Lecture 14 Outline

1. Convexity
2. Concave and Convex Functions
3. Cardinal and Ordinal Properties

Announcement:
- The last exam will be tomorrow at 10:30am, in WWPH 4716.
Convex Sets

- Sometimes, we know the second order conditions of a optimization problem are satisfied because the function has a particular shape.
  - For a function from reals to reals, if $f' = 0$ and $f'' < 0$ then the critical point is a maximum.
- To discuss a function's shape, one needs a well behaved domain.

**Definition**

A set $X \subset \mathbb{R}^n$ is convex if $\forall x, y \in X$ and $\forall \alpha \in [0, 1]$ we have

$$\alpha x + (1 - \alpha)y \in X$$

- Sets are either convex or not: there are no concave sets!
Convex and Concave Functions

**Definition**

Suppose $X$ is a convex subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is:

- **concave** if
  \[
  f(\alpha x + (1 - \alpha) y) \geq \alpha f(x) + (1 - \alpha) f(y)
  \]
  for all $\alpha \in [0, 1]$ and all $x, y \in X$;

- **strictly concave** if
  \[
  f(\alpha x + (1 - \alpha) y) > \alpha f(x) + (1 - \alpha) f(y)
  \]
  for all $\alpha \in (0, 1)$ and all $x, y \in X$ such that $x \neq y$;

- **convex** if
  \[
  f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y)
  \]
  for all $\alpha \in [0, 1]$ and all $x, y \in X$;

- **strictly convex** if
  \[
  f(\alpha x + (1 - \alpha) y) < \alpha f(x) + (1 - \alpha) f(y)
  \]
  for all $\alpha \in (0, 1)$ and all $x, y \in X$ such that $x \neq y$;

- **affine** if $f$ is concave and convex
  \[
  f(\alpha x + (1 - \alpha) y) = \alpha f(x) + (1 - \alpha) f(y)
  \]
  for all $\alpha \in [0, 1]$ and all $x, y \in X$. 
Remember from last week.

**Definition**

The upper contour set of \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) at \( c = f(x_0) \) for some \( x_0 \in \mathbb{R}^n \) is the set

\[
\{ x \in X : f(x) \geq c \}
\]

- Concavity and upper contour sets are related.

**Claim**

If \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is concave then \( \{ z \in \mathbb{R}^n : f(z) \geq c \} \) is convex.
**Concave Functions Have Convex Upper Contour Sets**

\[ f \text{ is concave } \Rightarrow \{ z \in \mathbb{R}^n : f(z) \geq c \} \text{ is convex.} \]

**Proof.**

Concavity means \( \forall \alpha \in [0, 1] \) we have \( f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y) \).

- Consider the points \( x, y \in \{ z \in \mathbb{R}^n : f(z) \geq c \} \), by definition
  \[ f(x) \geq c \quad \text{and} \quad f(y) \geq c \]

- Therefore, \( \forall \alpha \in [0, 1] \) we have
  \[ f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y) \quad \text{(by concavity)} \]
  \[ \geq \alpha c + (1 - \alpha)c \quad \text{(by the inequalities above)} \]
  \[ = c \]

- We conclude
  \[ \alpha x + (1 - \alpha)y \in \{ z \in \mathbb{R}^n : f(z) \geq c \} \]

proving the claim.
Concavity, Convexity, and Calculus

**Theorem**

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a $C^2$ function. Then,

- $f$ is concave $\iff D^2 f(x)$ is negative semi-definite for all $x$
- $f$ is concave $\iff f(y) - f(x) \leq Df(x) \cdot (y - x)$
- $f$ is convex $\iff D^2 f(x)$ is positive semi-definite for all $x$
- $f$ is convex $\iff f(y) - f(x) \geq Df(x) \cdot (y - x)$

- Check the proofs of these results (they are not difficult).
- These results have obvious consequences for the maximization (minimization) of $f$. 
**Theorem**

If $f$ is concave and $x^*$ is a critical point of $f$, then $x^*$ is a local maximizer.

If $f$ is convex and $x^*$ is a critical point of $f$, then $x^*$ is a local minimizer.

- This follows immediately from the sufficient conditions for unconstrained extrema and the results in the previous slide.
- When functions are concave (or convex) we also know whether or not a local optimum is also global.
Optima of Concave and Convex Functions

**Theorem**

If $f$ is concave and $x^*$ is a local maximizer of $f$, then $x^*$ is a global maximizer.  
If $f$ is convex and $x^*$ is a local minimizer of $f$, then $x^*$ is a global minimizer.

**Proof.**

Suppose $x^*$ is not a global maximizer of $f$.

- Then there is a point $\hat{x}$ such that 
  
  \[ f(\hat{x}) > f(x^*) \]

- but then, by concavity, for all $\alpha \in [0, 1]$
  
  \[ f(\alpha x^* + (1 - \alpha)\hat{x}) \geq \alpha f(x^*) + (1 - \alpha)f(\hat{x}) \]

- This contradicts that $x^*$ is a local max (why?).

- Do the other proof as exercise, and make sure you answer the “why?” part in detail.
Quasiconcavity

**Definition**

Suppose \( X \) is a convex subset of \( \mathbb{R}^n \). A function \( f : X \to \mathbb{R} \) is:

- quasiconcave if
  \[
  f(x) \geq f(y) \quad \Rightarrow \quad f(\alpha x + (1 - \alpha)y) \geq f(y) \quad \forall \alpha \in [0, 1]
  \]

- strictly quasiconcave if
  \[
  f(x) \geq f(y) \text{ and } x \neq y \quad \Rightarrow \quad f(\alpha x + (1 - \alpha)y) > f(y) \quad \forall \alpha \in [0, 1]
  \]

**An equivalent characterization**

A function \( f : \mathbb{R}^n \to \mathbb{R} \) is quasiconcave if and only if
\[
f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}
\]
for all \( \alpha \in [0, 1] \)

- Prove that the two definitions are equivalent.
Quasi-Concavity and Upper Contour Sets

**Theorem**

A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave if and only if the set

$$\{x \in \mathbb{R}^n : f(x) \geq a\}$$

is convex for all $a \in \mathbb{R}$.

