Linear Time-invariant Systems with Random Inputs
Mean of $Y(t)$

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} X(\tau)h(t - \tau)d\tau\right]$$

$$= \left[\int_{-\infty}^{\infty} h(t - \tau)E[X(\tau)]d\tau\right] = (h * \eta_X)(t)$$

Remarks

- Existence of $E[Y(t)]$ when $X(t)$ is wide sense stationary

$$E[Y(t)] = \eta_X \int_{-\infty}^{\infty} h(t - \tau)d\tau$$

$$|E[Y(t)]| \leq |\eta_X| \int_{-\infty}^{\infty} |h(\psi)|d\psi$$

existence of $E[Y(t)]$ requires $|E[Y(t)]| < M$, thus we need

$$\int_{-\infty}^{\infty} |h(\psi)|d\psi < L \text{ bounded}$$

or that the system be BIBO stable.
Autocorrelation of $Y(t)$

\[
R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty} h(t_2 - \tau)E[X(t_1)X(\tau)]d\tau
\]

\[
= \int_{-\infty}^{\infty} h(t_2 - \tau)R_{XX}(t_1, \tau)d\tau
\]

\[
= \int_{-\infty}^{\infty} h(\alpha)R_{XX}(t_1, t_2 - \alpha)d\alpha
\]

where we let $\alpha = t_2 - \tau$, $d\alpha = -d\tau$. Notice that the convolution is with respect to the second variable of the autocorrelation.

\[
R_{YY}(t_1, t_2) = E[Y(t_1)Y(t_2)] = \int_{-\infty}^{\infty} h(t_1 - \tau)E[X(\tau)Y(t_2)]d\tau
\]

\[
= \int_{-\infty}^{\infty} h(t_1 - \tau)R_{XY}(\tau, t_2)d\tau
\]

\[
= \int_{-\infty}^{\infty} h(\beta)R_{XY}(t_1 - \beta, t_2)d\beta
\]

where we let $\beta = t_1 - \tau$. Notice the convolution is with respect to the first variable of the autocorrelation.

Replacing $R_{XY}(.,.)$ in the last equation we get

\[
R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{XX}(t_1 - \beta, t_2 - \alpha)d\beta d\alpha
\]
Remarks

- $R_{YY}(t_1, t_2)$ can be obtained directly

$$R_{YY}(t_1, t_2) = \mathbb{E}[Y(t_1)Y(t_2)] = \mathbb{E} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_1 - \beta)h(t_2 - \alpha)X(\beta)X(\alpha) \right] d\alpha d\beta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t_2 - \alpha)h(t_1 - \beta)R_{XX}(\alpha, \beta) d\alpha d\beta$$

- Let $X(t)$ be strictly or w.s.s.

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha) R_{XX}(t_1, t_2 - \alpha) d\alpha \quad \tau = t_2 - t_1$$

$$= (h * R_{XX})(\tau)$$

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\alpha) R_{XY}(t_1 - \alpha, t_2) d\alpha$$

$$= \int_{-\infty}^{\infty} h(-\beta)R_{XY}(\tau - \beta) d\beta = h(-\tau) * R_{XY}(\tau)$$

so that

$$R_{YY}(t) = h(t) * h(-t) * R_{XX}(t)$$

- The above results can be extended to the covariance by letting $\tilde{Y}(t) = Y(t) - \eta_{Y}(t)$ and using

$$C_{YY}(t_1, t_2) = R_{\tilde{Y}\tilde{Y}}(t_1, t_2)$$
LTI Discrete-time Systems with Random Inputs

\[
Y(n) = \sum_k h(n-k)x(k) = \sum_k h(k)x(n-k)
\]

Mean

\[
E[Y(n)] = \sum_k h(n-k)E[X(k)]
\]

\(X(n)\) wide sense stationary

\[
E[Y(n)] = \sum_k h(k)E[X(n-k)] = \eta_X \sum_k h(k) = H(1)\eta_X
\]

\[
H(z) = \sum_k h(k)z^{-k}\big|_{z=1}
\]

