School of Information Sciences
University of Pittsburgh

TELCOM2125: Network Science and Analysis

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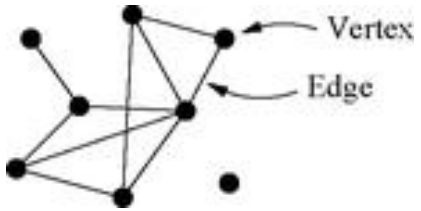
Figures are taken from:
M.E.J. Newman, “Networks: An Introduction”

What is a network?

- **A collection of points joined together in pairs by lines**
  - Points that will be joined together depends on the context
    - ✓ *Points* → Vertices, nodes, actors …
    - ✓ *Lines* → Edges



- **There are many systems of interest that can be modeled as networks**
  - Individual parts linked in some way
    - ✓ Internet
      - A collection of computers linked together by data connections
    - ✓ Human societies
      - A collection of people linked by acquaintance or social interaction

Why are we interested in networks?

- **While the individual components of a system (e.g., computer machines, people etc.) as well as the nature of their interaction is important, of equal importance is the *pattern* of connections between these components**
  - These patterns significantly affect the performance of the underlying system
    - ✓ The patterns of connections in the Internet affect the routes packets travel through
    - ✓ Patterns in a social network affect the way people obtain information, form opinions etc.

Why are we interested in networks?

- **Scientists in a wide variety of fields have developed tools for analyzing, modeling and understanding network structures**
  - Mathematical, statistical and computational tools
    - ✓ Identify the best connected node, the path between two nodes, predict how a process on a network (e.g., spread of a disease) will take place etc.

- **These tools work in an abstract level – not considering specific properties of the network examined**
  - General applicability to any system that can be represented as a network

The importance of structure

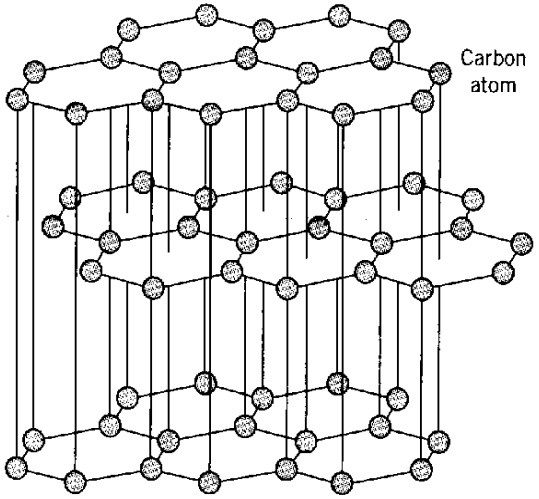
- **The value of a system does not only depend on its individual components**

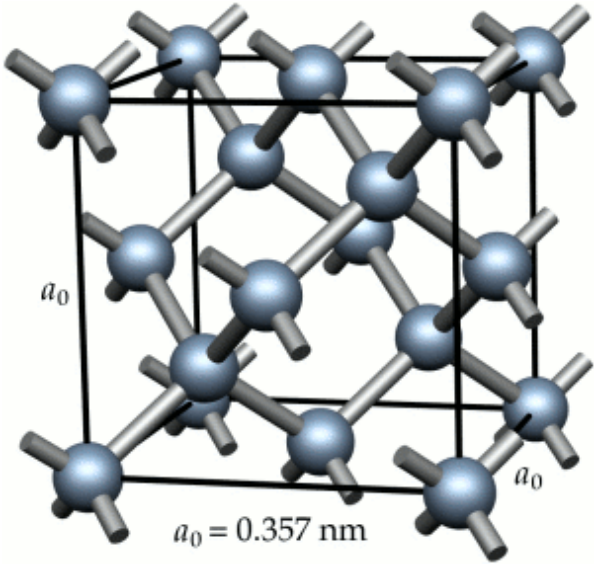
- **What is common in pencil and diamond?**
  - They are both made of carbon

The importance of structure

- **The value of a system does not only depend on its individual components**

- **What is different in pencil and diamond?**
  - Their structure!





Figures obtained from: http://batteryblog.ca and http://www.e6cvd.com

Examples of networks

- **The Internet**
  - Depending on the level of granularity we examine it vertices can be network devices (host machines and routers) or autonomous systems
  - Edges are physical links between vertices
  - Studying the Internet structure can help understand and improve the performance
    - ✓How do we route packets over the Internet?
    - ✓How resilient is the Internet in the failure of nodes/edges?
    - ✓Which edge's capacity should we boost?

Examples of networks

- **The World Wide Web**
  - Often is confused with the Internet but they are different!
  - The Web is a network of information stored in webpages
    - ✓ Nodes are the webpages
    - ✓ Edges are the hyperlinks between the pages
  - The structure of this network is one of the major factors that Google exploits in its search engine
  - Directed edges

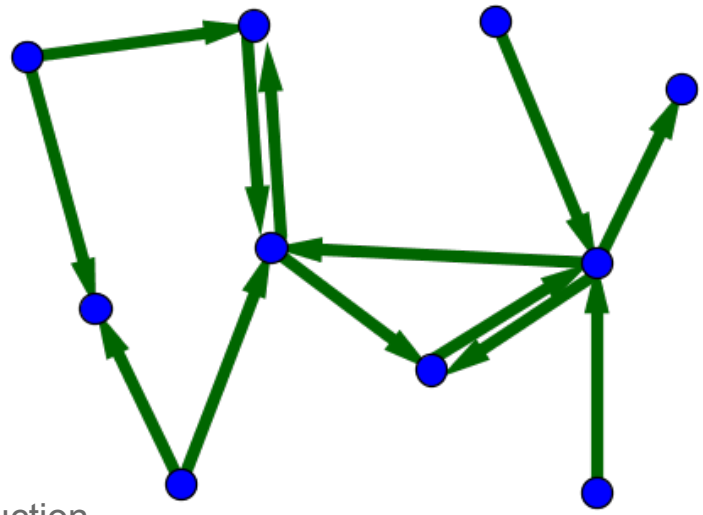
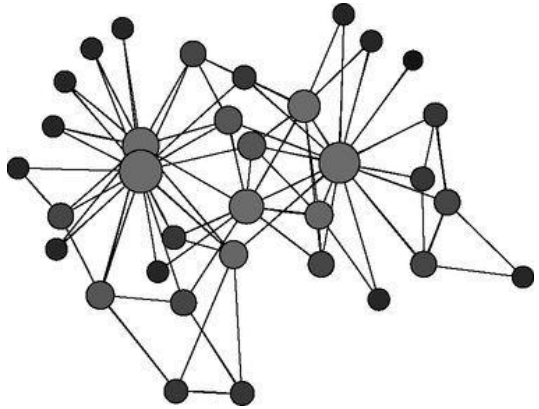


Figure obtained from: http://mathinsight.org/network_introduction

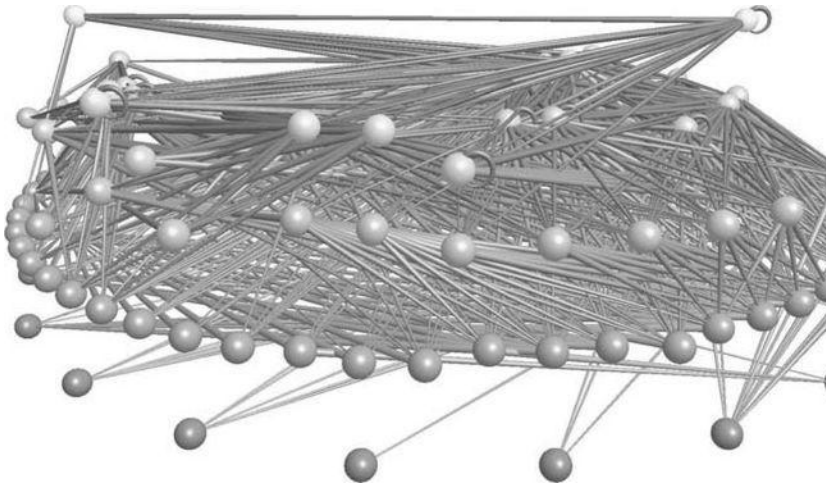
Examples of networks

- **Social networks**
  - Network of people
    - ✓Edges can represent friendships, relative relations, co-locations etc.
  - Long tradition in network analysis
  - Traditionally social network studies were based on small scale networks
    - ✓Online social media have provided network data on previously unreachable scale



