Problem: Which one of the following is a cyclic group? Give a generator for the group if it is cyclic, and if not, argue why (i.e. cannot be generated by one element).

(a) \((\mathbb{Q}\setminus\{0\}, \times)\)

Solution: \((\mathbb{Q}\setminus\{0\}, \times)\) is not cyclic. Let’s prove by contradiction. Suppose it is generated by \(a\), then \(a^n = -1\) for some \(n \in \mathbb{Z}\). This implies \(a = -1\). But \(<-1> = \{-1, 1\} \neq (\mathbb{Q}\setminus\{0\}, \times)\).

(b) Symmetry group of a square

Solution: \(D_4\) is not a cyclic group. \(|D_4| = 8\), but the order of every element is less or equal to 4.

(c) \((\mathbb{Z}^*_10, \times)\).

Solution: \(\mathbb{Z}^*_10 = \{1, 3, 7, 9\}\) is cyclic. 3 is a generator.

Problem: Give an isomorphism between \((\mathbb{Z}^*_8, \times)\) and the Klein group \(V\).

Solution: \(\mathbb{Z}^*_8 = \{1, 3, 5, 7\}\), \(V = \{1, a, b, c\}\). Drawing the multiplication table for \(\mathbb{Z}^*_8\) one checks that \(\phi(1) = 1, \phi(3) = a, \phi(5) = b, \phi(7) = c\) gives an isomorphism.

Problem: Verify that \((\mathbb{Z}^*_9, \times)\) is cyclic and find all its subgroups.

Solution: \(\mathbb{Z}^*_9 = \{1, 2, 4, 5, 7, 8\}\) is cyclic, since \(\mathbb{Z}^*_9 = \langle 2 \rangle\). Subgroups: \(\langle 1 \rangle, \{1, 8\}, \{1, 4, 7\}, \mathbb{Z}^*_9\).

Problem: Find all the elements of order 10 in \((\mathbb{Z}_{30}, +)\).

Solution: We know that the order of \(a\) is equal to \(30/d\) where \(d = (a, 30)\) (Theorem 6.14). Thus \(a\) is of order 10 if and only if \((a, 30) = 3\). Therefore, \(a = 3, 9, 21, 27\).

Problem: Let \(G = A_4\) the alternating group in 4 elements. Let \(\sigma = (134) \in A_4\) and \(H = \langle \sigma \rangle\). Find all the cosets of \(H\) in \(A_4\).
Solution: $\sigma$ has order 3 because it is a 3-cycle and hence $H = \{e, \sigma, \sigma^2\}$. $A_4$ has $4!/2 = 12$ elements. Hence $|A_4 : H| = 12/3 = 4$, i.e. there are 4 cosets. $H$ is always one of the cosets. We need to find 3 other cosets. In fact, one can verify that the four 3-cycles: $\sigma_1 = \sigma$, $\sigma_2 = (123)$, $\sigma_3 = (124)$ and $\sigma_4 = (234)$ give representatives for the four different cosets, that is, for any $i \neq j$ we have $\sigma_i\sigma_j^{-1} \notin H$. Thus the four cosets are: $\sigma_1H$, $\sigma_2H$, $\sigma_3H$ and $\sigma_4H$.

**Problem:** How many non-isomorphic abelian groups of order 180 are there? List all of them.

Solution: $180 = 2^23^25$.

- $\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$,
- $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$,
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

**Problem:** Does the symmetry group of a square (i.e. $D_4$) has a subgroup isomorphic to $V$? how about a subgroup isomorphic to $\mathbb{Z}_4$?

Solution: Consider the group consisting of vertical and horizontal reflections, rotation by $\pi$ and identity. This group is isomorphic to $V$. The group generated by rotation by $\pi/2$ is isomorphic to $\mathbb{Z}_4$.

**Problem:** Recall that $GL(2, \mathbb{R})$ denotes the group of all $2 \times 2$ invertible matrices (with real entries) and matrix multiplication. Is the set of all $2 \times 2$ invertible symmetric matrices (with real entries) a subgroup of $GL(2, \mathbb{R})$? A matrix $A$ is symmetric if it is symmetric with respect to its diagonal, in other words, $A = A^t$.