Autocorrelation

\[
R_{XY}(m,n) = E[X(m)Y(n)] = E\left[X(m)\sum_k h(k)X(n-k)\right]
\]

\[
= \sum_k h(k)R_{XX}(m,n-k)
\]

\[
R_{YY}(m,n) = E[Y(m)Y(n)] = \sum_k h(k)\sum_\ell h(\ell)R_{XX}(m-k,n-\ell)
\]

\[
= E\left[\sum_k h(k)X(m-k)Y(n)\right] = \sum_k h(k)R_{XY}(m-k,n)
\]
Special case: \( X(n) \) is w.s.s.

\[
\mu = m - n \implies \sum_k h(k)R_{XX}(\mu - k) = (h(\mu) * R_{XX}(\mu)) = R_{XY}(\mu)
\]

\[
\sum_{-\ell} h(\ell)R_{XY}(\rho - \ell) = h(-\rho) * R_{XY}(\rho) = R_{YY}(\rho)
\]

so that

\[
R_{YY}(\rho) = h(-\rho) * R_{XY}(\rho) = h(-\rho) * h(\rho) * R_{XX}(\rho)
\]

Differentiator

\[
X(t) \quad \frac{d(\cdot)}{dt} \quad \Rightarrow \quad Y(t) = \frac{dX(t)}{dt}
\]
\( Y(t) = dX(t)/dt \) defined in mean–square sense, find \( \eta_Y(t) \), \( R_{YY}(t_1, t_2) \). Is \( Y(t) \) w.s.s. if \( X(t) \) is w.s.s.?

\[
\eta_Y(t) = E[Y(t)] = E\left[ \frac{dX(t)}{dt} \right] = \frac{dE[X(t)]}{dt} = \frac{d\eta_X(t)}{dt}
\]

\[
R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = E\left[ X(t_1) \frac{dX(t_2)}{dt_2} \right] = \frac{dE[X(t_1)X(t_2)]}{dt_2} = \frac{dR_{XX}(t_1, t_2)}{dt_2}
\]

\[
R_{YY}(t_1, t_2) = E\left[ \frac{dX(t_1)}{dt_1} Y(t_2) \right] = \frac{dE[X(t_1)Y(t_2)]}{dt_1} = \frac{dR_{XY}(t_1, t_2)}{dt_1}
\]

So that

\[
R_{YY}(t_1, t_2) = \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}
\]

**Note** If we use

\[
R_{YY}(t_1, t_2) = E\left[ Y(t_1) \frac{dX(t_2)}{dt_2} \right] = \frac{dR_{YY}(t_1, t_2)}{dt_2}
\]

although correct, we cannot use equation \( R_{YY}(t_1, t_2) = dR_{XY}(t_1, t_2)/dt_1 \) to get \( R_{YY}(t_1, t_2) \).

If \( X(t) \) is w.s.s. then

\[ \eta_X(t) \text{ constant so } \eta_Y(t) = 0 \]

\[ R_{XX}(t_1, t_2) = R_{XX}(\tau) \quad \tau = t_2 - t_1 \]

\[ R_{XY}(t_1, t_2) = \frac{dR_{XX}(t_2 - t_1)}{dt_2} = \frac{dR_{XX}(\tau)}{d\tau} \frac{d\tau}{dt_2} \]

so \( R_{XY}(\tau) = \frac{dR_X(\tau)}{d\tau} \)

\[ R_{YY}(t_1, t_2) = \frac{dR_{XY}(t_2 - t_1)}{dt_1} = \frac{dR_{XY}(\tau)}{d\tau} \frac{d\tau}{dt_1} \]

so \( R_{YY}(\tau) = -\frac{dR_{XY}(\tau)}{d\tau} = -\frac{d^2 R_X(\tau)}{d\tau^2} \)
Moving averaging (MA) System

\[ Y(n) = X(n) - X(n-1) \]

Is \( Y(n) \) w.s.s. if \( X(n) \) is w.s.s.?