Example of networks

- **Biological networks**
  - Food webs
    - ✓ Ecological network
      - Vertices are species in an ecosystem
      - Directed edge from A to B, iff B eats A
  - Can help study many ecological phenomena, particularly concerning energy and carbon flows in ecosystems
    - ✓ Edges typically point from the prey to the predator, indicating the direction of the flow of energy
  - Metabolic networks
    - ✓ Protein-protein interactions

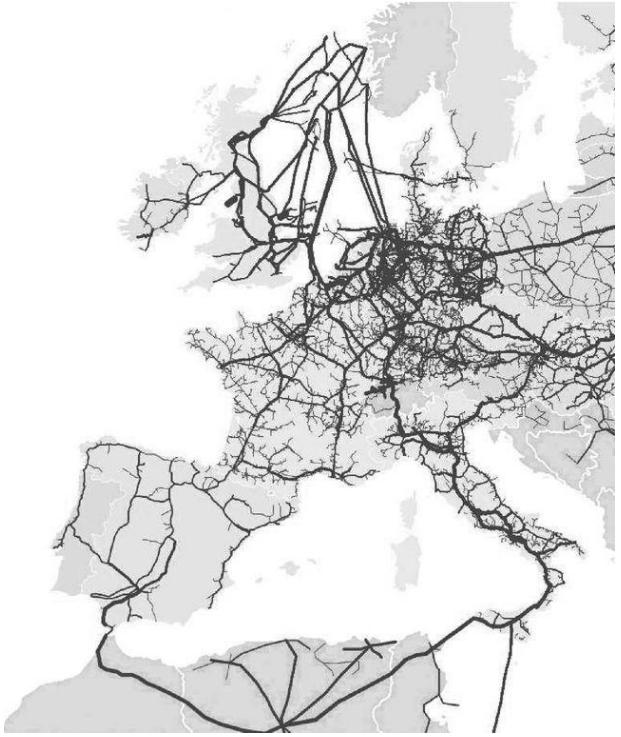


Example of networks

- **Transportation networks**
  - Airline routes, road and rail networks
    - ✓ Road networks (usually)
      - Vertices: road intersections
      - Edges: roads
    - ✓ Rail networks
      - Vertices: locations
      - Edges: between locations that are connected with a single train
      - More general bipartite networks
  - Network theory can be applied to identify the “optimal” structure of a transportation network for a pre-specified objective

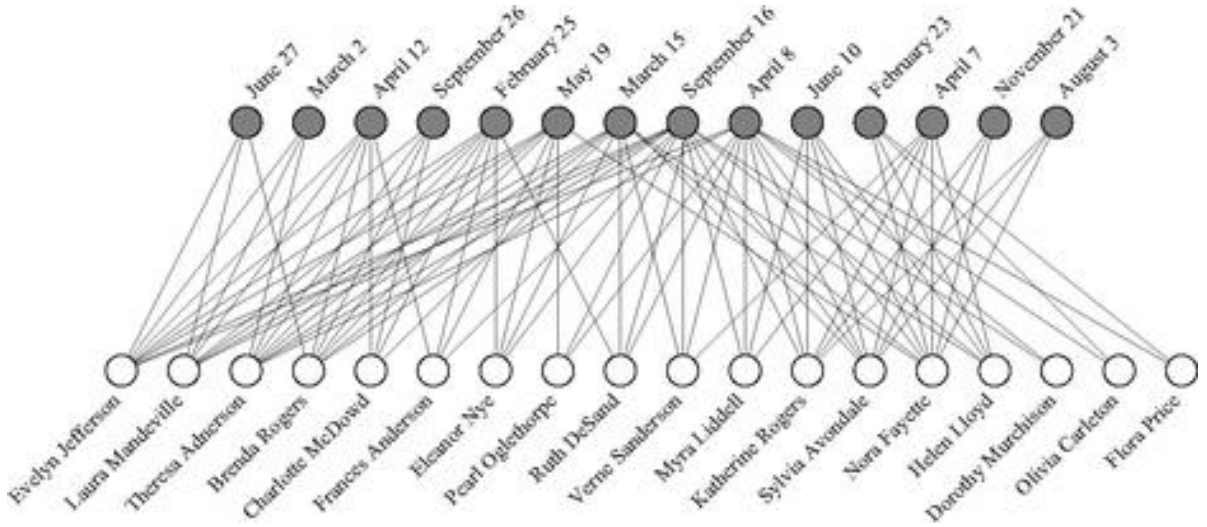
Example of networks

- **Delivery and distribution networks**
  - Gas pipelines, water and sewerage lines, routes used by the post office and package delivery and cargo companies
    - ✓Gas distribution network
      - Edges: pipelines
      - Vertices: intersections of pipelines
        - *Pumping, switching and storage facilities and refineries
  - River networks, blood vessels in animals and plants



Example of networks

- **Affiliation networks**
  - Two types of vertices
    - ✓ Connections allowed only among different types of vertices
  - E.g., board of directors of companies and their members, people and locations they visit etc.



Example of networks

- **Citation networks**
  - Vertices are papers
  - An edge exists from paper A to paper B, iff A cites B
  - Network analysis can be used to identify influential papers
    - ✓ Bibliometrics
- **Recommender networks**
  - Represent people's preferences for things
    - ✓ E.g., preference on certain products sold by a retailer

Synthesize

- **Many literature from different disciplines deals with networks**
  - Sociology
  - Computer Science
  - Math
  - Statistical Physics
  - Economics
  - …
- **What have we learned?**

Questions

- What can we do with network data?
- What can they tell us about the form and function of the system the network represents?
- What properties of the network systems can we measure or model?
- How are these properties related with the practical issues we care about?

Reasoning about networks

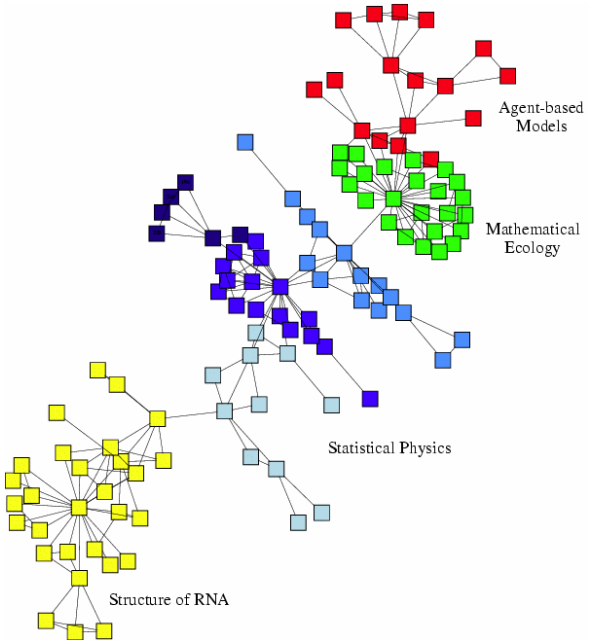
- **How do we reason about networks?**
  - Empirical: Study network data to find organizational principles
  - Mathematical models: Probabilistic, graph theory
  - Algorithms: analyzing graphs

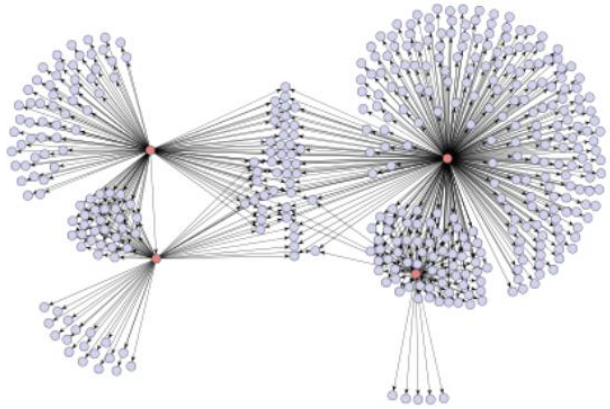
- **What do we hope to achieve from studying networks?**
  - Patterns and statistical properties of network data
  - Design principles and models
  - Understand why networks are organized the way they are
    - ✓ Predict behavior of networked systems

What do we study in networks?

- **Structure and evolution**
  - What is the structure of a network?
  - Why and how did it become to have such structure?

- **Processes and dynamics**
  - How do information disseminate?
  - How do diseases spread?





