Preference Relations and Choice Rules

Econ 2100, Fall 2017

Lecture 1, 31 August

Outline

1. Logistics
2. Introduction to Consumer Theory
3. Binary Relations
4. Preferences
5. Choice Correspondences
6. Revealed Preferences
Logistics

Instructor: Luca Rigotti
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Office Hours: Wednesday 1:30pm-3pm, or by appointment (send me an email)

Teaching Assistant: Kelly Hyde
Office: WWPH 4913, email: KDH50@pitt.edu
Office Hours: Tuesday 12:30-2:30pm.

Class Time and Location
Tuesday & Thursday, 9:00am to 10:15am in WWPH 4716

Recitation Times
Friday, 9-10:15am, in WWPH 4940.

Class Webpage http://www.pitt.edu/~luca/ECON2100/

Announcements
We will have a make-up lecture this Friday 9:00am to 10:15am in WWPH 4716
Logistics

Mas-Colell, Whinston and Green: *Microeconomic Theory.*
Other useful textbooks by Varian, Jehle and Reny, older Kreps.
My favorite source: Gerard Debreu: *Theory of Value.*

Grading

- **Problem Sets**: 10%
  Weekly, they will appear during lectures. **DO NOT** look for answers... These are for you to learn what you can and cannot do; we can help you get better only if we know your limits.
- **Midterm**: 30%
  In class, 12 October, includes materials up to previous class.
- **Final Exam**: 60%
  Cumulative, 3 hours long, Friday, 8 December.

- Working in teams strongly encouraged, but turn in problem sets individually.
  - Start working on exercises on your own, and then get together to discuss.
  - Typically, a Problem Set will be due at the beginning of each Tuesday lecture.
Goals of the Micro Theory Sequence

- Learn and understand the microeconomic theory every academic economist needs to know.
- Stimulate interest in micro theory as a field (some of you may want to become theorists).
- Enable you to read papers that use theory, and go to research seminars.
- ECON 2100 covers the following standard topics:
  - consumers: preferences, choices, utility function representation, utility maximization, demand theory, aggregation, decision making under uncertainty;
  - firms: production, profit maximization, aggregation;
  - general equilibrium theory: consumers and firms in competitive markets, Walrasian equilibrium, First and Second Welfare Theorems, existence of equilibrium, uniqueness of equilibrium, markets with uncertainty and time, Arrow-Debreu economies.
- In the Spring, Professor Sofia Moroni will cover game theory and information economics.
- Even if you are not enrolled in it, attend ECON 2010 as long as you can; Roee Teper will start with constrained optimization, a fundamental tool for this class.
- Questions? This will be fun: let’s go!
Imagine a consumer can choose objects from some abstract set. How will she do that?

We can think about it in three different ways:

1. We describe the consumer’s taste, her preferences, and those determine her choices.
2. We assume the consumer maximizes a utility function over her possible choices.
3. We characterize a choice functional describing how the consumer behaves coherently in each situation.

The first few lectures are about showing under what conditions these ways of describing consumers are equivalent.

Today we describe preferences and choices formally, and start talking about possible connections between them.

Mathematically, preferences are a special case of a binary relation.
Binary Relations

- A binary relation on some set is a collection of ordered pairs of elements of that set.

**Definition**

$R \subseteq X \times Y$ is a binary relation from $X$ to $Y$. We write “$xRy$” if $(x, y) \in R$ and “not $xRy$” if $(x, y) \notin R$. When $X = Y$ and $R \subseteq X \times X$, we say $R$ is a binary relation on $X$.

**Exercise**

Suppose $R, Q$ are two binary relations on $X$. Prove that, given our notation, the following are equivalent:

1. $R \subseteq Q$
2. For all $x, y \in X$, $xRy \Rightarrow xQy$.

**NOTE**

Items denoted Exercise are meant as practice for you, distinct from graded problem sets.
Examples of Binary Relations

A Function Is a Binary Relation

Suppose $f : X \rightarrow Y$ is a function from $X$ to $Y$.

- Then the binary relation $R \subseteq X \times Y$ defined by
  
  $xRy \Leftrightarrow f(x) = y$

  is the graph of $f$.

