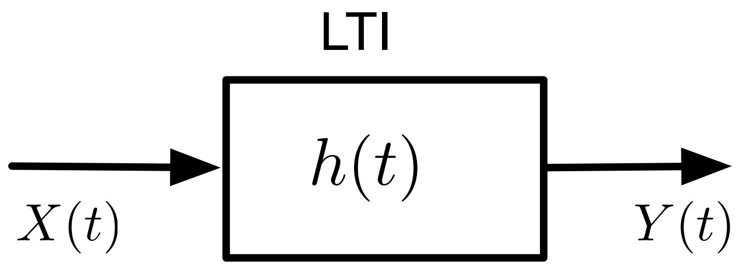
Linear Time-invariant Systems with Random Inputs



Input $X(t)$

output $Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau$ convolution integral

Mean of $Y(t)$

$$
\begin{aligned}
E[Y(t)] &= E\left[\int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau\right] \\
&= \left[\int_{-\infty}^{\infty} h(t-\tau)E[X(\tau)]d\tau\right] = (h * \eta_X)(t)
\end{aligned}
$$

Remarks

- Existence of $E[Y(t)]$ when $X(t)$ is wide sense stationary

$$
\begin{aligned}
E[Y(t)] &= \eta_X \int_{-\infty}^{\infty} h(t-\tau)d\tau \\
|E[Y(t)]| &\leq |\eta_X| \int_{-\infty}^{\infty} |h(\psi)|d\psi
\end{aligned}
$$

existence of $E[Y(t)]$ requires $|E[Y(t)]| < M$, thus we need

$$\int_{-\infty}^{\infty} |h(\psi)|d\psi < L \quad \text{bounded}$$

or that the system be BIBO stable.

Autocorrelation of $Y(t)$

$$\begin{aligned}
R_{XY}(t_1,t_2) &= E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} h(t_2-\tau)E[X(t_1)X(\tau)]d\tau \\
&= \int_{-\infty}^{\infty} h(t_2-\tau)R_{XX}(t_1,\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_1,t_2-\alpha)d\alpha
\end{aligned}$$

where we let $\alpha = t_2 - \tau$, $d\alpha = -d\tau$. Notice that the convolution is with respect to the second variable of the autocorrelation.

$$\begin{aligned}
R_{YY}(t_1,t_2) &= E[Y(t_1)Y(t_2)] = \int_{-\infty}^{\infty} h(t_1-\tau)E[X(\tau)Y(t_2)]d\tau \\
&= \int_{-\infty}^{\infty} h(t_1-\tau)R_{XY}(\tau,t_2)d\tau \\
&= \int_{-\infty}^{\infty} h(\beta)R_{XY}(t_1-\beta,t_2)d\beta
\end{aligned}$$

where we let $\beta = t_1 - \tau$. Notice the convolution is with respect to the first variable of the autocorrelation.

Replacing $R_{XY}(.,.)$ in the last equation we get

$$\boxed{R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(t_1-\beta,t_2-\alpha)d\beta d\alpha}$$

.

Remarks

- $R_{YY}(t_1,t_2)$ can be obtained directly

$$\begin{aligned}
R_{YY}(t_1,t_2) &= E[Y(t_1)Y(t_2)] = E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1-\beta)h(t_2-\alpha)X(\beta)X(\alpha)\right] d\alpha d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_2-\alpha)h(t_1-\beta)R_{XX}(\alpha,\beta)d\alpha d\beta
\end{aligned}$$

- Let $X(t)$ be strictly or w.s.s.

$$\begin{aligned}
R_{XY}(t_1,t_2) &= \int_{-\infty}^{\infty} h(\alpha)\underbrace{R_{XX}(t_1,t_2-\alpha)}_{R_{XX}(\tau-\alpha)}d\alpha \qquad \tau = t_2-t_1 \\
&= (h * R_{XX})(\tau)
\end{aligned}$$

$$\begin{aligned}
R_{YY}(t_1,t_2) &= \int_{-\infty}^{\infty} h(\alpha)\underbrace{R_{XY}(t_1-\alpha,t_2)}_{R_{XY}(\tau+\alpha)}d\alpha \\
&= \int_{-\infty}^{\infty} h(-\beta)R_{XY}(\tau-\beta)d\beta = h(-\tau) * R_{XY}(\tau)
\end{aligned}$$