Properties of networks

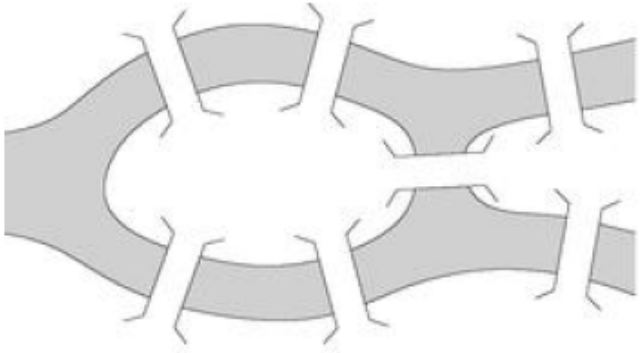
- **Network theory has developed a large number of tools that can be used to describe and understand networks**
  - Centrality: quantification of the importance of a vertex (or even an edge)
    - ✓ Various definitions that capture different aspects and can be useful in different contexts
      - Degree, eigenvector, Katz, PageRank etc.
  - Geodesic distance: minimum number of edges one would have to traverse in order to get from one vertex to the other
    - ✓ Implications on how fast *things* travel in the network

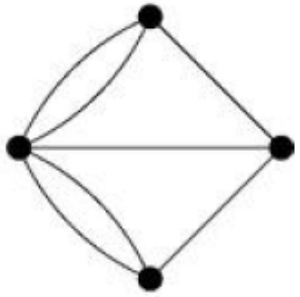
Properties of networks

- **Network theory includes also a number of concepts of practical importance**
  - The notion of hubs
    - ✓A small number of vertices with extremely high degree
    - ✓What are their implications in networks?
  - Small-world effect
    - ✓On average geodesic distances are much smaller compared to the size of the network
    - ✓Repercussions with regards to information diffusion
  - Communities in networks
    - ✓The way a network breaks to communities might reveal information for the network (e.g., an organization) that are not easy to see without network data

The tale of networks

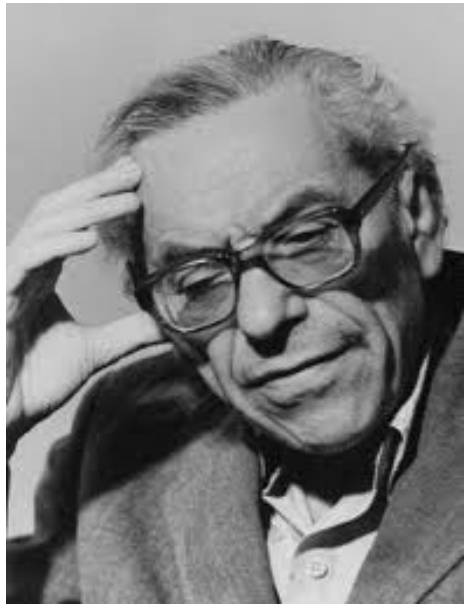
- **Seven bridges of Konigsberg**
  - Konigsberg was built on the banks of the river Pregel and on two islands that lie midstream
  - Seven bridges connected the land masses
  - Does there exist any walking route that crosses all seven bridges exactly once?
  - Leonhard Euler mapped the problem to a graph and proved that this is impossible
    - ✓ Foundations of graph theory





The tale of networks

- **Random networks**
  - Erdos extensively studied networks that form randomly
    - ✓Two vertices connect uniformly at random
  - Erdos realized that if networks develop randomly, they are highly efficient
    - ✓Even with few connections on average per link, the network can be connected with small paths
  - Erdos laid the foundations modern network theory



The tale of networks

- **The Milgram Small World Experiment: Six degrees of separation**
  - Randomly selected people living in Wichita, Kansas and Omaha, Nebraska
  - They were asked to get a letter to a stockbroker in Boston they had never met
    - ✓ Only the name and the occupation of the person in Boston was revealed
  - A large portion of these mails never reached the destination, but those who did reached it in a few hops
    - ✓ Funneling was also observed
    - ✓ Small paths exist, but it is hard to find them in a decentralized way

The tale of networks

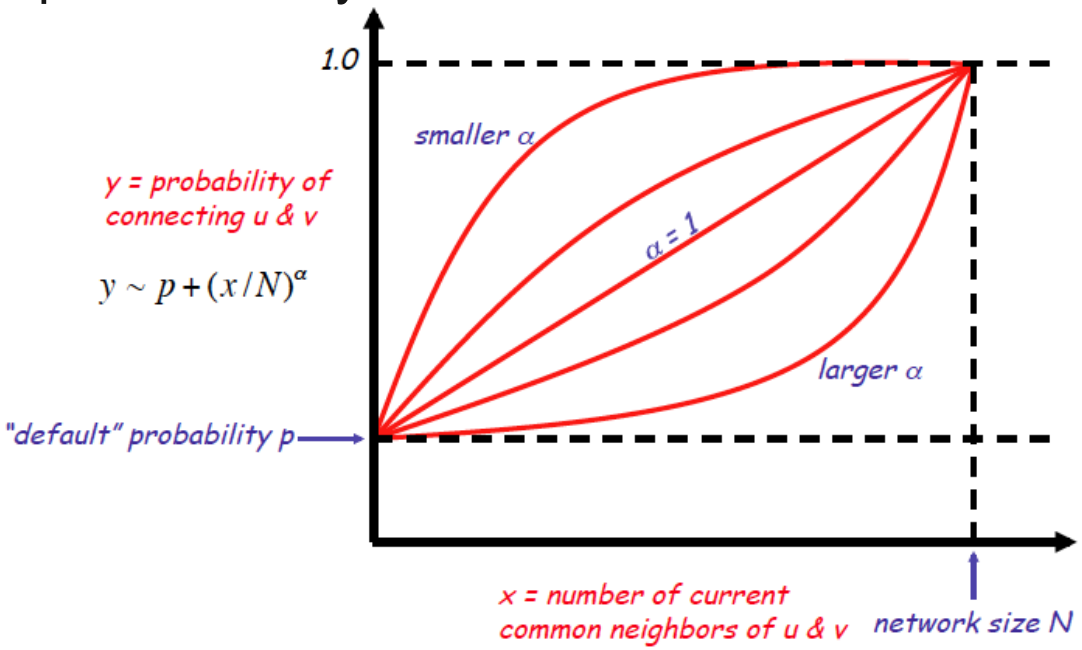
- **The strength of weak ties**
  - Granovetter studied the way that people find their jobs in communities around Boston
  - He found that in most of the cases it was not the closely related people that played a key role but rather some acquaintance
  - Our friends have more friends than we do and hence it is most probable that they will be of help
    - ✓ Also we are most probably exposed to the same information with our very close friends
  - Social capital

The tale of networks

- **Syncing behavior**
  - Watts and Strogatz started wondering about problems such as "How can Malaysian fireflies sync their behavior as if they were one giant organism?"
    - ✓ Is there a leader?
    - ✓ In what manner does the information travel across thousands or even millions of entities?
  - Watts strongly believed there is a strong connection with the six degrees phenomenon
    - ✓ Information seems to have an ability to travel across large populations fast

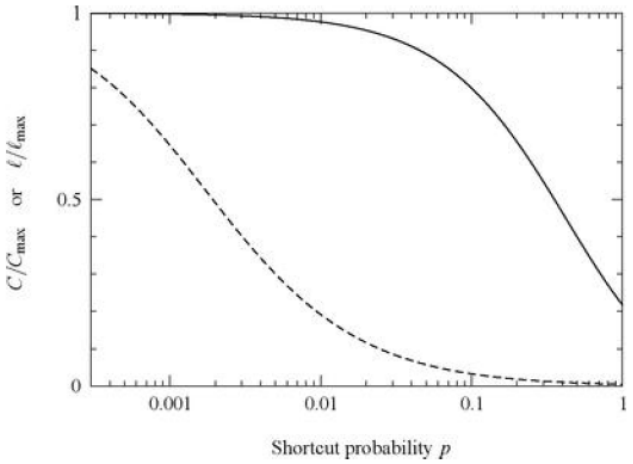
The tale of networks

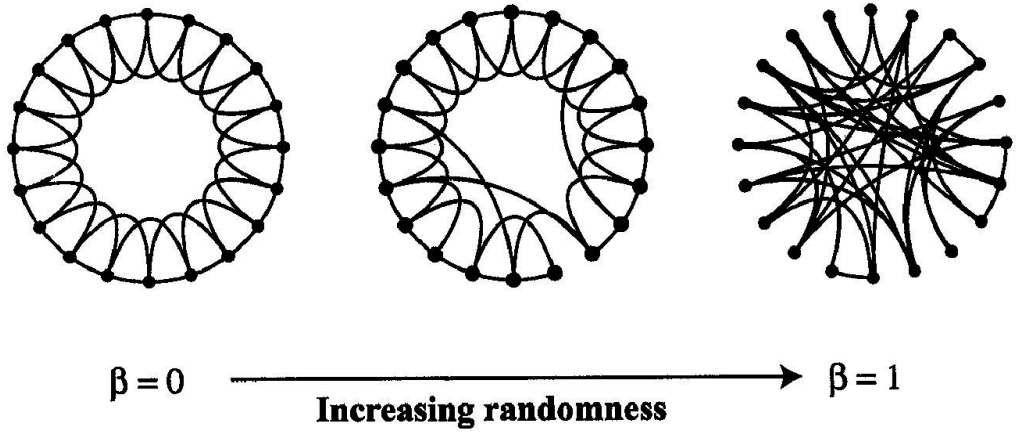
- **Alpha model**
  - How small world networks appear?
  - What are the rules that people follow when making friends?
    - ✓ People introduce their friends to each other
      - ○ The more common neighbors two vertices share, the more probable they are to connect



