Inventory Models
IE 1079/2079 Inventory Models - Varying Demands

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Class outline
- Homework 1 & 2 review
- Exam discussion
- Characteristics of inventory
- Inventory models - Varying demands

Homework 1 - Characterizing a problem
- Characterizing demand types
- Probabilistic demands - stochastic, drawn from a probability distribution. Contrast to Dynamic vs. Static. Static implies that the probability distribution of the demands does not change. Dynamic implies that the distribution changes over time, and the solution may include changes over time - example disaster relief operations
- Facilities - Capacitated vs incapacitated refers to limits on the facilities ability to meet a quantity of demand
- Facilities - General fulfillment vs nearest facility. Are you forced to use a given facility?

Homework 1 - Coding
- Comments are important. They are what allows you and anyone else to understand what you did a year from now.
- Identify variables, describe what the method does. Detail all branching (i.e. what does your 'if' statement do)
- For this course, if your comments are complete, that will get you a 'B' on homework problems involving coding.
- These problems are the same style as the midterm. A narrative description of objectives and constraints. You translate this into a math formulation.

- Problem 2 - The parameters given are badly scaled. You can solve as is, or you can replace the parameters with facility opening cost = 50, cost per unit demand per distance = 0.35. Or change cost per unit demand per distance = 0.35 (leaving Facility opening cost = 20) and transportation cost until the solution uses one less facility.

- Problem 3 - Drop. It requires proof techniques you may not have used in several years.

- Characterize problems - Take a scenario and categorize the location or inventory problem.

- Modeling - Take a written description of a problem (situation and objective), develop an LP or inventory model.

- LP constraints - From an English description of constraints, write math formulation.

- LP constraints - Explain constraints in a math LP model in English.

- Inventory - Newsboy problem

- Inventory models - Build a cost function

- One 3” x 5” index card

- Most of the exam is going to be modeling - i.e. few equations

- Answers for modeling questions are in math programming format - like you learned in linear programming

- I’m expecting you to use your index card for equations relating to inventory - i.e. newsboy models

- Types of inventory

- Purpose of inventory

- Characteristics of Inventory Systems

- Costs
Types of Inventory

- Raw materials - Resources required in production or processing activity of the firm.
- Components/subassemblies - Items that have not yet reached completion in the production process.
- Work-in-process (WIP) - Inventory waiting for processing or being processed. May include component inventories or raw materials.
- Finished goods - Final products of production processes.

Why hold inventory?

- Economies of scale - Because of setup cost, it is cheaper to set up a production run/order once, then hold items until then set up production for each item.
- Uncertainties - Buffer against uncertainty of demand.
- Speculation - If value of an item is expected to increase.
- Transportation - Goods are in transit because transit time is long.
- Smoothing - Storing inventory buffers production from the effect of demand variation.
- Logistics - Other reasons that are a result of business and operational considerations such as minimum order quantities and operational continuity.
- Control Costs - Larger buffers do not require as close management as inventories that are at the bare minimum.

Characteristics of Inventory Systems

- Constant vs. variable demand
- Known vs. random
- Lead time - How long is it from order to receipt of order?
- Review time - How often are inventory levels updated?
- Excess demand - What happens to demands that are not filled immediately from inventory?
- Changing inventory - Does the utility/usability of inventory change over time?

Inventory costs - Holding costs

- Cost of physical space
- Taxes and insurance costs
- Breakage, spilage, deterioration, obsolescence
- Opportunity costs of other investments (cost of capital)
### Inventory costs - Order costs

- Costs due to the inventory ordered or produced.
- Fixed - independent of size of orders.
- Variable - incurred on a per unit basis.
- Note: does not include overhead costs.

### Inventory costs - Penalty costs

- Shortage costs/stock-out costs
- Cost of not having sufficient stock on hand to satisfy a demand when it occurs.
- Question: does this result in back-order or is it lost?
- Loss of profit.
- Delay costs.
- Loss of good will.