- One way to think of a function is as a binary relation $R$ from $X$ to $Y$ such that: for each $x \in X$, there exists exactly one $y \in Y$ such that $(x, y) \in R$. 
Examples of Binary Relations

### Weakly greater than

Suppose $X = \{1, 2, 3\}$

- Consider the binary relation
  \[ R \subseteq \{1, 2, 3\} \times \{1, 2, 3\} \]
  defined as follows
  \[ R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}. \]
- $R$ is the binary relation “is weakly greater than,” or $\geq$.
- We can represent it graphically as:

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Examples of Binary Relations

Equal to and Strictly less than

Suppose \( X = \{1, 2, 3\} \).

- The following graphically represents the binary relation \( = \), or “is equal to:”

\[
\begin{array}{ccc}
3 & & \bullet \\
2 & \bullet & \\
1 & \bullet & \\
\hline
= & 1 & 2 & 3
\end{array}
\]

- The following would represent the binary relation \( < \) or “is strictly less than:”

\[
\begin{array}{ccc}
3 & \bullet & \bullet \\
2 & \bullet & \\
1 & \bullet & \\
\hline
< & 1 & 2 & 3
\end{array}
\]
Dual, Asymmetric, and Symmetric Components

**Definitions**

Given a binary relation $R$ on $X$.

- The **dual** of $R$, denoted $R'$, is defined by $xR'y$ if and only if $yRx$.
- The **asymmetric component** of $R$, denoted $P$, is defined by $xPy$ if and only if $xRy$ and not $yRx$.
- The **symmetric component** of $R$, denoted $I$, is defined by $xIy$ if and only if $xRy$ and $yRx$.

**Example**

Suppose $X = \mathbb{R}$ and $R$ is the binary relation $\geq$, or “weakly greater than.”

- The dual $R'$ is $\leq$ or “weakly less than,” because $x \geq y$ if and only if $y \leq x$.
- The asymmetric component $P$ is $>$ or “strictly greater than,” because $x > y$ if and only if $x \geq y$ and not $y \geq x$. (Verify this).
- The symmetric component of $I$ is $=$ or “is equal to,” because $x = y$ if and only if $x \geq y$ and $y \geq x$. 
Binary Relations: Examples

**Example**

Suppose $X = \{1, 2, 3\}$ and consider the following binary relation $R$:

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Then the following represent $R'$, $P$, and $I$:

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- **$P$**:
  
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- **$I$**:
  
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Properties of Binary Relations

**Definitions**

A binary relation \( R \) on \( X \) is:

- **complete** if, for all \( x, y \in X \), \( xRy \) or \( yRx \);
- **reflexive** if, for all \( x \in X \), \( xRx \);
- **irreflexive** if, for all \( x \in X \), \( \neg xRx \);
- **symmetric** if, for all \( x, y \in X \), \( xRy \) implies \( yRx \);
- **asymmetric** if, for all \( x, y \in X \), \( xRy \) implies \( \neg yRx \);
- **antisymmetric** if, for all \( x, y \in X \), \( xRy \) and \( yRx \) imply \( x = y \);
- **transitive** if, for all \( x, y, z \in X \), \( xRy \) and \( yRz \) imply \( xRz \);
- **negatively transitive** if, for all \( x, y, z \in X \), \( \neg xRy \) and \( \neg yRz \) imply \( \neg xRz \);
- **quasi-transitive** if, for all \( x, y, z \in X \), \( xPy \) and \( yPz \) imply \( xPz \);
- **acyclic** if, for all \( x_1, x_2, \ldots, x_n \in X \), \( x_1 P x_2, x_2 P x_3, \ldots \), and \( x_{n-1} P x_n \) imply \( x_1 R x_n \).
Exercise
Suppose $R$ is complete. Prove the following: If $R$ is transitive, then $R$ is quasi-transitive. If $R$ is quasi-transitive, then $R$ is acyclic.

Example
Suppose $X = R$.

- The binary relation $\geq$ is reflexive, complete, antisymmetric, transitive, and negatively transitive; $\geq$ is not asymmetric.
- The binary relation $>$ is irreflexive, asymmetric, antisymmetric, transitive, negatively transitive, quasi-transitive, and acyclic; $>$ is not reflexive, not complete, and not symmetric.
Equivalence Class

**Definition**

The equivalence class of \( x \in X \) is

\[
[x] = \{y \in X : xRy\}
\]

- Let \( X/R \) (the quotient of \( X \) over \( R \)) denote the collection of all equivalence classes.