The tale of networks

- **Beta model**
  - Alpha model showed that small world networks are possible
  - What is the meaning of parameter α?
  - Watts and Strogatz developed an even simpler model
    - ✓ You start from a regular lattice and you rewire edges
      - From order to randomness
      - People combine geographically contained connections with a few long-distance relations





The tale of networks

- **The fitness model**
  - Albert-Laszlo Barabasi and Reka Albert, realized that networks grow, and as they grow the nodes with more links get the bulk of the new links
    - ✓Preferential attachment
    - ✓As networks evolve hubs are formed
    - ✓Power law
  - Later Barabasi capitalized on the similarities that these models had with the Bose-Einstein equation
    - ✓Along with preferential attachment he introduced the attractiveness of a vertex

The future of the networks

- While most of the publicity of networks comes from social network analysis their applicability is virtually limitless

- Network theory is applied in finance, marketing, medicine, criminology, intelligence agencies…

- Just a sample:

http://www.ted.com/talks/nicholas_christakis_how_social_networks_predict_epidemics.html

In this class we study

- **Metrics for describing the structure of a network**
- **Random graph models**
- **Network formation models**
- **Processes in networks**
- **Combination of math and hands-on experience with software used for network analysis**
  - Hands-on experience will be at an individual's level effort

Course details

- **Highly technical course**
  - Linear algebra and probability knowledge is a minimum requirement
- **Some topics will be presented at a higher level**
- **Grade will be based on:**
  - Homework (30%)
  - Midterm (30%)
  - Final (30%)
  - Participation (10%)

http://www.sis.pitt.edu/~kpele/telcom2125_spring15.html

&

Courseweb

Textbook

- **There is one main book we will be using**
  - M.J.E. Newman, “Networks: An Introduction”, Oxford University Press (2010)
- **Other possible references**
  - D. Easley and J. Kleinberg, “Networks, Crowds and Markets: Reasoning About a Highly Interconnected World”, Cambridge University Press (2010)
  - M. Jackson, “Social and Economic Networks”, Princeton University Press (2010)
  - D.J. Watts, “Six Degrees: The Science of a Connected Age”, W.W. Norton & Company (2003)
  - Research papers (pointers will be provided)

Part 0.1: Review on Probability Theory

Figures taken from recitation slides CS224W (Stanford)

Fundamentals of probabilities

- **Sample space Ω : Set of all possible outcomes**
- **Event space $F = 2^{\Omega}$ (an event is a subset of the sample space)**
- **Probability measure: function $P : F \rightarrow \Re$ such that:**
  - $P(A) \geq 0, \quad \forall A \in F$
  - $P(\Omega) = 1$
  - For disjoint events A_i, $P(\bigcup_i A_i) = \sum_i P(A_i)$

Example

- **Consider throwing a dice twice:**
  - Sample space $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
  - Event space $F = 2^{\Omega}$
    - ✓ Sample events: let A be the event that the sum is even and let B be the event that we roll at least one 6
  - Probability measure: function P is simple counting in this discrete case
    - ✓ P(A) = 18/36=0.5
    - ✓ P(B) = 11/36

Union

- **For every two events A and B, the union of the two ("A or B") is:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  - E.g. continuing are above example

$$P(A \cup B) = 18/36 + 11/36 - 5/36 = 24/36 = 2/3$$

Conditional probability

Let A and B be two events. Then the conditional probability of A given B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

"What's the probability of A once we know B has happened?"

Rewriting gives us the useful product rule:

$$P(A \cap B) = P(A \mid B)P(B)$$

Independence

Two events are independent if:

$$P(A \cap B) = P(A)P(B)$$

Equivalently: $P(A \mid B) = P(A) \quad and \quad P(B \mid A) = P(B)$

Intuitively, knowing A doesn't tell you anything about B and vice-versa

But beware of relying on your intuition: rolling two dice (x_a and x_b), events x_a=2 and x_a+x_b=k are independent if k=7 and dependent otherwise!

Union bound

Recall that for any two events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If we are trying to upper bound the probability that A or B happens, the *worst case* is that A and B are disjoint (mutually exclusive) and hence, $P(A \cap B) = 0$

The very useful union bound states: Let A_i be some events (not necessarily independent). Then:

$$P\left(\bigcup_i A_i\right) \leq \sum_i P(A_i)$$

Bayes' rule

This is the most important rule of probability!

For two events A and B (with P(B)>0):

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Often used to used to update beliefs:

posterior = "support B provides for A" x "prior"

Example

If a person has malaria, there is a 90% chance that his/her test results are positive. However the test result are not very correct; there is a chance for 1% false positive. Also only 1% of the total population gets affected by Malaria. Now one person's test result came out as Positive. What's the odds that he will actually have Malaria?

M: event that person has malaria

B: event that blood test is positive

P(B|M) = 0.9

P(B|~M) = 0.01

P(M) = 0.01

Example

Bayes' rule: $$P(M \mid B) = \frac{P(B \mid M)P(M)}{P(B)}$$

What about P(B)? **Law of total probability**

$$P(B) = P(B \mid M)P(M) + P(B \mid \sim M)P(\sim M)$$

Substituting we finally get: P(M|B) = 0.476

How does this result change if you use a superior blood test that has only 0.1% false positives?

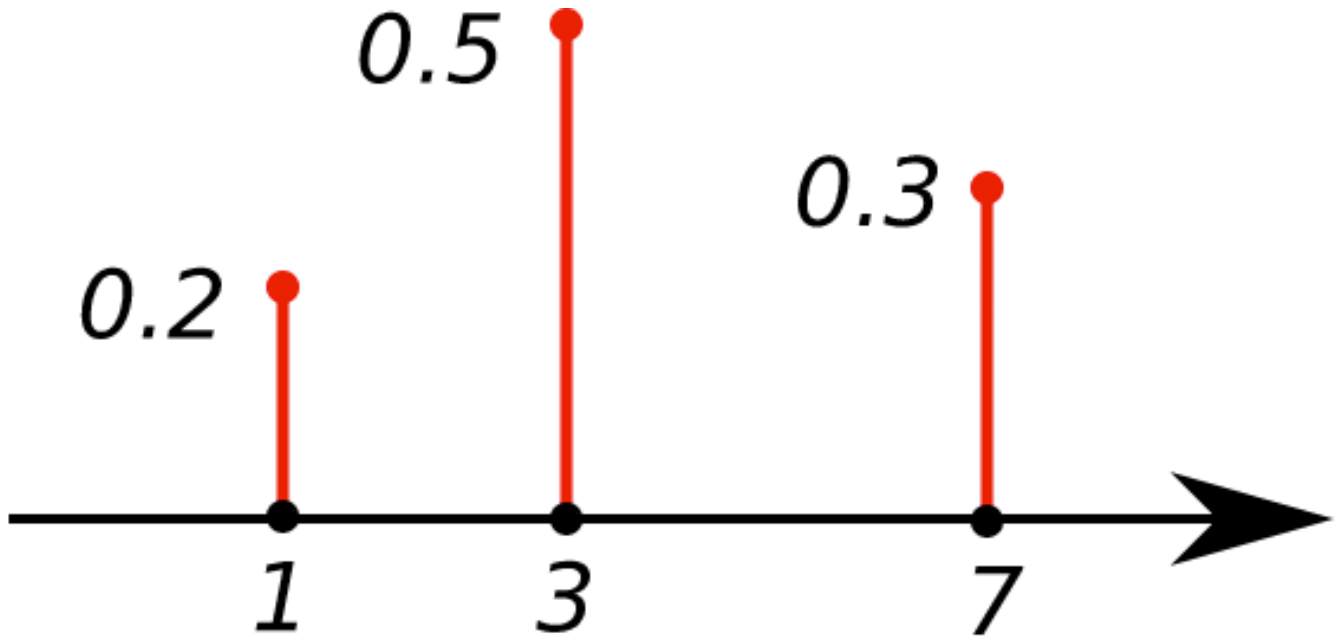
Random variables

- **A random variable is technically a real-valued function defined on the sample space** $X : \Omega \rightarrow \Re$
- **Probabilities of random variables come from underlying P function:** $P(X = k) = P(\{\omega \in \Omega \mid X(\omega) = k)$
  - It is called random variable because it is a variable that does not take on a single – deterministic – value, but it can take on a set of different values, each with an associated probability
  - Let X be a random variable (r.v.) that counts the number of 6's we roll in a 2 dice rolls
    - ✓ P(X=2)=P({6,6}) = 1/36
    - ✓ P(X=1)=P({1,6})+P({2,6})+P({3,6})+…+P({5,6})+P({6,1})+…=10/36
    - ✓ P(X=0)=?