### Inventory Models with Varying Demands

- Management of uncertainty determines the success of a firm.
- Sources of uncertainty
  - Consumer preferences and market trends
  - Availability
  - Cost of labor and resources
  - Uncertainty in resupply times
  - Uncertainty in weather affecting operations
  - Uncertainty of financial variables such as stock and commodity prices and interest rates
  - Uncertainty in demand

### Describing randomness

- You cannot predict the exact sales of an item in advance, but you can use past experience.
- Previous observations about random phenomenon can be used to estimate its probability distribution.
- The goal of properly quantifying uncertainty and the consequences of decisions is to build intelligent strategies.
- Objective: Minimize expected cost or Maximize expected profits.
Describing uncertainty in demand

- Demand can be described as having two parts
- \( D = D_{\text{Deterministic}} + D_{\text{Random}} \)
- Treat the model as deterministic if:
  - Variance in the \( D_{\text{Random}} \) component is small compared to \( D \)
  - Predictable variation is more important than the random variation (e.g. seasonality)
  - Problem structure is too complex to include an explicit representation of randomness in the model.

Inventory Control Models

- Periodic review - Inventory level is known at discrete points in time only. E.g. inventory taken at end of week or inventory is reported at end of day.
- Continuous review - Inventory level is known at all times. E.g. point of sale systems that transmit sales information instantly.
- One-period models - Objective is to balance costs of overage and underage.
- Multi-period models - Policy for use over many periods of changing demands. Note: very sophisticated, but not heavily used as usually there is a decision point at every period.

The Nature of Randomness

- Example: 100 points of demand data
- No obvious demand to the data, so it is difficult to predict the demand for any given week.
- But we can attempt to determine if there is a probability distribution that can be used to describe the demand for a week.

Nature of Randomness - Probability distribution

- Anyone want to guess?
- So we can model the demands using a probability distribution so our model can include effects of high and low events in about the right frequency
- Beware of negative values when there should not be any.
The Newsboy Model

- A single product is to be ordered at the beginning of a period and can only be used to satisfy demand during that period. Assume all relevant costs can be determined on the basis of ending inventory.
  - \( c_o \) = Cost per unit of inventory remaining at end of the period - Overage cost
  - \( c_u \) = Cost per unit of unsatisfied demand (i.e. negative inventory) - Underage cost
  - \( D \) = Demand with density function \( f(x) \) and c.d.f. \( F(x) \)
  - \( Q \) = Number of items to purchase at beginning of each inventory period

Develop the cost function

- Develop an expression of cost function incurred as a function of the demand variable \( D \) and the decision variable \( Q \)
- Determine the expected value of this expression with respect to the density function or the probability function of demand
- Determine the value of \( Q \) that minimizes the expected cost function
- Note: the basis is the cost compared to the potential of meeting all demand
- Two possibilities: \( D \geq Q, D < Q \)

Newsboy cost function

\[
G(Q, D) = c_o \max(0, Q - D) + c_u \max(0, D - q)
\]
\[
G(Q) = E(G(Q, D))
\]
\[
G(Q) = c_o \int_0^\infty \max(0, Q - x) f(x) dx + c_u \int_0^\infty \max(0, x - Q) f(x) dx
\]
\[
G(Q) = c_o \int_0^Q (Q - x) f(x) dx + c_u \int_Q^\infty (x - Q) f(x) dx
\]

Optimizing Newsboy cost function

- Determine the value of \( Q \) that minimizes the expected cost \( G(Q) \)
  \[
  \frac{dG(Q)}{dQ} = c_o \int_0^Q f(x) dx + c_u \int_Q^\infty -f(x) dx
  \]
  \[
  \frac{dG(Q)}{dQ} = c_o F(Q) + c_u (1 - F(Q))
  \]
- By taking the first derivative and setting this equal to 0 we get
  \[
  F(Q^*) = \frac{c_o}{c_o + c_u}
  \]
Newsboy Example

- Mac purchases magazine copies for $0.25 and sold them for $0.75
- Mac’s supplier pays Mac $0.10 for each unsold copy
- Demand is normally distributed with $\mu = 11.73$ and $\sigma = 4.74$
- Overage cost is $c_o = 0.25 - 0.10 = 0.15$
- Underage cost is the profit on a sale $c_u = 0.75 - 0.25 = 0.50$

Newsboy Example solution

- $F(Q^*) = \frac{c_u}{c_u + c_o} = 0.50/0.65 = 0.77$
- Based on the Normal distribution $z = 0.74$
- $Q = \sigma z + \mu = 4.74 \times 0.74 + 11.73$
- $Q = 15.24 \approx 15$