Solution: No. $(a \ b \ b' \ c')(a' \ b' \ c') = (aa' + bb' \ ab' + bc' \ ba' + cb' \ bb' + cc')$. In general $ab' + bc' \neq ba' + cb'$. So not closed under matrix multiplication.

**Problem:** Show that a group $G$ is abelian if and only if we have

$$(ab)^{-1} = a^{-1}b^{-1} \quad \forall a, b \in G.$$ 

Solution: Suppose $G$ is abelian that is $ab = ba$, then $(ab)^{-1} = (ba)^{-1} = a^{-1}b^{-1}$. Conversely, suppose $(ab)^{-1} = a^{-1}b^{-1}$. We know $(ab)^{-1} = b^{-1}a^{-1}$. So $a^{-1}b^{-1} = b^{-1}a^{-1} \forall a, b \in G$. Replacing $a$ with $a^{-1}$ and $b$ with $b^{-1}$ we conclude that $G$ is abelian.

**Problem:** Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 2 & 3 & 6 & 5 & 1 \end{pmatrix},$$

and $\tau = (21)(23)(45)(57)$. Write $\sigma$ and $\tau$ as product of disjoint cycles. Determine if they are odd or even.
Solution: $\sigma = (17)(243)(56)$. We multiply out $(12)(23)(45)(57)$ to write $\tau$ in the array form:

$$
\tau = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 1 & 5 & 7 & 6 & 4
\end{pmatrix}.
$$

Then $\tau = (123)(457)$. Since $(243)$ is even $\sigma$ is also even. $(123)$ and $(457)$ are even and hence $\tau$ is also even.

Problem: Let $G$ be the dihedral group $D_5$. Elements of $G$ are rotations and reflections. Let $r$ be the counter-clockwise rotation by $2\pi/5$ radian and $s$ be the reflection with respect to the $x$-axis. Recall that we have the relations: $rs = sr^{-1}, s^2 = 1$.

(i) Write the following elements of $G$ in terms of $r$ and $s$: (1) clockwise rotation by $4\pi/5$ radian. (2) Reflection with respect to the line $l$ which passes through the origin and makes an angle of $2\pi/5$ with the $x$-axis.

1) $r^{-2} = r^3$; 2) $r^2s$ (Note that we first apply $s$ then $r^2$.)

(ii) Simplify the following element and decide if it is a rotation or reflection: $r r s r^{-1} s r^{-1} s r^2$.

Using the relations $rs = sr^{-1}, s^2 = 1$ we have $rrsr^{-1}sr^{-1}sr^3sr^2 = r^3s^2r^{-4}s^2r^2 = r$, therefore, it is a rotation.

Problem: Let $\phi : G \to G'$ be an onto homomorphism (surjective). Show that if $G$ is abelian then $G'$ is abelian. Also show that if $G$ is cyclic then $G'$ is also cyclic.

Proof: Suppose $G$ is abelian. We want to show that $G'$ is abelian. Take $x', y' \in G'$. Since $\phi$ is onto we can find $x, y \in G$ with $\phi(x) = x'$, $\phi(y) = y'$. Now, $x'y' = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = y'x'$ as required.

Next let us show that if $G$ is cyclic then $G'$ is cyclic. Let $a$ be a generator for $G$, we show that $a' = \phi(a)$ is a generator for $G'$. Take $x' \in G'$. Then there is $x \in G$ with $\phi(x) = x'$. Since $a$ is a generator for $G$ we have $x = a^k$ for some $k \in \mathbb{Z}$. Then $x' = \phi(x) = \phi(a^k) = (\phi(a))^k = a^k$ which proves that $a'$ is a generator for $G'$.

Problem: Let $\phi : G \to G'$ be an onto homomorphism. Is it true that if $G'$ is cyclic then $G$ is also cyclic?

Solution: No. Example consider the map $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ given by $(x, y) \mapsto x$. One verifies that $\phi$ is a homomorphism. Now $\mathbb{Z}_2$ is cyclic while $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not.

