

**Department of Mathematics
University of Pittsburgh
MATH 0240 (Calculus III)**

Midterm 2, Fall 2014

Instructor: Kiumars Kaveh

Last Name:

Student Number:

First Name:

TIME ALLOWED: 50 MINUTES. TOTAL MARKS: 100
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.
PLEASE READ THROUGH THE ENTIRE TEST BEFORE STARTING
AND TAKE NOTE OF HOW MANY POINTS EACH QUESTION IS WORTH.
FOR FULL MARK YOU MUST PRESENT YOUR SOLUTION CLEARLY.

Question	Mark
1	/20
2	/20
3	/20
4	/20
5	/20
6	/1
TOTAL	/100 + 1

1. Using Lagrange's multipliers find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $g(x, y, z) = x^4 + y^4 + z^4 = 1$. Do this in the following two steps:

(a) [10 points] Write the Lagrange's equations.

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ 2x &= \lambda 4x^3 \\ 2y &= \lambda 4y^3 \\ 2z &= \lambda 4z^3\end{aligned}$$

(b) [10 points] Solve Lagrange's equations and find the maximum and minimum values.

$$\boxed{\begin{array}{l} \text{If } x \neq 0 \\ y \neq 0 \\ z \neq 0 \end{array}}$$

$$x^2 = \frac{1}{2\lambda}$$

$$y^2 = \frac{1}{2\lambda}$$

$$z^2 = \frac{1}{2\lambda}$$

$$\text{from } x^2 = \frac{1}{2\lambda} \geq 0 \Rightarrow \frac{1}{2\lambda} = \frac{1}{\sqrt{3}} \Rightarrow f(x, y, z) = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\boxed{\begin{array}{l} \text{If } x=0 \\ y \neq 0 \\ z \neq 0 \end{array}}$$

$$y^2 = \frac{1}{2\lambda} \text{ thus } x^4 + y^4 + z^4 = 1 \Leftrightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$$

$$z^2 = \frac{1}{2\lambda}$$

$$\text{thus } \left(\frac{1}{2\lambda}\right) = \pm \frac{1}{\sqrt{2}} \Rightarrow f(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1$$

$$\boxed{\begin{array}{l} \text{If } x=0 \\ y=0, z \neq 0 \end{array}}$$

$$\Rightarrow f(x, y, z) = 1$$

So the maximum is attained at $(\pm \frac{1}{4\sqrt{3}}, \pm \frac{1}{4\sqrt{3}}, \pm \frac{1}{4\sqrt{3}})$
and the value is: $\sqrt{3}$

The minimum is 1.

2.

- (a) [10 points] Represent the volume of the solid bounded between the paraboloid $x^2 + y^2 - 2z = 2$ and the sphere $x^2 + y^2 + z^2 = 25$ as an iterated integral in cylindrical coordinates (do not need to evaluate the integral).

The intersection: from: $x^2 + y^2 - 2z = 2$
 $\Rightarrow z = \frac{1}{2}(x^2 + y^2 - 2)$.

thus: $x^2 + y^2 + \frac{1}{4}(x^2 + y^2 - 2)^2 = 25$.

Let $r = \sqrt{x^2 + y^2}$,

$$r^2 + \frac{1}{4}(r^2 - 2)^2 = 25$$

$$\Rightarrow \int_0^{2\pi} \int_0^{2\sqrt{6}} \int_{\sqrt{25-r^2}}^{\sqrt{r^2-2}} \frac{1}{2}(r^2 - 2) dz dr d\theta$$

- (b) [10 points] Compute the volume of the solid which is the part of the interior of the sphere $x^2 + y^2 + z^2 = 16$ between the two planes $z = 0$ and $z = 2$. (Hint: cylindrical coordinates.)

If $z = 0 \Rightarrow$ the intersection circle is

$$x^2 + y^2 = 16$$

If $z = 2 \Rightarrow$ the intersection circle is

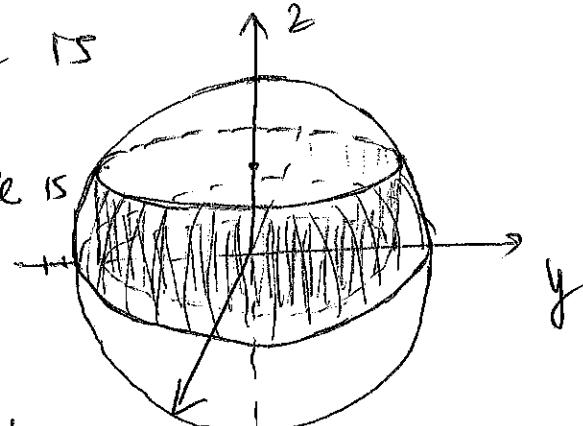
$$x^2 + y^2 = 12$$

Thus the volume equal:

$$\frac{1}{2} V(\text{sphere}) - V(\text{part of sphere above plane } z=2)$$

Using cylinder coordinates.

$$V(\text{part of sphere above } z=2) = \int_0^{2\pi} \int_0^{\sqrt{12}} \int_2^{\sqrt{16-r^2}} dz dr d\theta$$



$$= \int_0^{2\pi} \int_0^{\sqrt{12}} r(\sqrt{16-r^2} - 2) dr d\theta = \frac{20}{3} (2\pi) = \frac{40\pi}{3}$$

Thus the volume we need is

$$\frac{1}{2} \cdot \frac{4\pi(4)^3}{3} - \frac{400\pi}{3} = \frac{88\pi}{3}$$

3. [20 points] Use the spherical coordinates to evaluate the triple integral:

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} z\sqrt{x^2+y^2+z^2} dz dx dy.$$

lower bound for x : $x = 0$

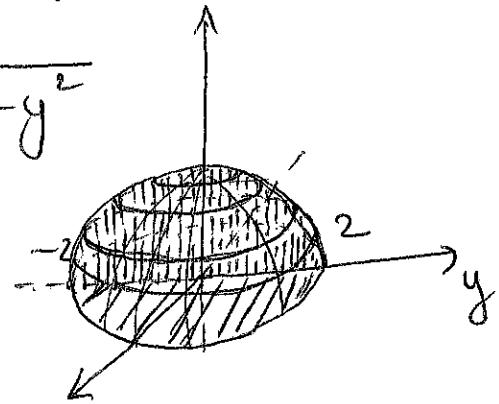
upper bound for x : $x = \sqrt{4-y^2}$

→ The region is a half of a disk

lower bound for z : 0

upper bound for z : $z = \sqrt{4-x^2-y^2}$

→ The volume is the front-half of the sphere
quarter $x^2+y^2+z^2=2$



⇒ The spherical coordinate is:

$$\begin{cases} x = \rho \cos \phi \sin \theta \\ y = \rho \cos \phi \cos \theta \\ z = \rho \sin \phi \end{cases}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 (\rho \sin \phi) \rho^4 \sin^2 \phi d\rho d\phi d\theta.$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{2^5}{5} \sin^2 \phi d\phi d\theta = \frac{2^5}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\phi}{2} - \frac{\sin(2\phi)}{4} \right) \Big|_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{2^5}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\pi}{4} d\theta = \frac{2^5 \pi^2}{20} = \frac{8}{5} \pi^2$$

4.[20 points] Find the work done by the force field

$$\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}},$$

on a particle that moves along the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + tk$ from $(1, 0, 0)$ to $(-1, 0, \pi)$.

$$A(1, 0, 0), \quad B(-1, 0, \pi)$$

The work done, by formula is:

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}$$

$$\text{At } (1, 0, 0): \begin{cases} \cos(t) = 1 \\ \sin(t) = 0 \\ t = 0 \end{cases} \Rightarrow t = 0.$$

$$\text{At } (-1, 0, \pi): \begin{cases} \cos(t) = -1 \\ \sin(t) = 0 \\ t = \pi \end{cases} \Rightarrow t = \pi.$$

$$\mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}.$$

$$\text{The work done: } \mathbf{F}(\mathbf{r}(t)) = \frac{\cos(t)\mathbf{i} + \sin(t)\mathbf{j} + tk}{(1+t^2)^{3/2}}$$

$$= \int_0^\pi \frac{-\sin(t)\cos(t) + \sin(t)\cos(t) + t}{(1+t^2)^{3/2}} dt$$

$$= \int_0^\pi \frac{t dt}{(1+t^2)^{3/2}}$$

$$= \frac{1}{2} \int_0^\pi \frac{d(t^2)}{(1+t^2)^{3/2}} = -\frac{1}{(1+t^2)^{1/2}} \Big|_0^\pi = 1 - \frac{1}{\sqrt{1+\pi^2}}$$

5. [20 points] Use the change of variable formula to evaluate the integral

$$\iint_R (x + x^2 + (x - y)^2) dA,$$

where R is the region bounded by the ellipse $2x^2 + y^2 - 2xy = 1$; $x = u$ and $y = u + v$.

Make substitution, we have: $2x^2 + y^2 - 2xy = 1$
 become: $2u^2 + (u+v)^2 - 2u(u+v) = 1$
 $2u^2 + u^2 + 2uv + v^2 - 2u^2 - 2uv = 1$
 $u^2 + v^2 = 1$

Compute Jacobian:

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

The inverse image of R is the arch: $u^2 + v^2 = 1$
 thus the integral is:

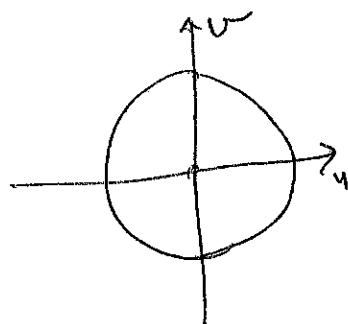
$$\begin{aligned} & \iint_{u^2 + v^2 \leq 1} (u+u^2 + (u-u-v)^2) \cdot |1| du dv \\ &= \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (u + u^2 + v^2) du dv \\ &= \int_{-1}^1 \left(v(u+u^2) + \frac{v^3}{3} \right) \Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du \\ &= \int_{-1}^1 2u(u+1)\sqrt{u^2+1} + \frac{2}{3}(1+u^2)(1+u^2) du \\ &= \int_{-1}^1 (u^2+1)^{3/2} du \end{aligned}$$

$$= \int_1^1 2u(1+u^2) \sqrt{1-u^2} du$$

$$= \int_{-1}^1 \left[u(1+u^2)u + \frac{\sqrt{3}}{3} \sqrt{1-u^2} \right] du$$

6. [1 point] Draw a Thanksgiving turkey!

Set: $\begin{cases} u = r \cos \theta \\ v = r \sin \theta \end{cases}$ we get:



$$\iint (u + u^2 + v^2) du dv$$

$u^2 + v^2 \leq 1$

$$= \int_0^{2\pi} \int_0^1 r(r \cos \theta + r^2) dr d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^3}{3} \cos \theta + \frac{r^4}{4} \right) \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{\cos \theta}{3} + \frac{1}{4} \right) d\theta$$

$$= \frac{\sin \theta}{3} \Big|_0^{2\pi} + \frac{\theta}{4} \Big|_0^{2\pi} = \frac{\pi}{2}$$

6.