Lot Size - Reorder Point

- For the Economic Lot Size Problem with Lead Times, solution is to order a quantity $Q$ when inventory reached the level $R$ that leads to the order being received when the inventory reaches 0
- $Q$ is now the amount ordered
- Continuous review, demand is random and stationary
- Fixed leadtime $\tau$ for placing the order
- Setup cost $K$ per order. Holding cost $h$ per unit per year
- Order cost $c$ per item. Stockout cost $p$ per unit of unmet demand

QR - Demand and decision variables

- Demand is a continuous random variable with pdf $f(x)$ and cdf $F(x)$
- $\mu = E(D)$ and $\sigma = \sqrt{\text{var}(D)}$ during lead time
- Rate of demand is $\lambda$ per year
- Decision variables $Q$ = lot size or order quantity
- $R$ = Reorder point
Inventory Models
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Preliminaries
Homework 1 & 2 review
Inventory Overview
Inventory Models - Varying Demands

QR - cost function

- Holding cost is proportional to inventory
  \[ h(Q/2 + R - \lambda \tau) \]
- Setup cost is based on the cycle time. So average is
  \[ (K\lambda)/Q \]
- Penalty cost is based on shortages, which is when the demand during lead time is greater then \( R \)
  \[ E(\max(D - R, 0)) = \int_R^\infty (x - R)f(x)dx = n(R) \]
- Stockouts per unit time are then \( n(R)\lambda / Q \)
- Ordering cost is based on the number ordered \( X \) cost per item
  \[ \text{Ordering cost} = \frac{\lambda c}{Q} \]
  Note, because it is independent of \( Q \) and \( R \), ignore this.

QR - Solution

Choose \( Q \) and \( R \) to optimize the total cost function.
\[ G(Q, R) = h(Q/2 + R - \lambda \tau) + (K\lambda)/Q + pn(R)\lambda / Q \]

To solve, iteratively solve the following two equations:
\[
Q = \sqrt{\frac{2\lambda(K + pn(R))}{h}} \\
1 - F(R) = Qh/p\lambda
\]

QR - Example

- For a shop (Harvey's) selling a gourmet mustard, mustard costs \( $ \) per jar and requires 6 month lead time for replenishment of stock.
- Interest rate = 20%. Loss of goodwill for not having the mustard is \( $ 25/jar \).
- Bookkeeping expenses for placing an order amount to $ 50.
- During the six month replenishment time, Harvey sells an average of 100 jars, with standard deviation of 25.
- What should Harvey's reorder policy be?

Harvey's

- First find reorder point \( R \) and lot size \( Q \). So start as if it were the economic lot size problem
- Note that yearly demand = 2 * six-month demand = 2 * 100 = 200

\[
Q_0 = \sqrt{\frac{2\lambda(K + \lambda \tau)}{h}} \\
= \sqrt{2 \times 50 \times 200 / (0.20 \times 10)} = 100 \\
R_0 = 1 - F(R_0) = \frac{Q_0 h}{p\lambda} \\
= 100 + 2/25 \times 200 = 0.04
\]
Then, find the $z$-value (Normal distribution) corresponding to $1 - F(R_0) = 0.04$

- $z = 1.75$
- Solve $R = \sigma z + \mu = 25 \times 1.75 + 100 = 144$

Next, solve for $n(R)$. Because this is the standard normal, $n(R)$ is computed using the standard loss function $L(z)$

$$L(z) = \int_{z}^{\infty} (t-z)\phi(t)dt$$

$$L(1.75) = 0.0162$$

$$n(R) = (\max(D - R, 0)) = \int_{R}^{\infty} (x-R)f(x)dx$$

$$= \sigma L \left(\frac{R - \mu}{\sigma}\right)$$

$$= \sigma L(z)$$

$$= (25 \times 0.0162) = 0.405$$

$$Q_1 = \sqrt{\frac{2\lambda [K + pn(R)]}{h}}$$

$$= \sqrt{\frac{2 \times 200}{2} \times [50 + 25 \times .405]}$$

$$= 110$$

- $Q_1 >> Q_0$ so iterate between solving for $R$ and $Q$ again.
- Stop when the next iteration of $Q$ is less than 1 from the previous iteration.