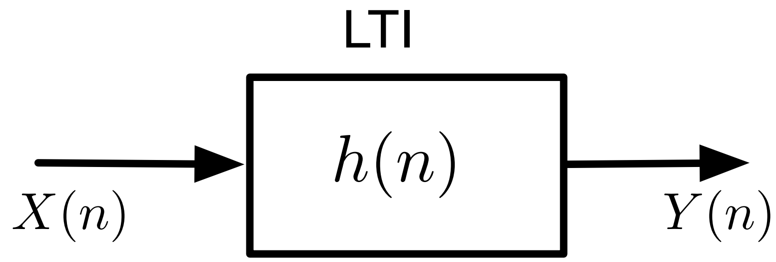
so that

$$R_{YY}(t) = h(t) * h(-t) * R_{XX}(t)$$

- The above results can be extended to the covariance by letting $\tilde{Y}(t) = Y(t) - \eta_Y(t)$ and using

$$C_{YY}(t_1,t_2) = R_{\tilde{Y}\tilde{Y}}(t_1,t_2)$$

LTI Discrete-time Systems with Random Inputs



$$\boxed{Y(n) = \sum_k h(n-k)x(k) = \sum_k h(k)x(n-k)}$$

Mean

$$E[Y(n)] = \sum_k h(n-k)E[X(k)]$$

$X(n)$ wide sense stationary

$$E[Y(n)] = \sum_k h(k)E[X(n-k)] = \eta_X \sum_k h(k) = H(1)\eta_X$$

$$H(z) = \sum_k h(k)z^{-k}|_{z=1}$$

Autocorrelation

$$\begin{aligned}
R_{XY}(m,n) &= E[X(m)Y(n)] = E\left[X(m)\sum_k h(k)X(n-k)\right] \\
&= \sum_k h(k)R_{XX}(m,n-k) \\
R_{YY}(m,n) &= E[Y(m)Y(n)] = \sum_k h(k)\sum_\ell h(\ell)R_{XX}(m-k,n-\ell) \\
&= E\left[\sum_k h(k)X(m-k)Y(n)\right] = \sum_k h(k)R_{XY}(m-k,n)
\end{aligned}$$

Special case: $X(n)$ is w.s.s.

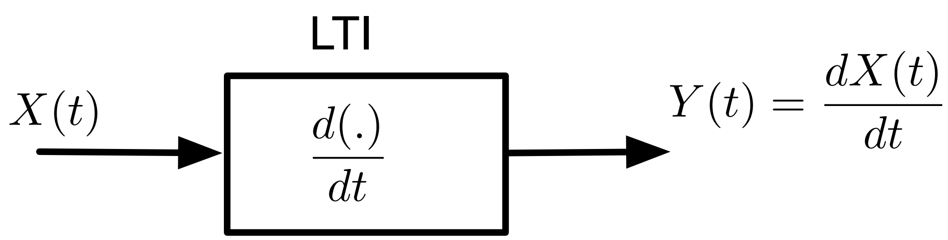
$$\mu = m - n \quad \Rightarrow \quad \sum_k h(k) R_{XX}(\mu - k) = (h(\mu) * R_{XX}(\mu)) = R_{XY}(\mu)$$

$$\sum_{-\ell} h(\ell) R_{XY}(\rho - \ell) = h(-\rho) * R_{XY}(\rho) = R_{YY}(\rho)$$

so that

$$R_{YY}(\rho) = h(-\rho) * R_{XY}(\rho) = h(-\rho) * h(\rho) * R_{XX}(\rho)$$

Differentiator



$Y(t) = dX(t)/dt$ defined in mean–square sense, find $\eta_Y(t)$, $R_{YY}(t_1, t_2)$. Is $Y(t)$ w.s.s. if $X(t)$ is w.s.s.?

$$
\begin{aligned}
\eta_Y(t) &= E[Y(t)] = E\left[\frac{dX(t)}{dt}\right] = \frac{dE[X(t)]}{dt} = \frac{d\eta_X(t)}{dt} \\
R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] = E\left[X(t_1)\frac{dX(t_2)}{dt_2}\right] = \frac{dE[X(t_1)X(t_2)]}{dt_2} = \frac{dR_{XX}(t_1, t_2)}{dt_2} \\
R_{YY}(t_1, t_2) &= E\left[\frac{dX(t_1)}{dt_1}Y(t_2)\right] = \frac{dE[X(t_1)Y(t_2)]}{dt_1} = \frac{dR_{XY}(t_1, t_2)}{dt_1}
\end{aligned}
$$

So that

$$
R_{YY}(t_1, t_2) = \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}
$$

Note If we use

$$
R_{YY}(t_1, t_2) = E\left[Y(t_1)\frac{dX(t_2)}{dt_2}\right] = \frac{dR_{YX}(t_1, t_2)}{dt_2}
$$

although correct, we cannot use equation $R_{YY}(t_1, t_2) = dR_{XY}(t_1, t_2)/dt_1$ to get $R_{YY}(t_1, t_2)$.