**Example**

Suppose \( X = \mathbb{R} \) and \( R \) is the binary relation \( \geq \):

\[
[7] = \{x \in \mathbb{R} : x \leq 7\}
\]
**Definition**

A binary relation $R$ on $X$ that is reflexive, symmetric, and transitive is called an equivalence relation.

- The following says that the equivalence classes of an equivalence relation form a partition of $X$: every element of $X$ belongs to exactly one equivalence class.

**Theorem**

Let $R$ be an equivalence relation on $X$. Then $\forall x \in X, \, x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

**Proof.**

Exercise.
Equivalence Relation and Equivalence Class

The following is an equivalence relation on $X = \{a, b, c, d\}$:

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- The equivalence classes of $R$ are $\{a, b\}$ and $\{c, d\}$, so the collection is $X/R = \{\{a, b\}; \{c, d\}\}$.

Exercise

Let $X$ be the set of all living people. Are the following relations on $X$ reflexive? symmetric? transitive? complete?

- “is married to” (assuming monogamy)
- “is the son or daughter of”
- “is an ancestor or descendant of”
- “is taller than”

For those that are equivalence relations, interpret $X/R$. 
Orders

**Definition**
A binary relation $R$ on $X$ is a **preorder** if $R$ is reflexive and transitive.

**Definition**
A binary relation $R$ on $X$ is a **weak order** if $R$ is complete and transitive.

**Definition**
A binary relation $R$ on $X$ is a **linear order** if $R$ is complete, transitive, and antisymmetric.

**Exercise**
Define the binary relation $\geq$ on $\mathbb{R}^2$ by: $x \geq y \iff x_1 \geq y_1$ and $x_2 \geq y_2$. Verify that $\geq$ is a preorder on $\mathbb{R}^2$. Verify that $\geq$ is not a weak order on $\mathbb{R}^2$.

**Exercise**
Define $\geq^\dagger$ on $\mathbb{R}^2$ by: $x \geq^\dagger y \iff x_1 > y_1$ or, $x_1 = y_1$ and $x_2 \geq y_2$. Verify that $\geq^\dagger$ is a linear order on $\mathbb{R}^2$ (this is sometimes called lexicographic ordering).
Upper and Lower Contour Sets

Definitions

Given a binary relation $R$ on $X$, the **upper contour set** of $x \in X$ is

$$\{y \in X : yRx\}.$$

Given a binary relation $R$ on $X$, the **lower contour set** of $x \in X$ is

$$\{y \in X : xRy\}.$$

Exercise

Suppose $R$ is a preorder on $X$. Prove that if $xRy$, then the lower contour set of $y$ is a subset of the lower contour set of $x$; that is, prove that

$$xRy \Rightarrow \{z \in X : yRz\} \subset \{z \in X : xRz\}.$$
Preferences Are Binary Relations

- Assume there is some abstract set $X$ of objects the consumer considers.
- The consumer’s taste, her preferences, describe how she ranks any two elements in $X$.
- In other words, preferences are described mathematically by a binary relation $\succeq$ on $X$.
- $x \succeq y$ reads: the decision maker (DM) weakly prefers $x$ to $y$; we also say $x$ is at least as good as $y$.
- Kreps: “A preference relation expresses the consumer’s feelings between pair of objects in $X$”.

Remark

For any $x, y \in X$, the consumer is willing to say which of the following holds:

1. $x \succeq y$ but not $y \succeq x$;
2. $y \succeq x$ but not $x \succeq y$;
3. $x \succeq y$ and $y \succeq x$;
4. neither $x \succeq y$ nor $y \succeq x$. 
Preference Relation

**Definition**

A binary relation \( \succeq \) on \( X \) is a **preference relation** if it is a weak order, i.e. complete and transitive.

- Transitivity can be thought of as implied by rationality in the sense of some sort of coherence of taste.
- Completeness, on the other hand, is harder to justify as a consequence of rationality: what is wrong in not being able to rank all possible pairs?
  - it is very powerful, as it rules out inability to rank (4. in the previous remark is not possible).
  - Because of this, all the other assumptions immediately become “global”.