Distributions

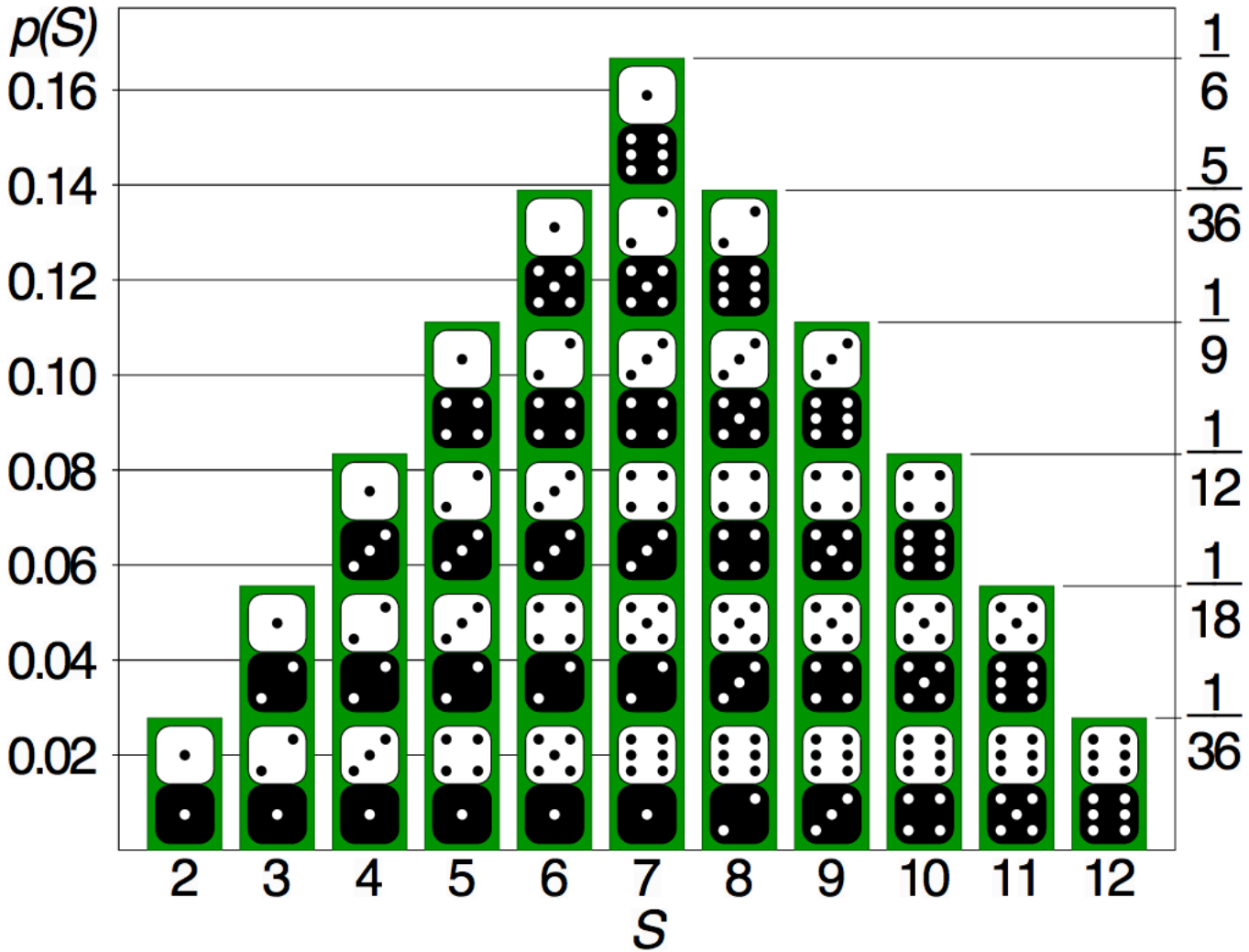
A probability mass function (pmf) assigns a probability to each possible value of a random variable (in the discrete case)

Example: “funny dice”



Distributions

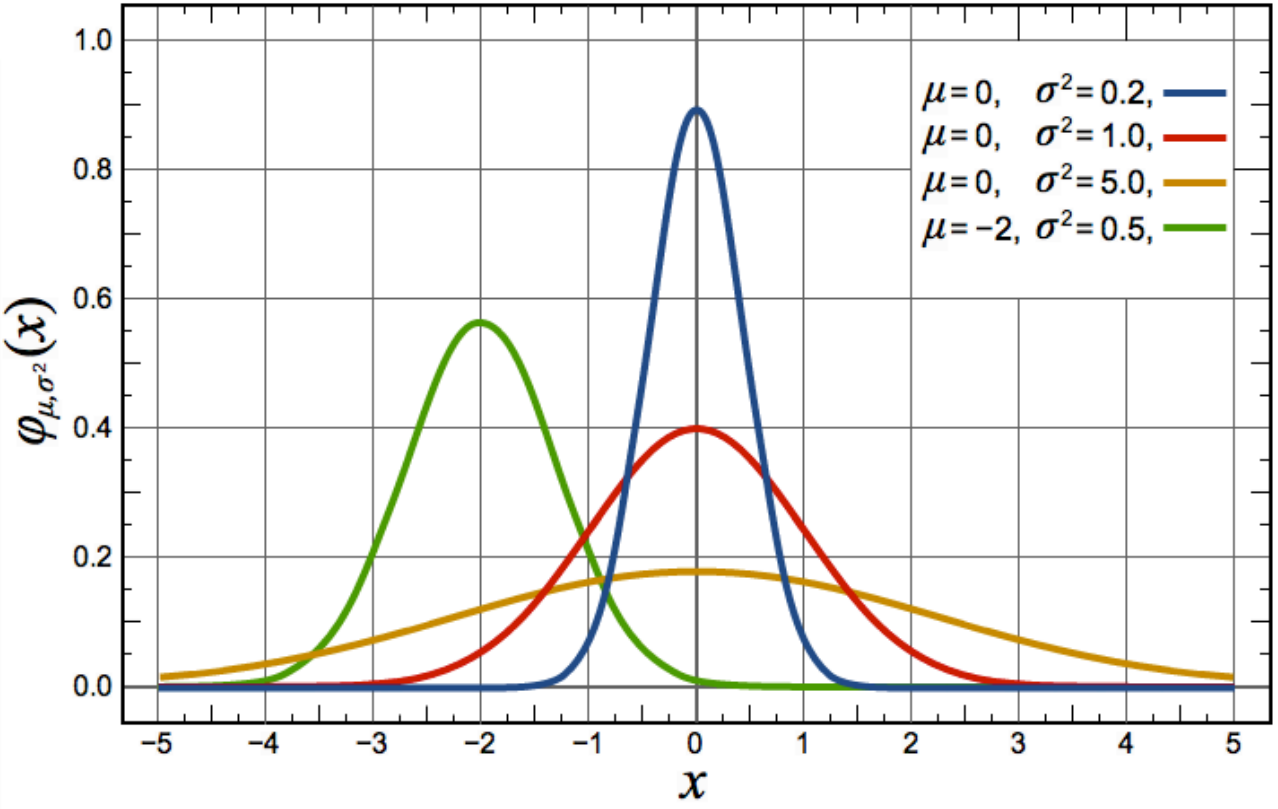
Distribution over sum of two dice rolls



Probability density functions

The PDF of a continuous random variable X describes the relative likelihood for X to take on a given value:

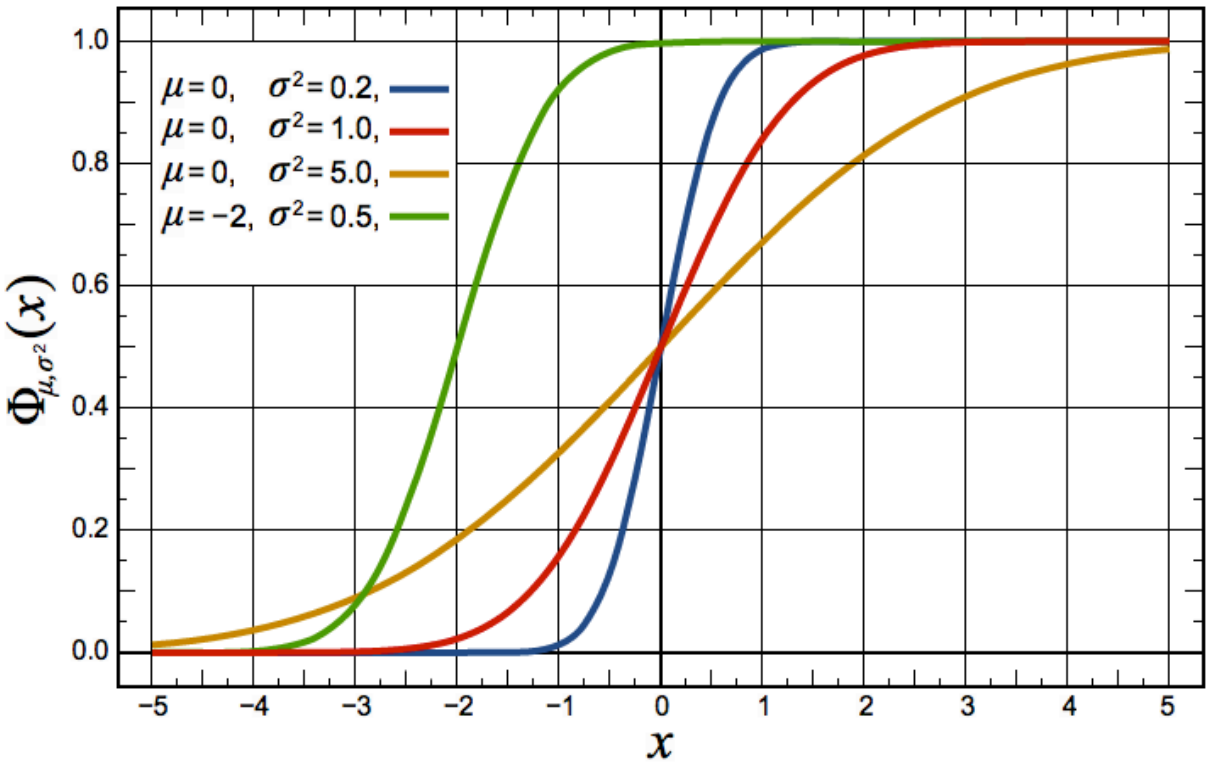
$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



Cumulative distribution

The CDF of a random variable X is:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$



Properties of distribution functions

CDF

$$0 \le F_X(x) \le 1$$

$$F_X \quad monotone \quad increa\sin g, \quad with \quad \lim_{x\to-\infty} F_X(x) = 0, \lim_{x\to\infty} F_X(x) = 1$$

PMF

$$0 \le p_X(x) \le 1$$

$$\sum_x p_X(x) = 1$$

$$\sum_{x\in A} p_X(x) = p_X(A)$$

PDF

$$f_X(x) \ge 0$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{x\in A} f_X(x)\,dx = P(X \in A)$$

Some common random variables

$\sim$ *Bernoulli*(p) $(0 \leq p \leq 1)$: $p_X(x) = \begin{cases} p & \text{x=1,} \\ 1-p & \text{x=0.} \end{cases}$

$\sim$ *Geometric*(p) $(0 \leq p \leq 1)$: $p_X(x) = p(1-p)^{x-1}$

$\sim$ *Uniform*(a, b) $(a < b)$: $f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$

$\sim$ *Normal*(μ, σ^2): $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

Expectation and variance

- **If the discrete random variable X has pmf p(x), then the expected value of X is:** $E[X] = \sum_x xp(x)$
- **For a continuous r.v. we have a similar result:** $E[X] = \int_{-\infty}^{\infty} xf(x)dx$
- **Expectation is linear:**

$$\forall a \in \Re \Rightarrow E[a] = a$$

$$E[ag(X) + bh(X)] = aE[g(X)] + bE[h(X)]$$

- $Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$
  - Variance is **not** linear

What is the expectation of a rolling dice?

Indicator variables

- **An indicator variable just indicates whether an event occurs or not**

$$I_A = \begin{cases} 1, & if \quad A \quad occurs \\ 0, & otherwise \end{cases}$$

- **They have a very useful property:**

$$E[I_A] = 1P(I_A = 1) + 0P(I_A = 0) = P(I_A = 1) = P(A)$$

Method of indicators

- **Suppose X is the number of events that occur among a collection of events $A_1, A_2,\ldots,A_n$. Let $X_1, X_2,\ldots,X_n$ be indicator variables for these events. Then,**

$$X = \sum_{i=1}^{n} X_i \Rightarrow E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P(A_i)$$

- **When to use: Useful when we need to find E[X] for a counting variable X, especially when we can break it down into counting the occurrences of events in a collection $A_1, A_2,\ldots,A_n$, where the probability $P(A_i)$ is easy to compute**
  - Note: The events A_i do not need be independent!

Method of indicators

- **Example: N professors are at a dinner and take a random coat when they leave. What is the expected number of professors with the right coat?**

Let G be the number of professors that get the right coat, and let G_i be an indicator function for the event that professor i gets his own coat. Then: $G=G_1+G_2+\ldots+G_n$

$$E[G] = E[G_1 + G_2 + \ldots + G_n] =$$
$$= E[G_1] + E[G_2] + \ldots + E[G_n] =$$
$$= 1/n + 1/n + \ldots. + 1/n = 1$$

Linearity of expectations does not assume independence !

Some useful inequalities

- **Markov's inequality**
  - X r.v. and a>0 $$P(X \geq a) \leq \frac{E[X]}{a}$$
  - Continuing our previous example we can see that the probability of at least 5 professors get the right coats is no higher than 20%
    - ✓ Regardless of N
- **Chebyshev's inequality**
  - X r.v. with finite, non-zero variance σ^2. For any real k>0

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

Chernoff bound

- **Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli with $P(X_i=1)=p_i$**
  - With $\mu = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} p_i$

$$P\left(\sum_{i=1}^{n} X_i \geq (1+\delta)\mu\right) \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}, \quad \forall \delta$$

  - Multiple variants of Chernoff bounds exist, which can be useful in different settings

Parameter estimation: Maximum likelihood

- Assume we have a parametrized distribution $f_X(x;\theta)$ and we do not know parameter θ (could be a vector as well)
- We observe IID samples $x_1, x_2, \ldots, x_n$
- Goal: Estimate θ
- The maximum likelihood estimator (MLE) is the value $\hat{\theta}$ that maximizes the likelihood of observing the data/ samples you observed

MLE example

- You flip a coin with unknown bias p of landing heads n times and get n_H heads and n_T tails. What is the MLE estimate for the coin's bias?

The likelihood of observing the data given a particular θ is:

$$P(D \mid \theta) = \theta^{n_H} (1-\theta)^{n_T}$$

Usually we work with the log-likelihood. That is,

$$\log(P(D \mid \theta)) = n_H \log\theta + n_T \log(1-\theta)$$

MLE example

- We want to maximize this last expression (which also maximizes the original likelihood – why?)

We take the derivative of the log-likelihood with respect to θ and set it equal to zero

$$\frac{d}{d\theta}\log(P(D\,|\,\theta)) = 0 \Rightarrow$$

$$\frac{d}{d\theta}\left[n_H \log\theta + n_T \log(1-\theta)\right] = 0 \Rightarrow$$

$$\frac{n_H}{\theta} - \frac{n_T}{1-\theta} = 0 \Rightarrow$$

$$\hat{\theta} = \frac{n_H}{n_H + n_T}$$

Closed form solutions are not always possible!