If $X(t)$ is w.s.s. then

$$
\begin{aligned}
&\eta_X(t) \text{ constant so } \ \eta_Y(t) = 0 \\
&R_{XX}(t_1, t_2) = R_{XX}(\tau) \qquad \tau = t_2 - t_1 \\
&R_{XY}(t_1, t_2) = \frac{dR_{XX}(t_2 - t_1)}{dt_2} = \frac{dR_{XX}(\tau)}{d\tau}\frac{d\tau}{dt_2} \\
&\text{so} \quad R_{XY}(\tau) = \frac{dR_X(\tau)}{d\tau} \\
&R_{YY}(t_1, t_2) = \frac{dR_{XY}(t_2 - t_1)}{dt_1} = \frac{dR_{XY}(\tau)}{d\tau}\frac{d\tau}{dt_1} \\
&\text{so} \quad R_{YY}(\tau) = -\frac{dR_{XY}(\tau)}{d\tau} = -\frac{d^2 R_X(\tau)}{d\tau^2}
\end{aligned}
$$

Moving averaging (MA) System

$$Y(n) = X(n) - X(n-1)$$

Is $Y(n)$ w.s.s. if $X(n)$ is w.s.s.?

Mean

$$E[Y(n)] = E[X(n)] - E[X(n-1)] = \eta_X(n) - \eta_X(n-1)$$

Autocorrelation

$$\begin{aligned} R_{XY}(m,n) &= E[X(m)Y(n)] = E[X(m)X(n) - X(m)X(n-1)] \\ &= R_{XX}(m,n) - R_{XX}(m,n-1) \\ R_{YY}(m,n) &= E[Y(m)Y(n)] = E[(X(m) - X(m-1))(X(n) - X(n-1))] \\ &= R_{XX}(m,n) - R_{XX}(m,n-1) - R_{XX}(m-1,n) + R_{XX}(m-1,n-1) \end{aligned}$$

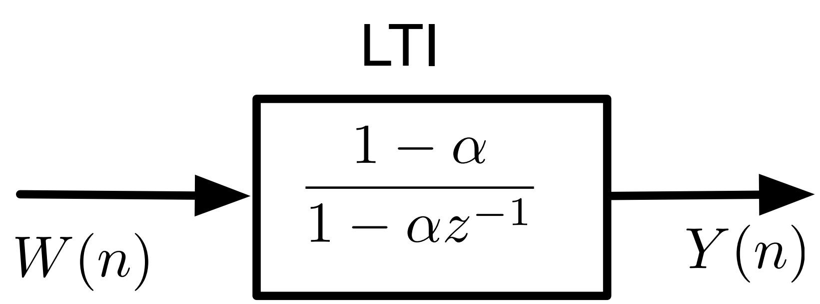
If $X(n)$ is w.s.s. then

$$\eta_Y(n) = 0$$
$$R_{XY}(n-m) = R_{XX}(n-m) - R_{XX}(n-1-m)$$
$$\ell = n-m, \quad \Rightarrow \quad R_{XY}(\ell) = R_{XX}(\ell) - R_{XX}(\ell-1)$$
$$R_{YY}(n-m) = R_{XX}(n-m) - R_{XX}(n-1-m) - R_{XX}(n-m+1) + R_{XX}(n-m)$$
$$\ell = n-m, \quad \Rightarrow \quad R_{YY}(\ell) = 2R_{XX}(\ell) - R_{XX}(\ell-1) - R_{XX}(\ell+1)$$

For the w.s.s. case, using that the impulse response of the MA system is $h(n) = \delta(n) - \delta(n-1)$ we have

$$\begin{aligned} R_{XY}(m) &= h(m) * R_{XX}(m) = R_{XX}(m) - R_{XX}(m-1) \\ R_{YY}(m) &= h(-m) * R_{XY}(m) = [\delta(m) - \delta(m+1)] * R_{XY}(m) = R_{XY}(m) - R_{XY}(m+1) \\ &= [R_{XX}(m) - R_{XX}(m-1)] - [R_{XX}(m+1) - R_{XX}(m)] \\ &= 2R_{XX}(m) - R_{XX}(m-1) - R_{XX}(m+1) \end{aligned}$$

Autoregressive (AR) System



$$Y(n) = \alpha Y(n-1) + (1-\alpha)W(n)$$

$W(n)$ is w.s.s.