Problem: What is the symmetry group of the following shapes: (1) Yin and Yang Buddhist symbol (ignoring the black-white color). (2) Star of David. (3) Red Cross.

Solution: 1) $\mathbb{Z}_2$, 2) $D_6$, 3) $D_4$. 

3
Problem: Let $G$ be any group and let $a$ be any element of $G$. Let $\phi : \mathbb{Z} \rightarrow G$ be defined by $\phi(n) = a^n$. Show that $\phi$ is a homomorphism.

Solution: We have $\phi(n+m) = a^{n+m} = a^n a^m = \phi(n) \phi(m)$ thus $\phi$ is a homomorphism. The image of $\phi$ is $\{a^n \mid a \in \mathbb{Z}\}$ which is the subgroup $\langle a \rangle$ generated by $a$. If $k = \text{ord}(a)$ then the kernel of $\phi$ is the subgroup $k\mathbb{Z}$. If $a$ has infinite order then $\ker(\phi) = \{0\}$.

Problem: Let $\phi : G \rightarrow G'$ be an onto homomorphism where $G, G'$ are finite groups. Show that $|G'|$ divides $|G|$.

Proof: Let $H$ be the kernel of $\phi$. By a theorem proved in class we know that $\phi(x) = \phi(y)$, for $x, y \in G$, if and only if $xH = yH$. Thus, as $\phi$ is onto, $|G'|$ is equal to the number of cosets of $H$, namely the index of $H$ in $G$. Hence $|G| = |H||G'|$ which proves the claim.

Problem: Find the order of the element $(3, 10)$ in $\mathbb{Z}_5 \times \mathbb{Z}_{18}$.

Solution: The order of 3 in $\mathbb{Z}_5$ is 5 since $(3, 5) = 1$. Also $(10, 18) = 2$ and thus the order of 10 in $\mathbb{Z}_{18}$ is $18/2 = 9$ (Theorem 6.14). Now the smallest number $k$ such that $k3 \equiv 0 \pmod{5}$ and $k10 \equiv 0 \pmod{18}$ is the least common multiple of 5 and 9 which is 45.

Problem: Give an example of each of the following. Justify your example briefly.

(a) A group $G$ with $|G| = 6$ and two subgroups $H$ and $K$ with $|H| = 2$ and $|K| = 3$ such that $G$ is not isomorphic to $H \times K$.

Solution: $G = S_3$, $H = \langle (12) \rangle$ and $K = \langle (123) \rangle$. Then $S_3$ is not isomorphic to $H \times K$ because $H$ and $K$ are cyclic and hence abelian, and thus $H \times K$ is also abelian, while $S_3$ is not abelian.

(b) A subgroup of $\mathbb{Z}_{10} \times \mathbb{Z}_{20}$ isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_2$.

Solution: Let $H < \mathbb{Z}_{10}$ be the subgroup generated by 2, and $K < \mathbb{Z}_{20}$ be the subgroup generated by 10. Then $H \cong \mathbb{Z}_5$, $K \cong \mathbb{Z}_2$ and $H \times K \cong \mathbb{Z}_5 \times \mathbb{Z}_2$.

(c) A subgroup of $S_4$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution: Subgroup generated by $\sigma = (12)$ and $\tau = (34)$. 

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Problem: Let $\phi : Z \rightarrow G$ be defined by $\phi(n) = a^n$. Show that $\phi$ is a homomorphism. Describe the image of $\phi$ and the kernel of $\phi$.

Solution: We have $\phi(n + m) = a^{n+m} = a^n a^m = \phi(n) \phi(m)$ thus $\phi$ is a homomorphism. The image of $\phi$ is $\{a^n \mid a \in \mathbb{Z}\}$ which is the subgroup $\langle a \rangle$ generated by $a$. If $k = \text{ord}(a)$ then the kernel of $\phi$ is the subgroup $k\mathbb{Z}$. If $a$ has infinite order then $\ker(\phi) = \{0\}$.