Part 0.2: Review on Linear Algebra

Figures taken from recitation slides CS224W (Stanford)

Matrices and vectors

- **Matrix is a rectangular array of numbers, e.g.,** $A \in \Re^{m \times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- **Vector: a matrix consisting of only one column, e.g.,** $x \in \Re^n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Matrix multiplication

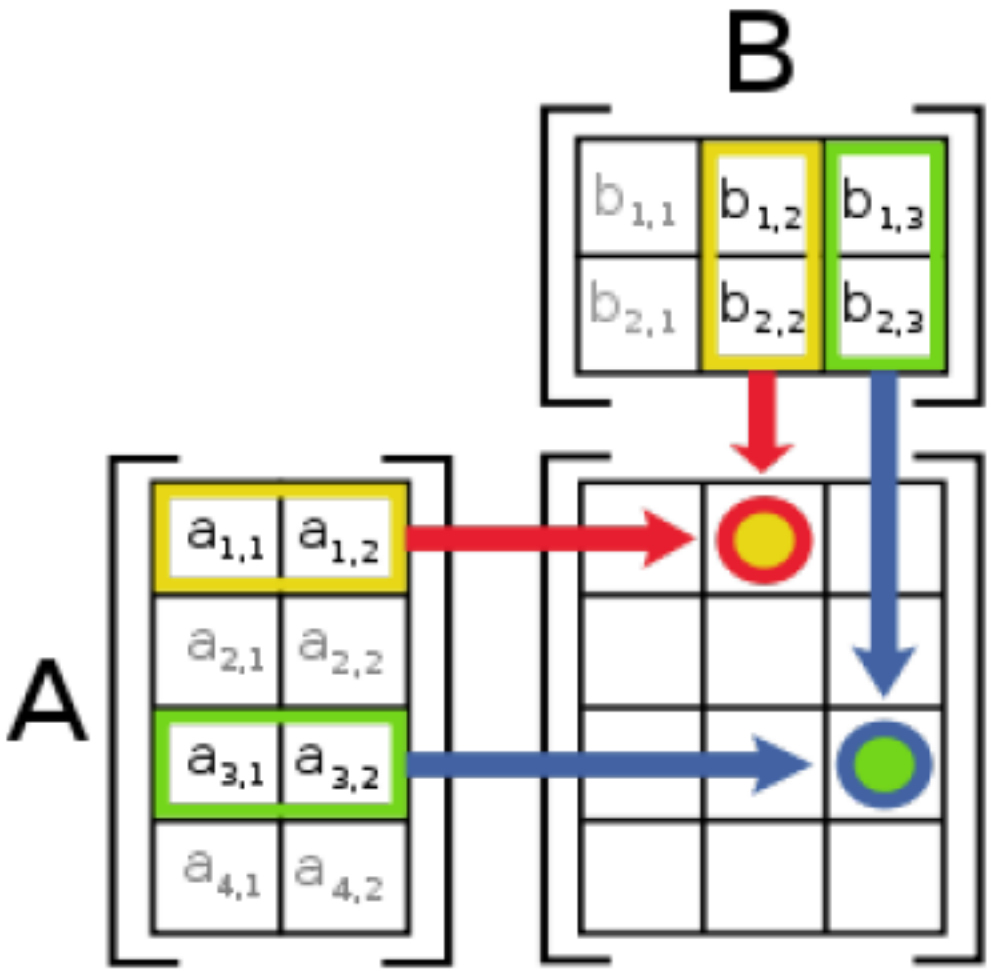
- **If** $A \in \Re^{m \times n}, B \in \Re^{n \times p}, C = AB$ **then** $C \in \Re^{m \times p}$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

- **Special cases**
  - Matrix – vector multiplication
  - Inner produce of two vectors
    - ✓ E.g., $x, y \in \Re^n$

$$x^T y = \sum_{i=1}^{n} x_i y_i \in \Re$$

Matrix multiplication



Properties of matrix multiplication & Operators

- **Associative: (AB)C=A(BC)**
- **Distributive: A(B+C)=AB+AC**
- **Non commutative: AB ≠ BA**

- **Transpose:** $A \in \Re^{m \times n}$ **then** $A^T \in \Re^{n \times m} : (A^T)_{ij} = A_{ji}$
- **Properties**
  - $(A^T)^T=A$
  - $(AB)^T=B^TA^T$
  - $(A+B)^T=A^T+B^T$

Identity matrix

- **Identity matrix:** $I_n \in \Re^{n \times n}$

$$I_n = \begin{cases} 1, & i = j \\ 0, & otherwise \end{cases}$$

$$\forall A \in \Re^{m \times n} : AI_n = I_m A = A$$

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, \; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; \cdots, \; I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Diagonal matrix

- **Diagonal matrix:** $D = diag(d_1, d_2, ..., d_n)$

$$D_{ij} = \begin{cases} d_i, & i = j \\ 0, & otherwise \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Other special matrices

- **Symmetric matrices: a square matrix A is symmetric if $A=A^T$**
- **Orthogonal matrix: a square matrix U is orthogonal if $UU^T=U^TU=I$**

Linear independence and rank

- **A set of vectors $\{x_1,x_2,\ldots,x_n\}$ is linearly independent if**

$$\nexists\{\alpha_1,\ldots,\alpha_n\}: \sum_{i=1}^{n} \alpha_i x_i = 0$$

- **Rank of an mxn matrix A is the maximum number of linearly independent columns (or equivalently, rows)**

- **Properties:**
  - $\text{rank}(A) \leq \min(m,n)$
  - $\text{rank}(A) = \text{rank}(A^T)$
  - $\text{rank}(AB) \leq \min\{\text{rank}(A),\text{rank}(B)\}$
  - $\text{rank}(A+B) \leq \text{rank}(A)+\text{rank}(B)$

Matrix inversion

- **If A is an nxn matrix with rank(A) = n, then the inverse of A, denoted with A^{-1} is the matrix that: $AA^{-1}=A^{-1}A=I$**

- **Properties**
  - $(A^{-1})^{-1}=A$
  - $(AB)^{-1}=B^{-1}A^{-1}$
  - $(A^{-1})^{T}=(A^{T})^{-1}$

- **The inverse of an orthogonal matrix is its transpose**

Eigenvalues and eigenvectors

- **Consider a real matrix nxn. $\lambda \in C$ is an eigenvalue of A with corresponding eigenvector $x \in C^n (x \neq 0)$ if**

$$Ax = \lambda x$$

- **Eigenvalues: the n possibly complex roots of the (characteristic) polynomial equation det(A-λI)=0**

$$A\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot (-3) \\ 1 \cdot 3 + 2 \cdot (-3) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

Eigenvalue and eigenvector properties

- **Usually eigenvectors are normalized to unit length**
- **If A is symmetric, then all the eigenvalues are real and the eigenvectors are orthogonal to each other**
  - Their inner product is zero
- $tr(A) = \sum_{i=1}^{n} \lambda_i$
- $\det(A) = \prod_{i=1}^{n} \lambda_i$
- $rank(A) = \left| \{ 1 \le i \le n \mid \lambda_i \ne 0 \} \right|$

Matrix eigendecomposition

- Consider an kxk matrix A, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and eigenvectors $x_1, x_2, \ldots, x_k$
- Also $P=[x_1|x_2|\ldots|x_k]$ and $D=\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Then:

$$
\begin{aligned}
AP &= A\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_k\end{bmatrix} \\
&= \begin{bmatrix}A\mathbf{X}_1 & A\mathbf{X}_2 & \cdots & A\mathbf{X}_k\end{bmatrix} \\
&= \begin{bmatrix}\lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & \cdots & \lambda_k\mathbf{X}_k\end{bmatrix} \\
&= \begin{bmatrix}\lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_k x_{k1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_k x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1k} & \lambda_2 x_{2k} & \cdots & \lambda_k x_{kk}\end{bmatrix} \\
&= \begin{bmatrix}x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{kk}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k\end{bmatrix} \\
&= PD,
\end{aligned}
$$

Matrix eigendecomposition

- Thus, $A = PDP^{-1}$
- Furthermore, by induction we can show that: $A^n=PD^nP^{-1}$
- The above matrix diagonalization is a special case of Singular Value Decomposition

Singular Value Decomposition

- **Singular Value Decomposition (SVD) is typically used for dimensionality reduction**
- **For instance let's consider the following matrix that represents some dataset:**

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}$$

  - The rank of this matrix is 2 (why?)
- **Why is "low" rank interesting?**
  - Each row of A is expressed as a 3-dimensional vector with the standard base vectors [1 0 0], [0 1 0] and [0 0 1]
  - However, the two linearly independent rows of A can form a new basis (i.e., [1 2 1] and [-2 -3 1]
  - Now each row has new coordinates [1 0], [0 1] and [1 -1] of lower dimensionality

Rank = "dimensionality"