If we let z^{-1} be equivalent to a delay then we have that the transfer function of the system is

$$H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} = (1-\alpha)\sum_{n=0}^{\infty} \alpha^n z^{-n}$$
$$h(n) = (1-\alpha)\alpha^n u(n)$$

The input/output difference equation is equivalent to

$$Y(n) = \sum_{k=0}^{\infty} h(k)W(n-k)$$

Then

$$\begin{aligned}
E[Y(n)] &= \sum_{k=0}^{\infty} h(k)E[W(n-k)] = \eta_W \sum_{k=0}^{\infty} h(k) = \eta_W H(1) \\
\underbrace{R_{WY}(m, m+m_0)}_{R_{WY}(m_0)} &= \sum_k h(k) \underbrace{R_{WW}(m, m+m_0-k)}_{R_{WW}(m_0-k)} \\
\underbrace{R_{YY}(m, m+m_0)}_{R_{YY}(m_0)} &= \sum_k \sum_\ell h(k)h(\ell)R_{WW}(m_0-k+\ell)
\end{aligned}$$

Suppose $W(n)$ is white noise

$$R_{WW}(m) = \delta(m)$$
$$R_{WY}(m) = \sum_k h(k)\delta(m-k) = h(m)$$
$$R_{YY}(m) = h(-m) * R_{WY}(m) = h(-m) * h(m)$$

Notice that $R_{WY}(m)$ is non-symmetric (zero for negative m) while $R_{YY}(m)$ is symmetric.

Difference equation for $R_{YY}(.)$ Consider the AR system

$$Y(n) = \alpha Y(n-1) + (1-\alpha)W(n) \tag{1}$$

such that if $W(n)$ is w.s.s. the output $Y(n)$ is also w.s.s. Multiply equation (1) by $Y(n+m)$ to get

$$\begin{aligned} &E[Y(n)Y(n+m)] = \alpha E[Y(n-1)Y(n+m)] + (1-\alpha)E[W(n)Y(n+m)] \\ &R_{YY}(m) = \alpha R_{YY}(m-1) + (1-\alpha)R_{WY}(n, m+n) \end{aligned}$$

if $W(n)$,$Y(n)$ are jointly wide sense stationary, i.e., $R_{WY}(n, m+n) = R_{WY}(m)$ then a difference equation to obtain the autocorrelation is

$$R_{YY}(m) = \alpha R_{YY}(m-1) + (1-\alpha)R_{WY}(m)$$

Continuous-time Stationary Processes
Autocorrelation: measures relation of $X(t)$ and $X(t+\tau)$ for a lag τ

