Department of Mathematics

University of Pittsburgh

MATH 0240 (Calculus III)

Midterm 1, Fall 2014

Instructor: Kiumars Kaveh

Last Name: **Student Number:**

First Name:

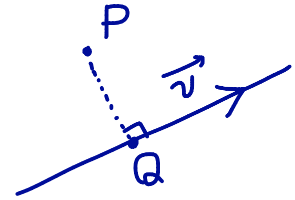
TIME ALLOWED: 50 MINUTES. TOTAL MARKS: 100
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.
PLEASE READ THROUGH THE ENTIRE TEST BEFORE STARTING AND TAKE NOTE OF HOW MANY POINTS EACH QUESTION IS WORTH.
FOR FULL MARK YOU MUST PRESENT YOUR SOLUTION CLEARLY.

Question	Mark
1	/20
2	/20
3	/20
4	/20
5	/20
6	/1
TOTAL	/100 + 1

1.[20 points] Find the point on the line $\mathbf{r}(t) = (t, 1-t, 2+t)$ which has minimum distance to the point $P = (1, 0, 2)$. Calculate this minimum distance. (Hint: if Q is a point on the line, look at the vector $\vec{PQ}$.)

The direction vector $\vec{v}$ of $\vec{r}(t) = \langle t, 1-t, 2+t \rangle$ is $\langle 1, -1, 1 \rangle$ (Coefficients of t)

If Q is the point on the line which is closest to $P = (1,0,2)$ then the vector $\vec{v}$ is orthogonal to $\vec{PQ}$:



Thus $\vec{v} \cdot \vec{PQ} = 0$, If $Q = (t, 1-t, 2+t)$ then $\vec{PQ} = (t, 1-t, 2+t) - (1,0,2) = (t-1, 1-t, t)$.

We have: $(1,-1,1) \cdot (t-1, 1-t, t) = 0 \Rightarrow (t-1) + (t-1) + t = 0 \Rightarrow 3t - 2 = 0 \Rightarrow t = \frac{2}{3} \Rightarrow$

$\vec{r}(t) = \left(\frac{2}{3}, \frac{1}{3}, \frac{8}{3}\right)$. $\quad \vec{PQ} = \left(\frac{-1}{3}, \frac{1}{3}, \frac{2}{3}\right) \quad d = |\vec{PQ}| = \sqrt{\frac{1+1+4}{9}} = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}$.

2. [20 points] Show that the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^6+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$. (Hint: Look at two different curves passing through the origin.)

Consider the curve $x^2 = y$. Clearly this curve passes through $(0,0)$, & moreover as $x \to 0$, $x^2 = y \to 0$.

Now, when $x \neq 0$ $f(x,x^2) = \frac{x^2 \cdot x^2}{x^6 + x^4} = \frac{x^4}{x^6+x^4}$. Thus $\lim_{x\to 0} f(x,x^2) = \lim_{x \to 0} \frac{x^4}{x^6+x^4} =$

$\lim_{x\to 0} \frac{x^4}{x^4(x^2+1)} = \lim_{x\to 0} \frac{1}{1+x^2} = 1$.

But $f(0,0) = 0 \neq 1$ so f not continuous at $(0,0)$.

(limit of f along some curve going to $(0,0)$ exists but is not equal to $f(0,0)$).

3. Consider the curve:

$$\mathbf{r}(t) = \sin(t)\mathbf{i} + \sin(t)\mathbf{j} + \sqrt{2}\cos(t)\mathbf{k}, \quad 0 \le t \le 2\pi$$

(a) [15 points] Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ to $\mathbf{r}(t)$.

$$r'(t) = (\cos(t),\ \cos(t),\ -\sqrt{2}\sin(t))$$

$$|r'(t)| = \sqrt{\cos^2(t) + \cos^2(t) + (-\sqrt{2})^2 \sin^2(t)} = \sqrt{2(\cos^2(t) + \sin^2(t))} = \sqrt{2}. \rightsquigarrow \text{Const. scalar}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \left(\frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}}\cos(t), -\sin(t)\right)$$

$$N(t) = \frac{T'(t)}{|T'(t)|} \qquad T'(t) = \left(\frac{-1}{\sqrt{2}}\sin(t), \frac{-1}{\sqrt{2}}\sin(t), -\cos(t)\right)$$

$$= \left(\frac{-1}{\sqrt{2}}\sin(t), \frac{-1}{\sqrt{2}}\sin(t), -\cos(t)\right) \qquad |T'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$$

(b) [5 points] Find the curvature κ at any point $\mathbf{r}(t)$.