**Question 1, Problem Set 1: Transitivity follows from weaker properties.**

Prove the following: if \( \succeq \) is asymmetric and negatively transitive, then \( \succeq \) is transitive.

**NOTE**

Items denoted Problem Set Questions will be handed in individually to Kelly’s mailbox, by 5pm on the due date.
Upper and Lower Contour Sets

**Definition**

The **upper contour set** of $x$ (denoted $\succeq (x)$) consists of the elements of $X$ that are weakly preferred to $x$ according to $\succeq$:

$$\succeq (x) = \{y \in X : y \succeq x\}$$

- These are all consumption bundles at least as good as $x$.

**Definition**

The **lower contour set** of $x$ (denoted $\preceq (x)$) consists of the elements of $X$ that $x$ is weakly preferred to according to $\preceq$:

$$\preceq (x) = \{y \in X : x \preceq y\}$$

- These are all consumption bundles that $x$ is at least as good as.
Preference Relations

Definitions

For any preference relation $\succeq$ on $X$,

- $\prec$ denotes the dual of $\succeq$, defined by
  \[ x \prec y \iff y \succ x; \]

- $\succ$ denote the asymmetric component of $\succeq$, defined by
  \[ x \succ y \iff [x \succeq y \text{ and not } y \succeq x]; \]

- $\sim$ denote the symmetric component of $\succeq$, defined by
  \[ x \sim y \iff [x \succeq y \text{ and } y \succeq x]. \]

- $x \succ y$ reads: DM strictly prefers $x$ to $y$;
- $x \sim y$ reads: DM is indifferent between $x$ and $y$. 
Exercise

Let $X = \{a, b, c\}$. Determine if the following binary relations are complete and/or transitive:

1. $\preceq = X \times X$;
2. $\preceq = \emptyset$;
3. $\preceq = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (b, c)\}$;
4. $\preceq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$;
5. $\preceq = \{(a, b), (b, c), (a, c)\}$.

Question 2, Problem Set 1.

Prove that if $\succeq$ is a preference relation (i.e. it is complete and transitive), then:

1. $\succeq$ is a preference relation;
2. $\succ$ is asymmetric and transitive;
3. $\sim$ is an equivalence relation.
4. $x \succeq y$ and $y \sim z$ imply $x \succeq z$;
5. $x \succeq y$ and $y \succ z$ imply $x \succ z$. 
A preference relation describes DM’s rankings of any two hypothetical pairs. Next, we describe the way in which the decision maker actually selects, chooses, from a given set of alternatives. Ideally, this should produce observable outcome that could be useful to test theories. After doing that, we will worry about the connection between preferences and choices. The mathematical object that describes choices is a correspondence.

**Definition**

A correspondence \( \varphi \) from \( X \) to \( Y \) is a mapping from \( X \) to \( 2^Y \); that is, \( \varphi(x) \subseteq Y \) for every \( x \in X \).
**Choice Rules**

- Items can be selected from some set of available objects.

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**Definition**

A choice rule for $X$ is a correspondence \[ C : 2^X \setminus \{\emptyset\} \to X \] such that \[ C(A) \subseteq A \] for all $A \subseteq X$.

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**Interpretation**

- **Subsets of** $2^X \setminus \{\emptyset\}$ **are “menus” or “budgets”**.
  - A particular $A \in 2^X \setminus \{\emptyset\}$ is interpreted as the set of available options (for example, affordable consumption).

- **Given a budget**, $C(A)$ is the set of options DM might choose from it.
  - If $C(A)$ has more than one element, she could choose any of them (but not all of them at once).

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**Remark**

- It is unclear whether $C(A)$ is actually observable. At best, one observes the decision maker choose an element of $C(A)$.

- So, one can **include** elements in the choice set from observation, but one cannot **exclude** them without making additional assumptions on $C(A)$. 


Choice Rules: An Example

Example

Let \( X = \{\text{apple, banana, carrot, dessert, elephant}\} \).