Mean

\[ E[Y(n)] = E[X(n)] - E[X(n-1)] = \eta_X(n) - \eta_X(n-1) \]

Autocorrelation

\[
\begin{align*}
R_{XY}(m, n) &= E[X(m)Y(n)] = E[X(m)X(n) - X(m)X(n-1)] \\
&= R_{XX}(m, n) - R_{XX}(m, n-1) \\
R_{YY}(m, n) &= E[Y(m)Y(n)] = E[(X(m) - X(m-1))(X(n) - X(n-1))] \\
&= R_{XX}(m, n) - R_{XX}(m, n-1) - R_{XX}(m-1, n) + R_{XX}(m-1, n-1)
\end{align*}
\]

If \( X(n) \) is w.s.s. then

\[
\begin{align*}
\eta_Y(n) &= 0 \\
R_{XY}(n-m) &= R_{XX}(n-m) - R_{XX}(n-1-m) \\
\ell = n-m, \quad &\Rightarrow \quad R_{XY}(\ell) = R_{XX}(\ell) - R_{XX}(\ell-1) \\
R_{YY}(n-m) &= R_{XX}(n-m) - R_{XX}(n-1-m) - R_{XX}(n-m+1) + R_{XX}(n-m) \\
\ell = n-m, \quad &\Rightarrow \quad R_{YY}(\ell) = 2R_{XX}(\ell) - R_{XX}(\ell-1) - R_{XX}(\ell+1)
\end{align*}
\]
For the w.s.s. case, using that the impulse response of the MA system is $h(n) = \delta(n) - \delta(n - 1)$ we have

\[
R_{XY}(m) = h(m) * R_{XX}(m) = R_{XX}(m) - R_{XX}(m - 1)
\]
\[
R_{YY}(m) = h(-m) * R_{XY}(m) = [\delta(m) - \delta(m + 1)] * R_{XY}(m) = R_{XY}(m) - R_{XY}(m + 1)
\]
\[
= [R_{XX}(m) - R_{XX}(m - 1)] - [R_{XX}(m + 1) - R_{XX}(m)]
\]
\[
= 2R_{XX}(m) - R_{XX}(m - 1) - R_{XX}(m + 1)
\]

Autoregressive (AR) System

![Autoregressive (AR) System](image-url)
\[ Y(n) = \alpha Y(n-1) + (1-\alpha)W(n) \]

\( W(n) \) is w.s.s.

If we let \( z^{-1} \) be equivalent to a delay then we have that the transfer function of the system is

\[
H(z) = \frac{1-\alpha}{1-\alpha z^{-1}} = (1-\alpha) \sum_{n=0}^{\infty} \alpha^n z^{-n}
\]

\[ h(n) = (1-\alpha)\alpha^n u(n) \]

The input/output difference equation is equivalent to

\[ Y(n) = \sum_{k=0}^{\infty} h(k) W(n-k) \]

Then

\[
E[Y(n)] = \sum_{k=0}^{\infty} h(k) E[W(n-k)] = \eta_W \sum_{k=0}^{\infty} h(k) = \eta_W H(1)
\]

\[
\begin{align*}
R_{WY}(m, m+M_0) &\equiv \sum_k h(k) R_{WY}(m, m+M_0-k) \\
R_{YY}(m, m+M_0) &\equiv \sum_k \sum_{\ell} h(k) h(\ell) R_{WW}(m_0-k + \ell)
\end{align*}
\]

Suppose \( W(n) \) is white noise

\[
R_{WW}(m) = \delta(m)
\]

\[
R_{WY}(m) = \sum_k h(k) \delta(m-k) = h(m)
\]

\[
R_{YY}(m) = h(-m) * R_{WY}(m) = h(-m) * h(m)
\]

Notice that \( R_{WY}(m) \) is non-symmetric (zero for negative \( m \)) while \( R_{YY}(m) \) is symmetric.
Difference equation for $R_{YY}(\cdot)$ Consider the AR system

$$Y(n) = \alpha Y(n-1) + (1-\alpha)W(n)$$

such that if $W(n)$ is w.s.s. the output $Y(n)$ is also w.s.s. Multiply equation (1) by $Y(n+m)$ to get

$$E[Y(n)Y(n+m)] = \alpha E[Y(n-1)Y(n+m)] + (1-\alpha)E[W(n)Y(n+m)]$$

$$R_{YY}(m) = \alpha R_{YY}(m-1) + (1-\alpha)R_{WY}(n,m+n)$$

if $W(n), Y(n)$ are jointly wide sense stationary, i.e., $R_{WY}(n,m+n) = R_{WY}(m)$ then a difference equation to obtain the autocorrelation is

$$R_{YY}(m) = \alpha R_{YY}(m-1) + (1-\alpha)R_{WY}(m)$$
Continuous-time Stationary Processes