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

Properties

- $R_X(\tau)$ is even function of lag τ

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)] = R_X(-\tau)$$

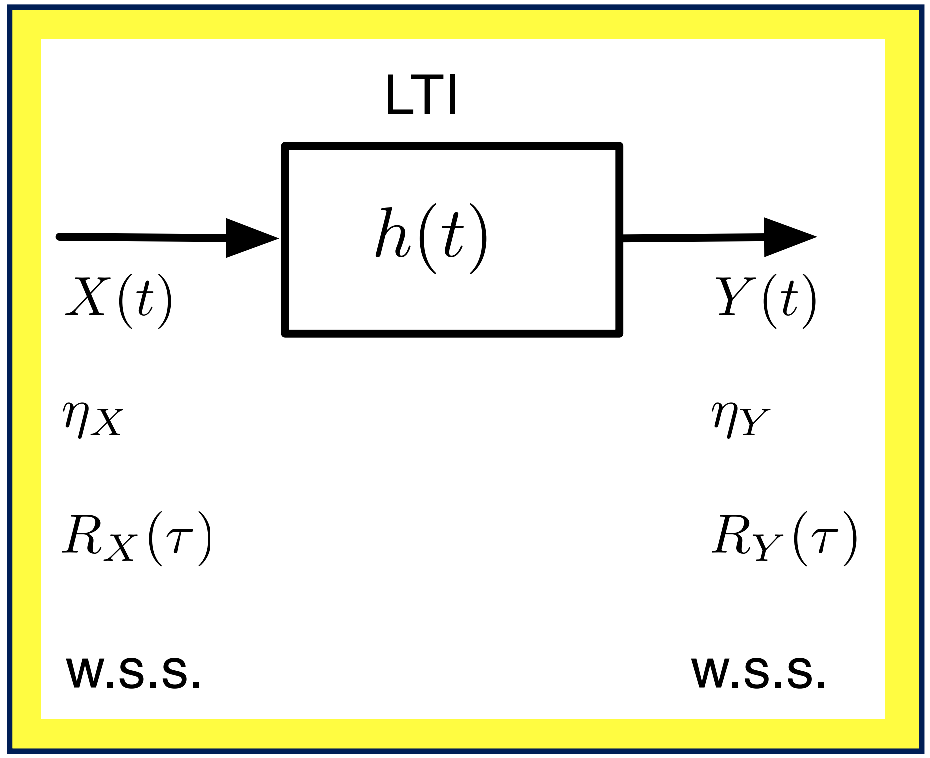
- $|R_X(\tau)| \leq R_X(0)$, indeed

$$\begin{aligned}&0 \leq E[(X(t+\tau) - X(t))^2] = E[X^2(t+\tau)] + E[X^2(t)] - 2E[X(t+\tau)X(t)] \\ &= 2R_X(0) - 2R_X(\tau) \quad \Rightarrow \quad R_X(0) \geq R_X(\tau)\end{aligned}$$

- If there is a $T > 0$ such that $R_X(0) = R_X(T)$ then $R_X(\tau)$ is periodic.
- $R_X(\tau)$ is a positive definite function.

Power Spectral Density — Continuous-time Random Processes
If $R_X(\tau)$ is the autocorrelation of a w.s.s. process $X(t)$ then $S_X(\Omega)$ (or $S_X(f)$, $\Omega = 2\pi f$) is the power spectral density of $X(t)$ and given by

$$\begin{aligned} S_X(\Omega) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j\Omega\tau} d\tau \\ R_X(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\Omega) e^{j\Omega\tau} d\Omega \\ &= \int_{-\infty}^{\infty} S_X(f) e^{j2\pi\tau} df \end{aligned}$$



Cross power spectral density If $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$ is the cross-correlation of jointly stationary processes $X(t)$ and $Y(t)$ then

$$S_{XY}(\Omega) = \mathcal{F}[R_{XY}(\tau)]$$

is the cross power spectral density.

Power Spectral Density — Discrete-time Random Processes
If $R_X(m)$ is the autocorrelation function of $X(n)$ then its power spectral density is

$$S_X(e^{j\omega}) = \sum_k R_X(m)e^{-j\omega m}$$

and

$$R_X(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(e^{j\omega})e^{j\omega m}d\omega$$

13

Properties of $S_X(\Omega)$
If $X(t)$ is a real-valued process

- $S_X(\Omega)$ is a real function

$$\begin{aligned} S_X(\Omega) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j\Omega\tau}d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau)\cos(\Omega\tau)d\tau - j\underbrace{\int_{-\infty}^{\infty} R_X(\tau)\sin(\Omega\tau)d\tau}_{0} \end{aligned}$$

- $S_X(\Omega)$ is an even function of Ω

$$S_X(\Omega) = S_X(-\Omega) \quad \text{because} \quad \cos(\Omega\tau) = \cos(-\Omega\tau)$$

(If $X(t)$ is not real-valued, then $S_X(\Omega)$ is not necessarily even.)

- $S_X(\Omega) \geq 0$, i.e., it has the positive characteristics of a power density function.

Remarks

- The Fourier transform cannot be applied directly to $X(t)$ because its FT would not exist.
- Similar properties for $S_X(e^{j\omega})$.

If $X(t)$, a w.s.s. random process, is the input of a LTI system with impulse response $h(t)$, the output $Y(t)$ is also w.s.s. random process with autocorrelation

$$R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau) \quad \text{and power spectral density}$$
$$S_Y(\Omega) = H(\Omega)^* H(\Omega) S_X(\Omega) = |H(\Omega)|^2 S_X(\Omega)$$

Remark

- For a discrete-time system

$$S_Y(e^{j\omega}) = |H(e^{j\omega})|^2 S_X(e^{j\omega})$$

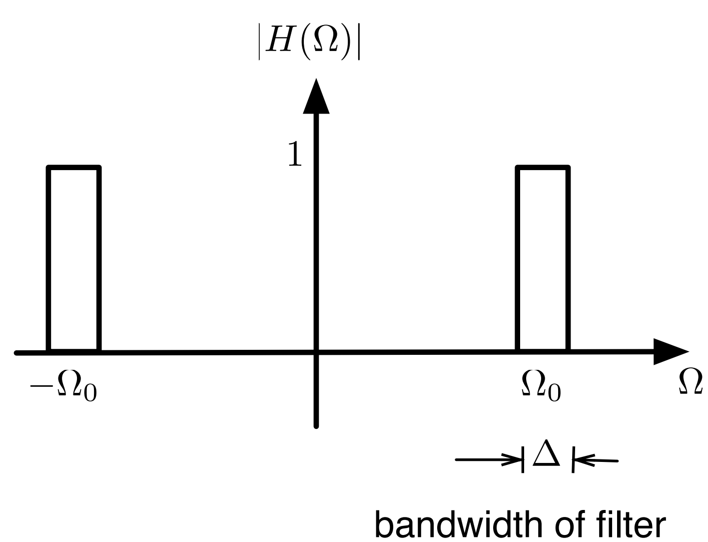
- For cross-correlation

$$\begin{aligned} R_{XY}(\tau) &= h(\tau) * R_X(\tau) \\ S_{XY}(\Omega) &= H(\Omega) S_X(\Omega) \end{aligned}$$