- In other words: the upper contour set of a quasiconcave function is a convex set, and if the upper contour set of some function is convex the function must be quasiconcave.
- Is this concavity?

**Example**

Suppose $f(x) = -x_1^2 - x_2^2$, draw the upper contour set.

**Example**

Suppose $g(x) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, draw the upper contour set.
Theorem

Let \( f : X \to \mathbb{R} \), where \( X \subset \mathbb{R}^n \) is open, be a \( C^1 \) function. Then, \( f \) is quasiconcave if and only if

\[
f(y) \geq f(x) \quad \Rightarrow \quad Df(x)(y - x) \geq 0
\]

Prove this.

Is this concavity?
Quasi-Concavity and Maximization

- Quasi concave functions have nice properties for maximization.

**Theorem**

Suppose \( f : X \to \mathbb{R} \) attains a maximum on \( X \). Then:

1. If \( f \) is quasi-concave, the set of maximizers is convex.
2. If \( f \) is strictly quasi-concave, the maximizer of \( f \) is unique.

- Prove this as an homework.
- Notice this does not guarantee that a solution exists.
- The (strict) quasi-concavity assumption plays a crucial role in economics as it tells us a lot about the solution of (constrained) optimization problems.
Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if and only if

$$\{ x \in \mathbb{R}^n : f(x) \leq a \}$$

is convex for all $a \in \mathbb{R}$.

- The lower contour set is a convex set.
  - Quasiconcavity is equivalent to the upper contour set being convex.
- Can you figure out theorems equivalent to the ones on the previous slides for quasiconvex functions?
Identifying Quasiconcave or Quasiconvex Functions

1. Solve for the level sets and graph a selection of them.
2. Decide by inspection which side of the plotted level set is the upper contour set and which side is the lower contour set.
3. Are the inspected upper or lower contour sets convex?
Relationship between Concavity and Quasiconcavity

- $f$ concave implies the upper contour set is convex.
- $f$ quasi-concave is equivalent to the upper contour set being convex.

**Theorem**

If $f : \mathbb{R}^n \to \mathbb{R}$ is concave, then $f$ is quasiconcave.
If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then $f$ is quasiconvex.

**Example**

$e^x$ is quasiconcave but not concave. In fact it is also convex and quasiconvex.

**Theorem**

Any increasing or decreasing function is both quasiconvex and quasiconcave.
Global Properties

REMARK

- Quasiconcavity and Quasiconvexity are **global properties** of a function.
- Unlike continuity, differentiability, concavity and convexity (of functions), they are not defined at a point.

- Next, we try to distinguish systematically between global and “non global” properties.
Loosely: A concept will be thought of as ordinal if “order" is all that matters.

- The statement $f(x) > f(y)$ is ordinal.
- “Greater than” is an ordinal concept since it does not matter how much greater one value is, it just matters that it is greater.

**Definition**

Given an $f : \mathbb{R}^n \to \mathbb{R}$ we say that property (*) is ordinal if

$f : \mathbb{R}^n \to \mathbb{R}$ has property (*)

and

$\Rightarrow g \circ f$ has property (*)

$g : \mathbb{R} \to \mathbb{R}$ is strictly increasing
Examples

Concavity is not ordinal

Let \( f : (0, \infty) \to \mathbb{R} \) be defined as \( f(x) = x^{\frac{1}{2}} \). This is a concave function.

- Notice that
  \[
  x > y \implies f(x) > f(y)
  \]
- Now take \( g \) as follows
  \[
  g(y) = y^4
  \]
- Then
  \[
  g \circ f(x) = x^2
  \]
- And again we have that
  \[
  x > y \implies g \circ f(x) > g \circ f(y)
  \]
- But \( x^2 \) is a convex function.
- Therefore, concavity is not an ordinal property.
Examples

Quasiconcavity is Ordinal

Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is quasiconcave. That is, the set \( \{ x \in \mathbb{R}^n : f(x) \geq c \} \) is convex \( \forall c \in \mathbb{R} \).

- Take a strictly increasing function
  \[ g : \mathbb{R} \to \mathbb{R} \]

- The following 2 sets are equivalent
  \[ \{ x \in \mathbb{R}^n : g \circ f(x) \geq c \} \iff \{ x \in \mathbb{R}^n : f(x) \geq g^{-1}(c) \} \]

  why?

- therefore the upper contour set remains convex

- Thus quasiconcavity is an ordinal property.
Cardinal Properties

**Definition**

Given an \( f : \mathbb{R}^n \to \mathbb{R} \) we say that property (*) is cardinal if

\[
f : \mathbb{R}^n \to \mathbb{R} \text{ has property (*)}
\]

and

\[
\Rightarrow g \circ f \text{ has property (*)}
\]

\( g : \mathbb{R} \to \mathbb{R} \) is strictly increasing affine function

- Remember, a function is affine if
  \[
f (\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y)
\]
  for all \( \alpha \in [0, 1] \) and all \( x, y \in X \).

- A ordinal property is also cardinal (obvious), while a cardinal property is not necessarily ordinal.