Autocorrelation: measures relation of $X(t)$ and $X(t + \tau)$ for a lag $\tau$

$$R_X(\tau) = E[X(t)X(t + \tau)]$$

Properties

- $R_X(\tau)$ is even function of lag $\tau$
  $$R_X(\tau) = E[X(t)X(t + \tau)] = E[X(t + \tau)X(t)] = R_X(-\tau)$$

- $|R_X(\tau)| \leq R_X(0)$, indeed
  $$0 \leq E[(X(t + \tau) - X(t))^2] = E[X^2(t + \tau)] + E[X^2(t)] - 2E[X(t + \tau)X(t)]$$
  $$= 2R_X(0) - 2R_X(\tau) \Rightarrow R_X(0) \geq R_X(\tau)$$

- If there is a $T > 0$ such that $R_X(0) = R_X(T)$ then $R_X(\tau)$ is periodic.
- $R_X(\tau)$ is a positive definite function.

Power Spectral Density — Continuous-time Random Processes

If $R_X(\tau)$ is the autocorrelation of a w.s.s. process $X(t)$ then $S_X(\Omega)$ (or $S_X(f)$, $\Omega = 2\pi f$) is the power spectral density of $X(t)$ and given by

$$S_X(\Omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\Omega\tau} d\tau$$

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\Omega) e^{j\Omega\tau} d\Omega$$

$$= \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f} df$$
Cross power spectral density If $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$ is the cross-correlation of jointly stationary processes $X(t)$ and $Y(t)$ then

$$S_{XY}(\Omega) = \mathcal{F}[R_{XY}(\tau)]$$

is the cross power spectral density.

**Power Spectral Density — Discrete-time Random Processes**

If $R_X(m)$ is the autocorrelation function of $X(n)$ then its power spectral density is

$$S_X(e^{j\omega}) = \sum_k R_X(m)e^{-j\omega m}$$

and

$$R_X(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(e^{j\omega})e^{j\omega m} d\omega$$
Properties of $S_X(\Omega)$
If $X(t)$ is a real-valued process

- $S_X(\Omega)$ is a real function

\[
S_X(\Omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\Omega\tau}d\tau
= \int_{-\infty}^{\infty} R_X(\tau)\cos(\Omega\tau)d\tau - j\int_{-\infty}^{\infty} R_X(\tau)\sin(\Omega\tau)d\tau
\]

- $S_X(\Omega)$ is an even function of $\Omega$

\[
S_X(\Omega) = S_X(-\Omega) \quad \text{because} \quad \cos(\Omega\tau) = \cos(-\Omega\tau)
\]
(If $X(t)$ is not real-valued, then $S_X(\Omega)$ is not necessarily even.)

- $S_X(\Omega) \geq 0$, i.e., it has the positive characteristics of a power density function.

Remarks

- The Fourier transform cannot be applied directly to $X(t)$ because its FT would not exist.

- Similar properties for $S_X(e^{j\omega})$.

If $X(t)$, a w.s.s. random process, is the input of a LTI system with impulse response $h(t)$, the output $Y(t)$ is also w.s.s. random process with autocorrelation

\[
R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau) \quad \text{and power spectral density}
S_Y(\Omega) = H(\Omega)^*H(\Omega)S_X(\Omega) = |H(\Omega)|^2S_X(\Omega)
Remark

- For a discrete-time system

\[ S_Y(e^{j\omega}) = |H(e^{j\omega})|^2 S_X(e^{j\omega}) \]

- For cross-correlation

\[ R_{XY}(\tau) = h(\tau) * R_X(\tau) \\
S_{XY}(\Omega) = H(\Omega)S_X(\Omega) \]

- Physical significance of \( S_X(\Omega) \)