$$\kappa(t) \stackrel{\text{def.}}{=} \frac{d|T|}{ds} \stackrel{\text{Chain rule}}{=} \frac{|T'(t)|}{|r'(t)|} = \frac{1}{\sqrt{2}}.$$

4. [20 points] Find local maximum, local minimum and saddle point(s) of the function

$$f(x, y) = x^4 + y^3 - 3y + 4x + 5.$$

$$\begin{cases} f_x = 4x^3 + 4 \\ f_y = 3y^2 - 3 \end{cases}$$

$f_{xx} = 12x^2 \quad f_{xy} = f_{yx} = 0$

$f_{yy} = 6y$

Solving this system we get $x = -1$, $y = \pm 1$, i.e. there are two critical points $(-1,-1)$ & $(-1,+1)$.

$D = \det \begin{bmatrix} 12 & 0 \\ 0 & -6 \end{bmatrix} = 12 \times (-6) < 0 \Rightarrow (-1,-1)$ saddle point.

$D = \det \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix} = 12 \times 6 > 0 \Rightarrow (-1,1)$ local max or min.

Since $f_{xx} > 0 \Rightarrow (-1,1)$ is a local min.

5.

(a) [10 points] Find the derivative of the function $f(x, y, z) = (y\ln(z)) + x^2$ in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{3}}(1, -1, 1)$ at the point $(2, 1, 1)$.

$\nabla f = (f_x, f_y, f_z) = \left(2x, \ln z, \frac{y}{z}\right)$. $\nabla f(2,1,1) = \left(2\cdot 2, \ln(1), \frac{1}{1}\right) = (4, 0, 1)$.

dot product

$D_{\vec{u}} f(2,1,1) = (4,0,1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}} + \frac{0}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{5}{\sqrt{3}}$.

(b) [10 points] Consider the quadric surface:

$$(x-1)^2 + 3y^2 - z^2 = 1.$$

By looking at the cross sections/traces (along xy, or yz or xz coordinate planes) determine what type it is (i.e. an ellipsoid, hyperboloid, elliptic paraboloid, hyperbolic paraboloid or a cone). Next find the equation of tangent plane to this surface at the point $(1, -1, \sqrt{2})$.

(limit case of hyperbola)

$x=0$ $\quad 3y^2 - z^2 = 0 \Rightarrow (\sqrt{3}y - z)(\sqrt{3}y + z) = 0 \rightsquigarrow$ two lines

$x=1$ $\quad 3y^2 - z^2 = 1 \Rightarrow$ hyperbola

In fact $\quad 3y^2 - z^2 = $ Const. is hyperbola (or in esp. case of $x=0$ two lines).

$y =$ Const. $\quad (x-1)^2 - z^2 = (1-3y^2) \rightsquigarrow$ Const. which is equ. of hyperbolas in xz-plane.

Cross sections in xy & xz planes are hyperbolas.

It is a hyperboloid (in fact, of one sheet).

Equ. of tangent plane to a level surface $f(x,y,z) =$ Const. at (x_0, y_0, z_0) is:

dot product

$$\nabla f(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0, \text{ i.e.:}$$

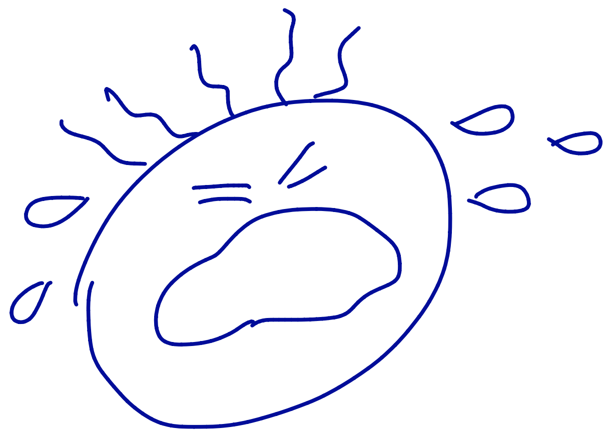
$$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0.$$

$$\nabla f = (2(x-1), 6y, -2z) \quad \nabla f(1,-1,\sqrt{2}) = (0, -6, -2\sqrt{2}).$$

$$(0,-6,-2\sqrt{2}) \cdot (x-1, y+1, z-\sqrt{2}) = -6(y+1) - 2\sqrt{2}(z-\sqrt{2}) = 0.$$

Simplifying we get: $\quad -6y - 2\sqrt{2}z = 6 - 4 = 2.$

6. [1 point] Draw a cartoon showing yourself writing this test!



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