Continuous Covering Location Problems

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A location problem: Where are we going to put the thing(s)?
What places are available?
On what basis do we choose?

Taxonomy of Facility Location Problems

- Nature of the network
- Nature of facilities
- Nature of demand
- Objective and setting
### Locational space

- Which places are available?
- Planer
- Network
- Discrete

### Continuous problem characteristics

- Planer (or along one or $n$-dimensions)
- Cannot give an exhaustive list of potential site locations
- Site generating

### Site generating

- Where do candidate sites come from?
- Typically, described by some polygon
- Described by Geographic Information System (GIS)
- Forbidden regions/Barriers
- Examples?

### Goal

- Single or multi-objective
- Pareto efficient solutions or optimal?
- How do you define optimal?
- Math model solutions are a starting point. Details that are not captured in the model can also be important.
Distance measures

- Euclidean Distance
  \[ d^\text{Euc}(X, P) = \sqrt{(a-x)^2 + (b-y)^2} \]
- Rectangular
  \[ d^\text{Rect}(X, P) = |(a-x)| + |(b-y)| \]
- Rectangular max
  \[ d^\text{Max}(X, P) = \max\{(a-x)| + |(b-y)|\} \]
- p-norm
  \[ d^p(X, P) = \sqrt[p]{(a-x)^p + (b-y)^p} \]
- Hexagonal (for mapping)
  Minkowski distances (satisfy the triangle inequality)

Distance complications

- Barriers
- Terrain - anything that affects speed
- Current/wind - anything that adds a constant vector to velocity

Objectives

Objectives in a location problem will include a summary of the effects of distance.

- Pull - Objectives are improved when distance is decreased
- Push - Objectives improved when distance increases

Full Covering Problems

- Locate a facility to cover a region
- Examples
  **Full covering model** Given the points \( P_m(m \in M) \) in the plane we must find the circular ball with minimum radius covering them all; its centre is then the optimal site.
Convex: C is said to be convex if, for all x and y in C and all t in the interval [0,1], the point \((1 - t)x + ty\) is in C.

Convex hull: the boundary of the minimal convex set containing a given non-empty finite set of points in the plane. Unless the points are collinear.

Full Covering - Convex Hull

Elzinga-Hearn method

- Initialize: Pick any two demand points
- Handle two points
  - Let C be the circle defined the first two demand points
  - If C covers all demand points, C is the MCC: Stop
  - If C does not cover all demand points, then add any demand point outside C and proceed with these three demand points.

Unconstrained Full Covering

- Find the minimal covering circle (MCC) that covers the points.
- At least two demand points \(P_m\) lie on the MCC.
- If there are only two demand points \(P_j\) and \(P_k\) on the MCC, they form a diameter of the MCC.
- If three or more demand points are on the MCC (which is then fully determined), three of these points form an acute triangle.
- Any circle satisfying one of the previous two properties and which covers all demands points is the MCC

Elzinga-Hearn method cont.

- Handle three points
  - Determine if the triangle formed by the three points is an acute triangle. If not, drop the point at the obtuse angle and proceed with the remaining points.
  - Let C be the circle defined by the demand points
  - If C covers all demand points, C is the MCC: Stop
  - Otherwise, add any demand point \(P\) which is outside C
  - Drop one of the former three demand points chosen by figure.
  - Repeat
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MCC Discussion

• Note that for each step, the circle gets larger, so it will complete.
• Certain steps are those that you do not know how to do. Search for code or a library that can solve those steps and include them in your program. Remember to cite your sources.
• Another method uses Voronoi diagrams, which involves identifying the regions where each given point is the ‘extreme’ point. This requires knowledge of computation geometry methods (or use of software libraries).
• Etzinga-Hearn is relatively fast in practice.

Constrained Continuous Covering

Constrained Minimum Covering Circle In the presence of a feasible region \( S \), the constrained euclidean distance min-max single facility location problem (i.e. find the smallest covering circle with center in \( S \) (CMCC)), maybe solved using the properties:

- Either the CMCC is equal to the MCC, if the solution to the MCC lies in \( S \).
- Or the optimal site (center of circle) must be either
  - A point of \( S \) closest to some destination \( P_i \), and \( P_i \) lies on the CMCC
  - the point of intersection of the bisector of two destinations \( P_i \) and \( P_j \) with the boundary of \( S \), closest to \( P_i \) and \( P_j \), and then both \( P_i \) and \( P_j \) lie on the CMCC.

Finding the CMCC

1. Solve the MCC. If the center of \( C \) is in \( S \), stop: \( C \) is the CMCC.
2. Try out all the points \( P_i \) in turn. For each one, calculate \( X \) in \( S \) that is closest to \( P_i \). Construct a circle \( C \) with center \( X \) and radius \( d_{Euc}(X, P_i) \). If any circle \( C \) is a covering circle, stop: \( C \) is the CMCC.
3. Try out all pairs \( P_i, P_j \). For each one construct the bisector of \( P_i, P_j \). Let \( X \) be the point of intersection with the boundary of \( S \) (if it exists). Take the circle \( C \) centered at \( X \) with radius \( d_{Euc}(X, P_i) \). If \( C \) is a covering circle, save \( C \) as a candidate CMCC.
4. If this method did not stop in Steps 1 or 2, the smallest candidate \( C \) found in step 3 is the CMCC.