\( S_X(\Omega) \) is the distribution of the power over frequency

\[ E[Y^2(t)] = R_{YY}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\Omega)e^{j0}d\Omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\Omega)|H(\Omega)|^2d\Omega \]

Let \( H(s) \) be the transfer function of an ideal bandpass filter with frequency response

\[ |H(\Omega)| \]

![Diagram showing the transfer function of an ideal bandpass filter with frequency response](image)
\[ S_Y(\Omega) = S_X(\Omega)|H(\Omega)|^2 \]

\[ \approx \begin{cases} 
S_X(\Omega_0) & |\Omega \pm \Omega_0| \leq \Delta/2 \\
0 & \text{otherwise}
\end{cases} \]

We thus have

\[ E[Y^2(t)] = R_Y(0) = 2\Delta S_X(\Omega_0) \]

where the units of \( \Delta \) are rad/sec and those of \( R_Y(0) \) are power, so that \( S_X(.) \) has as units power/(rad/sec) or power density over frequency. Notice also that

\[ E[Y^2(t)] = 2\Delta S_X(\Omega_0) \geq 0 \]

indicating that as a density function \( S_X(\Omega_0) \geq 0 \).
Other properties of $S_X(\Omega)$

- Let $Y(t) = aX_1(t) + bX_2(t)$ where $X_i(t), \ i = 1, 2$ are orthogonal w.s.s.
  \[
  R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[(aX_1(t) + bX_2(t))(aX_1(t+\tau) + bX_2(t+\tau))]
  = a^2 R_{X_1}(\tau) + b^2 R_{X_2}(\tau)
  
  S_Y(\Omega) = a^2 S_{X_1}(\Omega) + b^2 S_{X_2}(\Omega)
  \]

- Let $Y(t) = \frac{dX(t)}{dt}$, which can be thought of $X(t)$ being the input of a LTI system with $H(\Omega) = j\Omega$ then
  \[
  S_Y(\Omega) = |j\Omega|^2 S_X(\Omega) = \Omega^2 S_X(\Omega)
  \]
  This is equivalent to using the derivative property of the Fourier transform
  \[
  R_X(\tau) \leftrightarrow S_X(\Omega) \\
  \frac{d^2 R_X(\tau)}{dt^2} \leftrightarrow (j\Omega)^2 S_X(\Omega) = -\Omega^2 S_X(\Omega) \\
  R_Y(\tau) = -\frac{d^2 R_X(\tau)}{dt^2} \leftrightarrow \Omega^2 S_X(\Omega) = S_Y(\Omega)
  \]

- Consider the modulation process: $X(t)$ input w.s.s. process, modulates a complex exponential $e^{j\Omega_0 t}$ so that the output is
  \[
  Y(t) = X(t)e^{j\Omega_0 t}
  \]
  which is a complex process
  \[
  R_Y(\tau) = E[Y(t)Y^*(t+\tau)] = E[X(t)X(t+\tau)e^{j\Omega_0(t-t-\tau)}] \\
  = R_X(\tau)e^{-j\Omega_0 \tau}
  \]
  so that
  \[
  S_Y(\Omega) = S_X(\Omega + \Omega_0)
  \]
  i.e., shifted in frequency to $\Omega_0$. $S_Y(\Omega)$ is not even because $Y(t)$ is complex.
• If the modulation is done with a sinusoid,

\[
Y(t) = X(t) \cos(\Omega_0 t) \quad \Omega_0 \text{ constant}
\]

\[
R_Y(\tau) = 0.5R_X(\tau)e^{-j\Omega_0 \tau} + 0.5R_X(\tau)e^{j\Omega_0 \tau}
\]

\[
S_Y(\Omega) = 0.5S_X(\Omega + \Omega_0) + 0.5S_X(\Omega - \Omega_0)
\]

• Let \(X(t)\) be zero-mean w.s.s. white noise so that

\[
E[X(t)] = 0
\]

\[
R_X(\tau) = \sigma_X^2 \delta(\tau)
\]

\[
S_X(\Omega) = \sigma_X^2
\]

i.e., just like white light, the spectrum of white noise has all possible frequencies.
Calculation of $R_X(\tau)$ from $S_X(\Omega)$