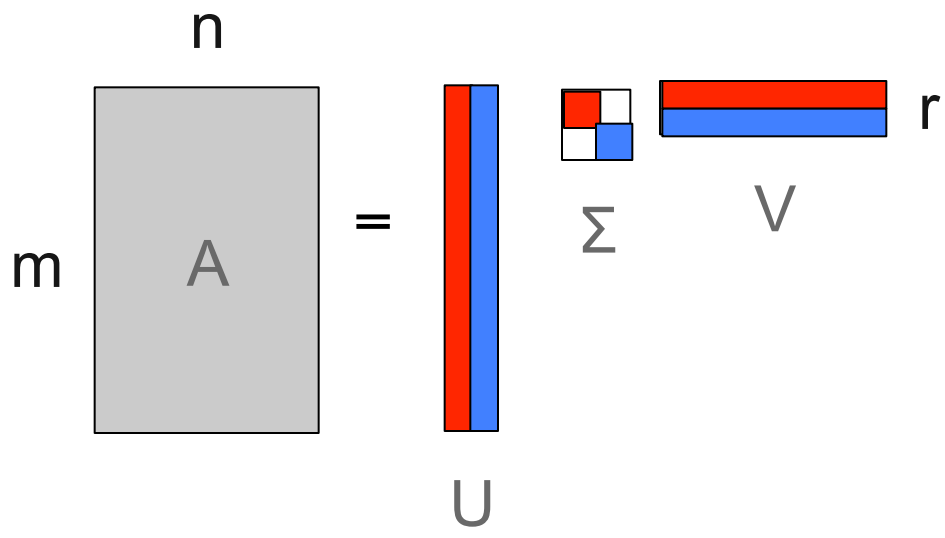
Singular Value Decomposition

$$A_{[m\times n]} = U_{[m\times r]}\Sigma_{[r\times r]}\left(V_{[n\times r]}\right)^T$$

- **A: Input (data) matrix**
  - m x n matrix (e.g., m users, n products)
- **U: Left singular vectors**
  - m x r matrix (e.g., m users, n 'concepts')
- **Σ: Singular values**
  - r x r diagonal matrix (strength of each 'concept')
    - ✓(r: rank of matrix A)
- **V: Right singular vectors**
  - n x r matrix (n products, r 'concepts')

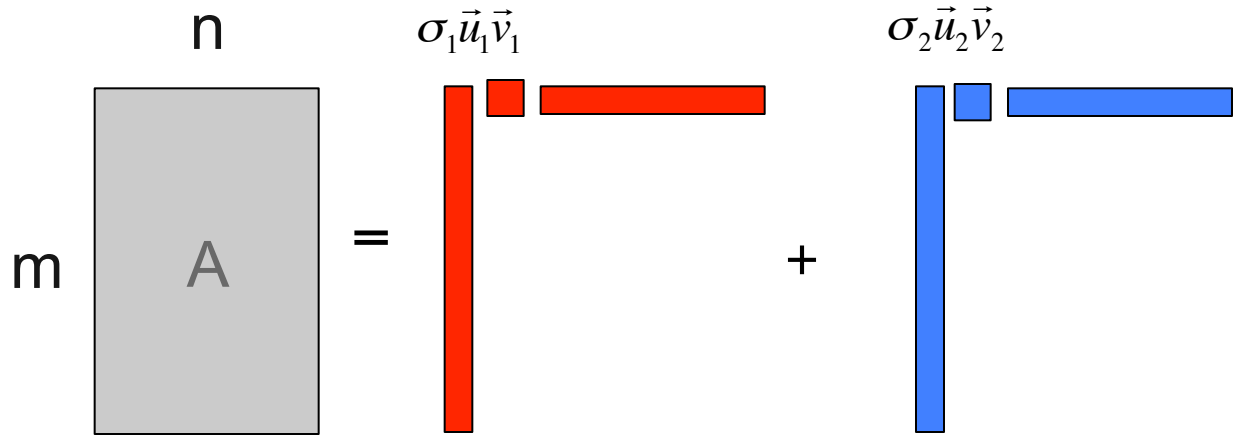
Singular Value Decomposition

$$A \approx U\Sigma V^T = \sum_i \sigma_i \vec{u}_i \circ \vec{v}_i^T$$



Singular Value Decomposition

$$A \approx U\Sigma V^T = \sum_i \sigma_i \vec{u}_i \circ \vec{v}_i^T$$



σ_i : *scalar*

$\vec{u}_i$: *vector*

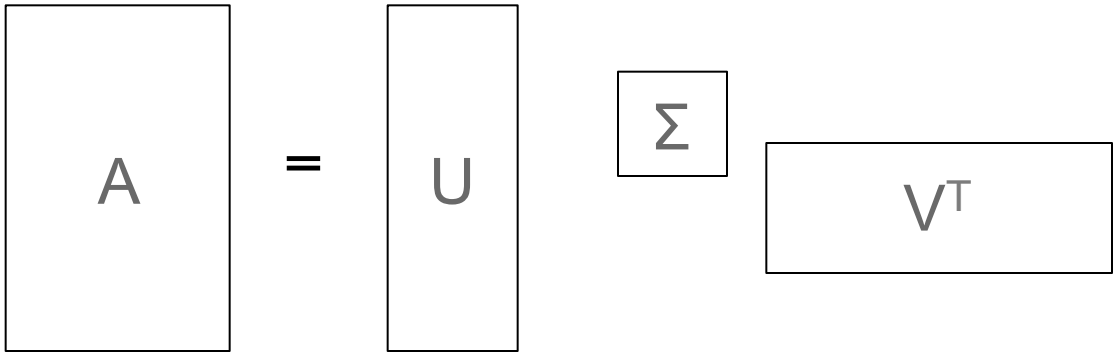
$\vec{v}_i$: *vector*

SVD Theorem

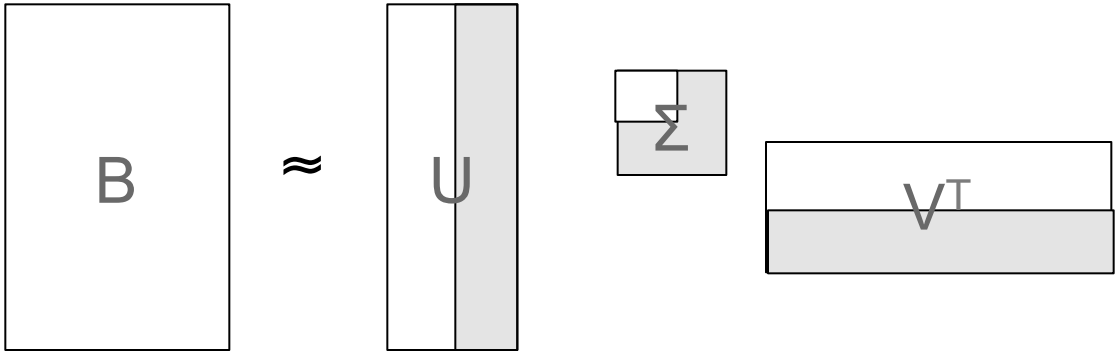
- **It is <u>always</u> possible to decompose a real matrix A into $A = U\Sigma V^T$, where:**
  - U, Σ, V: unique
  - U, V: column orthonormal
    - ✓ $U^TU = I$ and $V^TV = I$
  - Σ: diagonal
    - ✓ Entries are positive and sorted in decreasing order ($\sigma_{11} \geq \sigma_{22} \geq \ldots \geq 0$)

Best low rank approximation

- **How do we do dimensionality reduction?**
  - By setting the k smallest singular values to 0
    - ✓ This essentially removes the contribution of the corresponding left and right singular vectors as well



B is the best approximation of A (in the sense of the Frobenius norm)



Best low rank approximation

- **Theorem:**

  Let A = $U\Sigma V^T$ where Σ: $\sigma_1 \geq \sigma_2 \geq \ldots$, and rank(A)=r. Then $B=USV^T$ is the best rank-k approximation to A where: S diagonal r x r matrix where $s_i=\sigma_i$ (i=1,…,k) else $s_i=0$

- **What does "best" mean?**
  - B is a solution to: $\min_B \|A - B\|_F$ where rank(B) = k

- **How many singular values to keep?**
  - Heuristic: keep 80%-90% of the energy ($= \sum \sigma_i^2$)

Relationship between SVD & eigendecomposition

- **SVD gives: $A = U\Sigma V^T$**
- **Eigendecomposition gives: $A = PDP^{-1}$**
  - A needs to be symmetric for eigendecomposition
- **U, V, P are orthonormal**
- **D and Σ are diagonal**

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T = U\Sigma V^T (V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$$

$$A^T A = V\Sigma^T U^T (U\Sigma V^T) = V\Sigma\Sigma^T V^T$$

The above equations show how to compute the SVD of a matrix through eigendecomposition

$D = \Sigma\Sigma^T$, U is the eigenvectors of AA^T and V is the eigenvectors of A^TA