Discussion on CMCC

• How to identify the point of intersection of a line and a polygon.
• Computation geometry
• Spatially enabled databases
• ST_Intersection(geometry, geometry) function in SFSQL
• You do not have to solve the CMCC, but you should know how to draw figures to explain the method.
• There can also be a constraint that some regions are restricted.
**Motivation**

- What if the number of facilities and their covering distance is fixed?
- Not able to cover all demand points.
- Therefore, the goal is to cover as many demand points as possible.

Maximal Covering Problem: Find the (center $X$ of a) circular disk of radius $r$ covering the largest possible weight.

$$\begin{align*}
\max & \sum_i C_i w_i \\
\text{s.t. } & C_i = \begin{cases} 
0 & \text{if } d_{Euc}(X, P_i) > r \\
1 & \text{if } d_{Euc}(X, P_i) \leq r
\end{cases}
\end{align*}$$

**Finding boundary points**

- For every pair $P_i, P_j$ where $d_{Euc} \leq r$, find the points that form an equilateral triangle.

**Find maximal covering**

- Draw circles around each demand point of radius $r$
- For each region defined by the circles, identify the weight covered.
- Regions with the highest coverage are the solution regions.
- Optimal regions are either full disks or disk boundaries.
- Therefore, check only demand points or disk boundary points.
- How do you find disk boundary points?

**Varying radius**

- Minimal quantile location problem: given a desired coverage, what is the minimum radius that still allows to obtain the coverage, and where should the centre be placed.
- Optimal region decreases as the radius decreases.
- At some point, the last region reduces to a point, beyond which two radius disks are touching, or three disks have only one point in common.
- So, only points that are a midpoint between two demands, or at the same distance from three demand points.
Solving minimal quantile location problem

- Take all pairs or triplets of demand points, and construct the smallest enclosing disk $C$ and calculate the weight of covered demand points. Keep only the disks that have sufficient coverage to be the candidate set.
- The retained disk with the smallest radius is the optimal minimal quantile solution.
- Note that this method finds the minimal quantile location problem for all potential radii.

Solving the Empty covering problem

- Create the (Closest Point) Voronoi diagram.
- The solution must lie either on a corner point of the feasible region, at the intersection of a Voronoi node and the feasible region boundary, or at the intersection of two Voronoi edges.

Empty covering problems

- Where to locate an undesirable facility
- Examples?
  - Site, within a feasible region, that is farthest possible from any sensitive places.

Minimal Covering Problem

- Minimal Covering: For a given radius $r$, what is the smallest possible coverage achievable?
- Maximal radius: For a given maximal level of coverage, what is the largest radius possible?
- Solution: Same as maximal covering problem, except pick regions that are Uncovered.
Minimal covering

- Similar to Maximal radius problem
- The candidate circles are those with one of the following criteria
  - 1 demand point, and the center is a corner of the feasible region \( S \)
  - 2 demand points, and center is on the boundary of \( S \)
  - 3 (or more) demand points forming an acute triangle

Push-Pull Covering Models

- Models that combine qualities of both push and pull models.
- Multiple objectives
- Fix a bound (limit) on one objective, and optimize the other
- Find all non-dominated solutions (Pareto optimal) for the bicriterial problem.

Positioning Models

- In marketing, brand positioning
- Each brand is characterized by \( n \) quantitative attributes.
- These attributes are an \( n \)-dimensional coordinate system.
- Customer groups are also represented on same system, with weights.
- Assume customers purchase brand “closest” to them.
- Where to add a new brand is a maximal covering problem.

Multiple Facility Covering Location Models

- When there are multiple facilities, the question is which interactions between demand points and facilities need to be accounted for in the model.
- Allocations between facilities and demand points may be fixed or allocated as part of the model.
- Multi-facility models have facilities assigned to demands. Location-allocation models both site facilities and assign them to demands.
- Facilities are already assigned to demands, but there is interaction between facilities.
- **Pull models** Minimize the maximum weighted distance/cost among facility-demand pairs and inter-facility pairs.
- **Push models** Maximize the minimum weighted distance/cost between facility/fixed-point pairs and inter-facility pairs.

**p-center Covering**

- Cover all demand points using p circles (ball) of minimal equal radius.
- Problem in 2 dimensions is \( NP \)-hard.
- These problems are solved in polynomial time in relation to \( \mathcal{O} \) -the number of demand points, but exponential in \( p \).
- Similar to the MCC, candidate points are at the center of the smallest covering circle of some set of demand points. Identify these points like for MCC, and choose \( p \) of these. Find the set \( p \) with the minimum-maximum radius circle.
- Usually solved using heuristics.
- Note that for the \( p \)-center, only one facility is actually at \( r \), there may be multiple solutions as the other \( p - 1 \) locations are not constrained.

**Other variants**

- **\( p \)-Maximal Covering** Cover maximal possible weight of demand points using \( p \) disks of given radius.
- **\( p \)-Dispersion** Locate \( p \) points as far apart from each other within the feasible region.
- Continuous \( p \)-covering Cover all points of an area by \( p \) circles of the smallest radius.

**Extensive Facility Covering Location Models**

- Facilities can be lines, line segments, circles, or grids
- Line to approximate two-dimensional points AKA linear regression
- When distances are not-Euclidean (i.e. no such thing as orthogonal) other techniques are needed to get closest distances.
- Book covers, this course will not.