- Physical significance of $S_X(\Omega)$
  $S_X(\Omega)$ is the distribution of the power over frequency

$$\begin{aligned} E[Y^2(t)] &= R_{YY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\Omega)e^{j0}d\Omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\Omega)|H(\Omega)|^2 d\Omega \end{aligned}$$

  Let $H(s)$ be the transfer function of an ideal bandpass filter with frequency response



$$
\begin{aligned}
S_Y(\Omega) &= S_X(\Omega)|H(\Omega)|^2 \\
&\approx \begin{cases} S_X(\Omega_0) & |\Omega \pm \Omega_0| \leq \Delta/2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

We thus have

$$E[Y^2(t)] = R_Y(0) = 2\Delta S_X(\Omega_0)$$

where the units of Δ are rad/sec and those of $R_Y(0)$ are power, so that $S_X(.)$ has as units power/(rad/sec) or power density over frequency. Notice also that

$$E[Y^2(t)] = 2\Delta S_X(\Omega_0) \geq 0$$

indicating that as a density function $S_X(\Omega_0) \geq 0$.

Other properties of $S_X(\Omega)$

- Let $Y(t) = aX_1(t) + bX_2(t)$ where $X_i(t)$, $i = 1, 2$ are orthogonal w.s.s.

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t+\tau)] = E[(aX_1(t) + bX_2(t))(aX_1(t+\tau) + bX_2(t+\tau))] \\ &= a^2 R_{X_1}(\tau) + b^2 R_{X_2}(\tau) \\ S_Y(\Omega) &= a^2 S_{X_1}(\Omega) + b^2 S_{X_2}(\Omega) \end{aligned}$$

- Let $Y(t) = \frac{dX(t)}{dt}$, which can be thought of $X(t)$ being the input of a LTI system with $H(\Omega) = j\Omega$ then

$$S_Y(\Omega) = |j\Omega|^2 S_X(\Omega) = \Omega^2 S_X(\Omega)$$

This is equivalent to using the derivative property of the Fourier transform

$$\begin{aligned} R_X(\tau) &\quad\leftrightarrow\quad S_X(\Omega) \\ \frac{d^2 R_X(\tau)}{dt^2} &\quad\leftrightarrow\quad (j\Omega)^2 S_X(\Omega) = -\Omega^2 S_X(\Omega) \\ R_Y(\tau) = -\frac{d^2 R_X(\tau)}{dt^2} &\quad\leftrightarrow\quad \Omega^2 S_X(\Omega) = S_Y(\Omega) \end{aligned}$$

- Consider the modulation process: $X(t)$ input w.s.s. process, modulates a complex exponential $e^{j\Omega_0 t}$ so that the output is

$$Y(t) = X(t)e^{j\Omega_0 t}$$

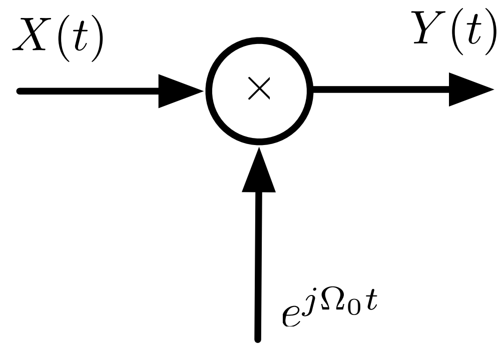
which is a complex process

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y^*(t+\tau)] = E[X(t)X(t+\tau)e^{j\Omega_0(t-t-\tau)}] \\ &= R_X(\tau)e^{-j\Omega_0\tau} \end{aligned}$$

so that

$$S_Y(\Omega) = S_X(\Omega + \Omega_0)$$

i.e., shifted in frequency to Ω_0. $S_Y(\Omega)$ is not even because $Y(t)$ is complex.