Remember that $R_X(\tau) = R_X(-\tau)$, i.e., even function of $\tau$

\[
S_X(\Omega) = S_X(s)|_{s=j\Omega}
\]

\[
S_X(s) = \int_{-\infty}^{\infty} R_X(\tau)e^{-st}d\tau = \int_{-\infty}^{0} R_X(\tau)e^{-st}d\tau + \int_{0}^{\infty} R_X(\tau)e^{-st}d\tau
\]

\[
R_X(\tau)u(\tau) \quad \text{causal component of } R_X(\tau)
\]

\[
R_X(\tau)u(-\tau) \quad \text{anti-causal component of } R_X(\tau)
\]

we have

\[
S^{-}(s) = \int_{-\infty}^{0} R_X(\tau)e^{-st}d\tau = \int_{0}^{\infty} R_X(t)e^{st}dt = S^{+}(-s)
\]

so that we have the following Fourier pairs

\[
S_X(\Omega) = S^{+}(s) + S^{+}(-s) \Leftrightarrow R_X(\tau) = R_X(\tau)u(\tau) + R_X(\tau)u(-\tau)
\]

Example: first-order differential equation

\[
Y^{(1)}(t) + \alpha Y(t) = X(t) \quad \alpha > 0, -\infty < t < \infty
\]

$X(t)$ is zero mean, unit variance stationary process. Calculate $S_Y(\Omega)$ and $R_Y(\tau)$.

Since $\eta_x = 0$, then $C_X(\tau) = R_X(\tau) = \delta(\tau)$ and $S_X(\Omega) = 1$. The spectral density of the output is

\[
S_Y(\Omega) = |H(j\Omega)|^2S_X(\Omega) = \left| \frac{1}{\alpha + j\Omega} \right|^2 = \frac{1}{\alpha^2 + \Omega^2}
\]

because the spectrum of $Y(t)$ has lost some of the higher frequency components, $Y(t)$ is called colored or brown noise.
To find $R_Y(\tau)$ we let $s = j\Omega$ ($\Omega = s/j$ and $\Omega^2 = -s^2$) so that

$$S_Y(s) = \frac{1}{\alpha^2 - s^2} = \frac{1}{(s + \alpha)(\alpha - s)} = \frac{A}{s + \alpha} + \frac{B}{\alpha - s}$$

where the pole in the left-hand s-plane corresponds to a causal component and the second term with pole in the right-hand s-plane corresponds to an anticausal component.

$$A = S_Y(s)(s + \alpha)|_{s = -\alpha} = \frac{1}{2\alpha}$$

$$S_Y^+(s) = \frac{1/(2\alpha)}{s + \alpha} \Rightarrow R_Y(\tau)u(\tau) = \frac{1}{2\alpha} e^{-\alpha\tau} u(\tau)$$

By symmetry, $R(-\tau) = R(\tau)$ so that

$$R_Y(\tau) = \frac{1}{2\alpha} e^{-\alpha|\tau|}$$

To find the cross power density $S_{XY}(\Omega)$ we have

$$S_{XY}(\Omega) = \mathcal{F}[h(\tau) \ast R_X(\tau)] = H(\Omega) S_X(\Omega) = H(\Omega)$$

$$= \frac{1}{\alpha + j\Omega}$$

and

$$R_{XY}(\tau) = e^{-\alpha\tau} u(\tau)$$

which is not symmetric, and causal.
Example: Second-order system The input/output equation is given by

\[ Y^{(2)}(t) + 3Y^{(1)}(t) + 2Y(t) = 5X(t) \]

\( X(t) \) is stationary, white noise with zero mean, unit variance. Find \( R_Y(\tau) \)

\[
S_Y(s) = H(s)H(-s) = \frac{5}{s^2 + 3s + 2} \frac{5}{s^2 - 3s + 2}
\]

\[ s^2 + 3s + 2 = (s + 1)(s + 2) \]

\[
S_Y(s) = \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{C}{s - 1} + \frac{D}{s - 2}
\]

\[ A = S_Y(s)(s + 1)|_{s=-1} = \frac{25}{6} \]

\[ B = S_Y(s)(s + 2)|_{s=-2} = \frac{-25}{12} \]

thus we have

\[ R_Y(\tau) = \frac{25}{6}(e^{-|\tau|} - 0.5e^{-2|\tau|}) \]

Example: Analog averager Let the output of an analog averager be

\[ Y(t) = \frac{1}{T} \int_{t-T}^{t} X(\tau)d\tau \]

where the input \( X(t) \) has an autocorrelation function \( R_X(\tau) = \sigma_X^2 \delta(\tau) \). Determine \( R_Y(\tau) \) and \( S_Y(\Omega) \).