- If \( C(\{a, b, c\}) = \{a, b\} \), DM could choose either the apple or the banana from a basket containing an apple, a banana, and a carrot;
- This is not interpreted as meaning the decision maker will consume both the apple and the banana.
- By definition, this means she will consume only one between the apple and the banana, but we do not know which one.
**Induced Choice Rules**

**Definition**

Given a binary relation $\succsim$, the **induced choice rule** $C_\succsim$ is defined by

$$C_\succsim(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}.$$ 

- This is a natural method to construct a choice rule from a binary relation: DM chooses something she prefers to all other available alternatives.
- This definition answers one of our questions: the induced choice rule gives a choice procedure that is consistent with a given preference relation.

**Example**

Let $X = \{a, b, c\}$ and let $\succsim = \{(a, a), (b, b), (c, c), (a, b), (b, c), (c, a)\}$. What is $C_\succsim(\{a, b, c\})$?

**Example**

Suppose $X = \{1, 2, 3, \ldots\}$ and consider $\geq$, $\leq$, and $>$. Then:

- $C_\geq(A) = \max A$ if $A$ is finite and $C_\geq = \emptyset$ if $A$ is infinite.
- $C_\leq(A) = \min A$ for all sets $A$.
- $C_>(A) = \emptyset$ for all $A$. 
Non Empty Choice Rules

**Definition**
The choice rule $C$ is **non-empty** if $C(A) \neq \emptyset$ for all non-empty $A \subseteq X$.

**Question 3, Problem Set 1.**
Prove that if $\succsim$ is a preference relation, then $C_{\succsim}(A) \neq \emptyset$ whenever $A$ is finite.

- Given a preference relation, the corresponding induced choice rule is non-empty on finite menus.

**Question 4, Problem Set 1.**
Prove that $\succsim$ is complete and acyclic (but not necessarily transitive) if and only if $C_{\succsim}(A) \neq \emptyset$ whenever $A$ is finite.
Observable Choices

**Definition**

Given a binary relation $\succeq$, the **induced choice rule** $C_\succeq$ is defined by

$$C_\succeq(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}.$$  

- This definition starts with a binary relation and derives a choice set.
- A preference relation describes the decision maker’s taste. It is **not** observable.
- Therefore, the induced choice rule is also unobservable: the only way an observer can know $C_\succeq$ is to know $\succeq$.  

### GOAL

Observe DM’s behavior and use these observations to learn about her preferences.

- This is the opposite direction of induced choice.
- One starts with a choice rule and then deduces whether or not that particular choice rule could have been induced by some preference relation.
  - This is similar to the idea of finding a theory which is consistent with some data.
- One wants to find some possible rationale behind particular choice patterns.
- Naturally, this is not possible for an arbitrary choice rule. Only some choice procedures are consistent with rationality.
- The next step is to define a choice rule that could come from some preference.
Rationalizable Choice Rules

**Definition**
A choice rule $C$ is rationalized by $\succsim$ if $C = C_{\succsim}$ and $\succsim$ is a preference relation.

This is a choice rule that behaves as if it maximizes some complete and transitive binary relation among the available alternatives.

**Definition**
A choice rule $C$ is rationalizable if there exists a preference relation $\succsim$ such that $C = C_{\succsim}$.

If a choice rule is not rationalizable, there is no hope of learning whether or not choices are consistent with some preference.

If a choice rule is rationalizable, one may still not be able to identify the preferences behind it because there may be many possible $\succsim$ that are consistent with it.

Next, define the preference relation consistent with some given choice rule.
**Definition**

Given a choice rule $C$, its *revealed preference relation* $\succeq_C$ is defined by

$$x \succeq_C y \text{ if there exists some } A \text{ such that } x, y \in A \text{ and } x \in C(A).$$

- $x \succeq_C y$ reads “$x$ is revealed preferred to $y$”.
- The idea is that if DM chooses $x$ when $y$ is available one can say that $x$ is revealed to be weakly preferred to $y$.
- Observing choices and the menus they come from, one builds the preference relation $\succeq_C$.
  - These are the preferences consistent with particular choice behavior.

**Remark**

The definition says that there exists some menu including $x$ and $y$ where $x$ is chosen, not that $x$ is chosen in all menus including $x$ and $y$.

- The “revealed” preferences do not necessarily reflect DM’s levels of happiness or well-being; they only reflect what DM decided to do.
  - If we observe her order a burger at a restaurant, we say she reveals to prefer the burger to a salad, but that is not necessarily the same as saying she thinks the burger tastes better.
Proposition

If $C$ is rationalized by a complete and transitive $\succeq$, then $\succeq = \succeq_C$.