- If the modulation is done with a sinusoid,

$$
\begin{aligned}
Y(t) &= X(t)\cos(\Omega_0 t) \qquad \Omega_0 \;\; \text{constant} \\
R_Y(\tau) &= 0.5 R_X(\tau) e^{-j\Omega_0 \tau} + 0.5 R_X(\tau) e^{j\Omega_0 \tau} \\
&= 0.5 R_X(\tau)\cos(\Omega_0 \tau)
\end{aligned}
$$

$S_Y(\Omega) = 0.5 S_X(\Omega + \Omega_0) + 0.5 S_X(\Omega - \Omega_0)$

- Let $X(t)$ be zero-mean w.s.s. white noise so that

$$
\begin{aligned}
& E[X(t)] = 0 \\
& R_X(\tau) = \sigma_X^2 \delta(\tau) \\
& S_X(\Omega) = \sigma_X^2
\end{aligned}
$$

i.e., just like white light, the spectrum of white noise has all possible frequencies.

Calculation of $R_X(\tau)$ from $S_X(\Omega)$
Remember that $R_X(\tau) = R_X(-\tau)$, i.e., even function of τ

$$\begin{aligned} S_X(\Omega) &= S_X(s)|_{s=j\Omega} \\ S_X(s) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-s\tau}d\tau = \underbrace{\int_{-\infty}^{0} R_X(\tau)e^{-s\tau}d\tau}_{S^-(s)=\mathcal{L}[R_X(\tau)u(-\tau)]} + \underbrace{\int_{0}^{\infty} R_X(\tau)e^{-s\tau}d\tau}_{S^+(\Omega)==\mathcal{L}[R_X(\tau)u(\tau)]} \end{aligned}$$

$R_X(\tau)u(\tau)$ causal component of $R_X(\tau)$
$R_X(\tau)u(-\tau)$ anti-causal component of $R_X(\tau)$

we have

$$S^-(s) = \int_{-\infty}^{0} R_X(\tau)e^{-s\tau}d\tau = \int_{0}^{\infty} R_X(t)e^{st}dt = S^+(-s)$$

so that we have the following Fourier pairs

$$S_X(\Omega) = S^+(s) + S^+(-s) \quad \leftrightarrow \quad R_X(\tau) = R_X(\tau)u(\tau) + R_X(\tau)u(-\tau)$$

Example: first-order differential equation

$$Y^{(1)}(t) + \alpha Y(t) = X(t) \qquad \alpha > 0, -\infty < t < \infty$$

$X(t)$ is zero mean, unit variance stationary process. Calculate $S_Y(\Omega)$ and $R_Y(\tau)$.

Since $\eta_x = 0$, then $C_X(\tau) = R_X(\tau) = \delta(\tau)$ and $S_X(\Omega) = 1$. The spectral density of the output is

$$S_Y(\Omega) = |H(j\Omega)|^2 S_X(\Omega) = \left|\frac{1}{\alpha + j\Omega}\right|^2 = \frac{1}{\alpha^2 + \Omega^2}$$

because the spectrum of $Y(t)$ has lost some of the higher frequency components, $Y(t)$ is called colored or brown noise.

To find $R_Y(\tau)$ we let $s = j\Omega$ ($\Omega = s/j$ and $\Omega^2 = -s^2$) so that

$$S_Y(s) \quad = \quad \frac{1}{\alpha^2 - s^2} = \frac{1}{(s+\alpha)(\alpha - s)} = \frac{A}{s+\alpha} + \frac{B}{\alpha - s}$$

where the pole in the left-hand s-plane corresponds to a causal component and the second term with pole in the right-hand s-plane corresponds to an anticausal component.

$$A = S_Y(s)(s+\alpha)|_{s=-\alpha} = \frac{1}{2\alpha}$$

$$S_Y^+(s) = \frac{1/(2\alpha)}{s+\alpha} \quad \Rightarrow \quad R_Y(\tau)u(\tau) = \frac{1}{2\alpha}e^{-\alpha\tau}u(\tau)$$

By symmetry, $R(-\tau) = R(\tau)$ so that

$$R_Y(\tau) = \frac{1}{2\alpha}e^{-\alpha|\tau|}$$

To find the cross power density $S_{XY}(\Omega)$ we have

$$\begin{aligned} S_{XY}(\Omega) \quad &= \quad \mathcal{F}[h(\tau) * R_X(\tau)] = H(\Omega)S_X(\Omega) = H(\Omega) \\ &= \quad \frac{1}{\alpha + j\Omega} \end{aligned}$$

and

$$R_{XY}(\tau) = e^{-\alpha\tau}u(\tau)$$

which is not symmetric, and causal.