Impulse response: by change of variable \( \mu = t - \tau \) we get

\[ Y(t) = \frac{1}{T} \int_{0}^{T} X(t-\mu)d\mu \]

so that the impulse response is \( h(t) = (1/T)(u(t) - u(t-T)) \)
\[ R_Y(\tau) = h(-\tau) * \underbrace{h(\tau) * R_X(\tau)}_{h(\tau) \sigma_X^2 \delta(\tau) = \sigma_X^2 h(\tau)} \]
\[ = \sigma_X^2 h(\tau) * h(-\tau) \]
\[ = \begin{cases} \left( \frac{\sigma_X^2}{T} \right)(1 - |\tau|/T) & |\tau| \leq T \\ 0 & |\tau| > T \end{cases} \]
\[ = \frac{\sigma_X^2}{T} \left[r(\tau + T) - 2r(\tau) + r(\tau - T)\right] \]

To compute the power spectral density \( S_Y(\Omega) \), take the second derivative of \( R_Y(\tau) \) which gives

\[
\frac{d^2 R_Y(\tau)}{d\tau^2} = \frac{\sigma_X^2}{T^2} \left[\delta(\tau + T) + \delta(\tau - T) - 2\delta(\tau)\right]
\]
so that

\[
(j\Omega)^2 S_Y(\Omega) = \frac{2\sigma_X^2}{T^2} (\cos(\Omega \tau) - 1)
\]

\[
S_Y(\Omega) = \frac{2\sigma_X^2}{T^2} \frac{1 - \cos(\Omega \tau)}{\Omega} = \sigma_X^2 \left[\sin(\Omega T/2) / \Omega T/2\right]^2
\]

which is a real, positive even function.
Discrete-time Stationary Processes

\[ X(n), \text{ w.s.s. process} \]
\[ E[X(n)] = m_X \]
\[ S_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-jk} \]
\[ R_X(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X((e^{j\omega})d\omega \quad \omega \text{ rad} \]

Discrete-time White Noise

\[ X(n), \text{ w.s.s. process} \]
\[ E[X(n)] = 0 \]
\[ R_X(k) = \sigma_X^2 \delta(k) = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ S_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \sigma_X^2 \delta(k) = \sigma_X^2 \quad -\pi \leq \omega \leq \pi \]

Notice the difference with the continuous-time white noise where \( R_X(\tau) = \sigma_X^2 \delta(\tau) \) cannot be define at \( \tau = 0 \) because of \( \delta(\tau) \). The power density \( S_X(e^{j\omega}) \) is defined for all possible discrete frequencies \( \omega \).
Example: Discrete-time moving average

\[ Y(n) = X(n) + \alpha X(n - 1) \]

\( X(n) \) is white noise with zero mean and variance \( \sigma^2 \). Find \( E[Y(n)] \), \( R_Y(k) \) and \( S_Y(e^{j\omega}) \).

\[
E[Y(n)] = E[X(n)] + \alpha E[X(n - 1)] = 0
\]

\[
R_Y(k) = E[Y(n)Y(n + k)] = E[(X(n) + \alpha X(n - 1))(X(n + k) + \alpha X(n + k - 1))]
= (1 + \alpha^2)R_X(k) + \alpha R_X(k + 1) + \alpha R_X(k - 1)
= \begin{cases} 
(1 + \alpha^2)\sigma^2 & k = 0 \\
\alpha\sigma^2 & k = 1, -1 \\
0 & \text{otherwise}
\end{cases}
\]

The power density is then

\[
S_Y(e^{j\omega}) = (1 + \alpha^2)\sigma^2 + 2\alpha\sigma^2 \cos(\omega)
\]