10, 10) \succ (15, 5, 9) \succ (9, 25, 7.5) \succsim (10, 10, 10)$$

- These observations are not consistent with utility/preference maximization theory: they violate transitivity.

Directly Revealed Preference

Definitions

Suppose $\{x^j, p^j, w^j\}_{j=1}^N$ is a finite set of demand data that satisfy $p^j \cdot x^j \leq w^j$.

- The directly revealed weak preference is defined by
$$x^j \succsim^R x^k \quad \text{if} \quad p^j \cdot x^k \leq w^j$$
- The directly revealed strict preference is defined by
$$x^j \succ^R x^k \quad \text{if} \quad p^j \cdot x^k < w^j$$

- These are similar to the revealed preference given by a choice rule:
  - if x^k was affordable, but the consumer chose x^j instead, she revealed that x^j is at least as good a choice as x^k.
  - if x^k was strictly affordable, but the consumer chose x^j instead, she revealed that x^j is a strictly better choice than x^k.

Remark: The strict relation is derived from observation

- $\succ^R$ is defined directly; it does not necessarily correspond to the asymmetric component of $\succsim^R$.

Indirectly Revealed Preference

Definitions

Suppose $\{x^j, p^j, w^j\}_{j=1}^N$ is a finite set of demand data that satisfy $p^j \cdot x^j \leq w^j$.

- The indirectly revealed weak preference is defined by

$$x^j \succsim^I x^k \quad \begin{array}{c} \text{if there exists} \\ x^{i_1}, x^{i_2}, \dots, x^{i_m} \\ \text{such that} \end{array} \quad \begin{array}{c} x^j \succsim^R x^{i_1}, \\ x^{i_1} \succsim^R x^{i_2}, \\ \dots \\ x^{i_m} \succsim^R x^k \end{array}$$

- The indirectly revealed strict preference is defined by

$$x^j \succ^I x^k \quad \begin{array}{c} \text{if there exists} \\ x^{i_1}, x^{i_2}, \dots, x^{i_m} \\ \text{such that} \end{array} \quad \begin{array}{c} x^j \succsim^R x^{i_1}, \\ x^{i_1} \succsim^R x^{i_2}, \\ \dots \\ x^{i_m} \succsim^R x^k \end{array} \quad \begin{array}{c} \text{and one of the relations} \\ \\ \text{in the chain is strict} \end{array}$$

- $\succ^I$ is also defined directly from observation, and maybe different than the asymmetric component of $\succsim^I$.

Exercise

Verify that $x^j \succsim^R x^k$ implies $x^j \succsim^I x^k$.
Verify that if $x^j \succsim^R x^k$ and $x^k \succ^I x^i$, then $x^j \succ^I x^i$.