Example: Second-order system The input/output equation is given by

$$Y^{(2)}(t) + 3Y^{(1)}(t) + 2Y(t) = 5X(t)$$

$X(t)$ is stationary, white noise with zero mean, unit variance. Find $R_Y(\tau)$

$$S_Y(s) = H(s)H(-s) = \frac{5}{s^2+3s+2}\frac{5}{s^2-3s+2}$$
$$s^2+3s+2 = (s+1)(s+2)$$
$$S_Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-1} + \frac{D}{s-2}$$
$$A = S_Y(s)(s+1)|_{s=-1} = \frac{25}{6}$$
$$B = S_Y(s)(s+2)|_{s=-2} = \frac{-25}{12}$$

thus we have

$$R_Y(\tau) = \frac{25}{6}(e^{-|\tau|} - 0.5e^{-2|\tau|})$$

Example: Analog averager Let the output of an analog averager be

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(\tau)d\tau$$

where the input $X(t)$ has an autocorrelation function $R_X(\tau) = \sigma_X^2\delta(\tau)$. Determine $R_Y(\tau)$ and $S_Y(\Omega)$.

Impulse response: by change of variable $\mu = t - \tau$ we get

$$Y(t) = \frac{1}{T}\int_{0}^{T} X(t-\mu)d\mu$$

so that the impulse response is $h(t) = (1/T)(u(t) - u(t-T))$

$$
\begin{aligned}
R_Y(\tau) &= h(-\tau) * \underbrace{h(\tau) * R_X(\tau)}_{h(\tau)*\sigma_X^2\delta(\tau)=\sigma_X^2 h(\tau)} \\
&= \sigma_X^2 h(\tau) * h(-\tau) \\
&= \begin{cases} (\sigma_X^2/T)(1-|\tau|/T) & |\tau| \le T \\ 0 & |\tau| > T \end{cases} = \frac{\sigma_X^2}{T}[r(\tau+T) - 2r(\tau) + r(\tau-T)]
\end{aligned}
$$

To compute the power spectral density $S_Y(\Omega)$, take the second derivative of $R_Y(\tau)$ which gives

$$
\frac{d^2 R_Y(\tau)}{dt^2} = \frac{\sigma_X^2}{T^2}[\delta(\tau+T) + \delta(\tau-T) - 2\delta(\tau)]
$$

so that

$$
\begin{aligned}
(j\Omega)^2 S_Y(\Omega) &= \frac{2\sigma_X^2}{T^2}(\cos(\Omega\tau) - 1) \\
S_Y(\Omega) &= \frac{2\sigma_X^2}{T^2}\frac{1-\cos(\Omega\tau)}{\Omega} = \sigma_X^2\left[\frac{\sin(\Omega T/2)}{\Omega T/2}\right]^2
\end{aligned}
$$

which is a real, positive even function.

Discrete-time Stationary Processes

$$
\begin{aligned}
& X(n), \quad \text{w.s.s. process} \\
& E[X(n)] = m_X \\
& S_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j\omega k} \\
& R_X(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X((e^{j\omega}) d\omega \qquad \omega \text{ rad}
\end{aligned}
$$

Discrete-time White Noise

$$
\begin{aligned}
& X(n), \quad \text{w.s.s. process} \\
& E[X(n)] = 0 \\
& R_X(k) = \sigma_X^2 \delta(k) = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & \text{otherwise} \end{cases} \\
& S_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \sigma_X^2 \delta(k) = \sigma_X^2 \qquad -\pi \leq \omega \leq \pi
\end{aligned}
$$

Notice the difference with the continuous-time white noise where $R_X(\tau) = \sigma_X^2 \delta(\tau)$ cannot be define at $\tau = 0$ because of $\delta(\tau)$. The power density $S_X(e^{j\omega})$ is defined for all possible discrete frequencies ω.

Example: Discrete-time moving average

$$Y(n) = X(n) + \alpha X(n-1)$$

$X(n)$ is white noise with zero mean and variance σ^2. Find $E[Y(n)]$, $R_Y(k)$ and $S_Y(e^{j\omega})$.

$$\begin{aligned}
E[Y(n)] &= E[X(n)] + \alpha E[X(n-1)] = 0 \\
R_Y(k) &= E[Y(n)Y(n+k)] = E[(X(n) + \alpha X(n-1))(X(n+k) + \alpha X(n+k-1))] \\
&= (1+\alpha^2)R_X(k) + \alpha R_X(k+1) + \alpha R_X(k-1) \\
&= \begin{cases} (1+\alpha^2)\sigma^2 & k = 0 \\ \alpha\sigma^2 & k = 1, -1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The power density is then

$$S_Y(e^{j\omega}) = (1+\alpha^2)\sigma^2 + 2\alpha\sigma^2 \cos(\omega)$$