Proof.

Let $\succeq$ be a preference relation which rationalizes $C$; that is, for all $A \subseteq X$:

$$C(A) = C_{\succeq}(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}$$

- Suppose $x \succeq y$. We need to show $x \succeq_C y$.
  - Since $\succeq$ is complete and transitive, we have $x \succeq x$.
  - Thus $x \succeq z$ for all $z \in \{x, y\}$, so $x \in C_{\succeq}(\{x, y\})$.
  - Since $C = C_{\succeq}$, this implies $x \in C(\{x, y\})$.
  - Therefore $x$ is revealed to be preferred to $y$ and $x \succeq_C y$.

- Suppose $x \succeq_C y$. We need to show $x \succeq y$.
  - By definition, there exists some set $A$ with $x, y \in A$ and $x \in C(A)$.
  - Then $x \in C_{\succeq}(A)$ because $C = C_{\succeq}$.
  - By definition, $x \succeq z$ for all $z \in A$.
  - But $y \in A$, so $x \succeq y$. 

\qed
Revealed Preferences Are Rationalizable Preferences

**Proposition**

If $C$ is rationalized by $\succeq$, then $\succeq = \succeq_C$.

Now the proof is done, what does this mean?

**Remark**

The only preference relations that can rationalize $C$ are revealed preference relations.

To check whether or not a choice rule is rational, all one needs to do is check whether or not it acts as if it were “maximizing” its revealed preference relation.

Contrapositively, if $\succeq_C$ does not rationalize $C$, then no other preference relation will rationalize $C$.

This imposes restrictions on DM’s behavior: from some observed choices we can deduce what other (unobserved) choices will have to be.
Restrictions on Choice Rules

Example

Let \( X = \{a, b, c\} \) and let \( C \) be a rationalizable choice rule such that

\[
C(\{a\}) = \{a\}, \quad C(\{b\}) = \{b\}, \quad C(\{c\}) = \{c\}, \quad \text{and} \quad C(\{b, c\}) = \{b\}.
\]

Can one predict what \( C(\{a, b, c\}) \) looks like with the help of the previous result?

Claim: \( C(\{a, b, c\}) = \{a\} \).

- Note that \( a \succ_C a, \ a \succ_C b, \) and \( a \succ_C c \), so \( a \in C_{\succ_C} (\{a, b, c\}) = C(\{a, b, c\}) \).
- Since \( b \notin C(\{a, b\}) \), \( b \) must fail to be preferred to some element of \( \{a, b\} \), either \( \neg (b \succeq a) \) or \( \neg (b \succeq b) \).
  - But we have \( b \succeq b \) by completeness, therefore we must have \( \neg (b \succeq a) \).
- Similarly, we know \( \neg (b \succeq_C a) \) because \( b \notin C(\{a, b\}) \), so \( b \notin C_{\succeq_C} (\{a, b, c\}) \).
- Finally, we know \( \neg (c \succeq_C a) \) because \( c \notin C(\{a, c\}) \), so \( c \notin C_{\succeq_C} (\{a, b, c\}) \).
- Collecting these findings, \( C_{\succeq_C} (\{a, b, c\}) = \{a\} \).
- Since \( C \) is rationalizable, \( C(A) = C_{\succeq_C} (A) \) for all \( A \subseteq X \).
- Therefore \( C(\{a, b, c\}) = \{a\} \).
1. Prove that if $\succsim$ is asymmetric and negatively transitive, then $\succsim$ is transitive.

2. Prove that if $\succsim$ is a preference relation (i.e. it is complete and transitive), then:
   - $\succsim$ is a preference relation;
   - $\succ$ is asymmetric and transitive;
   - $\sim$ is an equivalence relation;
   - $x \succsim y$ and $y \sim z$ imply $x \succsim z$; and
   - $x \succsim y$ and $y \succ z$ imply $x \succ z$.

3. Prove that if $\succsim$ is a preference relation, then $C_{\succsim}(A) \neq \emptyset$ whenever $A$ is finite.

4. Prove that $\succsim$ is complete and acyclic (but not necessarily transitive) if and only if $C_{\succsim}(A) \neq \emptyset$ whenever $A$ is finite.