



Quiz tomorrow → triple integrals
(general region + cylindrical
→ spherical)

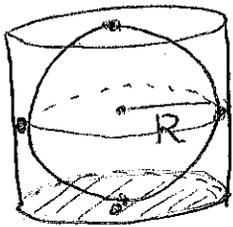
 → vol. of sphere $\frac{4\pi}{3} R^3$ calculated using SSS in sph. coord.

Coming soon! Calculating line & surface integral.

We will see surface area of sphere is $4\pi R^2$

(interestingly $(\frac{4\pi}{3} R^3)' = 4\pi R^2$, or $\frac{4\pi}{3} R^3 = \int 4\pi R^2 dR$)
w.r.t. R

in general the philosophy is $\int \text{area} = \text{volume}$.



Vol. of cylinder = $2R \cdot \pi R^2 = 2\pi R^3$

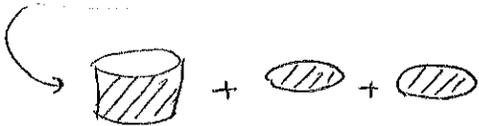
Vol. of sphere = $\frac{4\pi}{3} R^3$

~~Ratio~~ $\frac{\text{vol. of sphere}}{\text{Vol. of cylinder}} = \boxed{\frac{2}{3}}$

Surface area of sphere

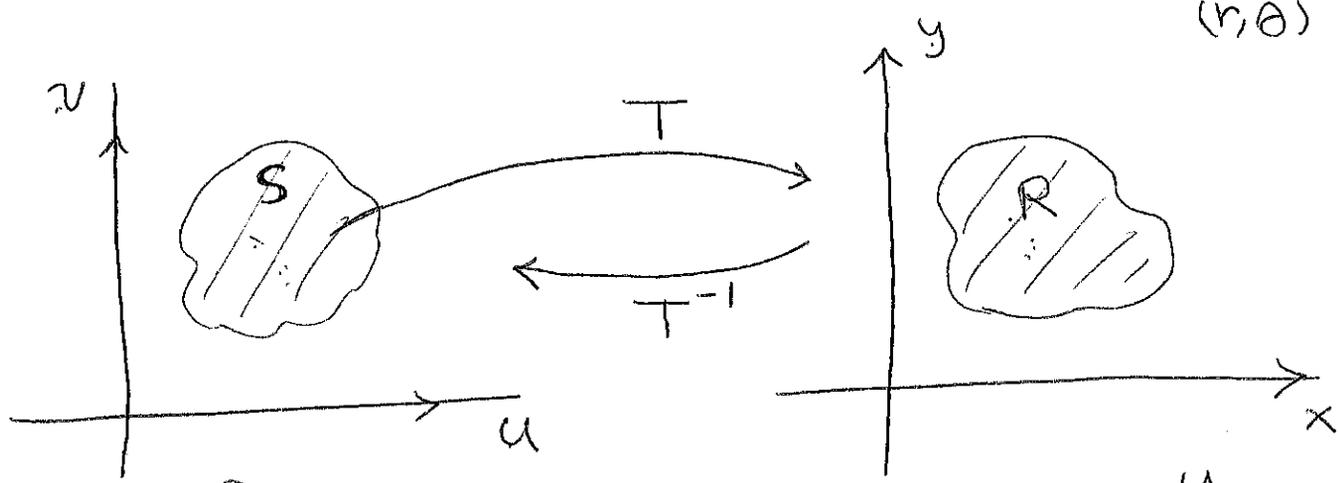
Surface area of cylinder

$\frac{4\pi R^2}{2R(2\pi R) + 2\pi R^2} = \frac{4\pi R^2}{6\pi R^2} = \boxed{\frac{2}{3}}$



12.8 Change of variable for integrals (multiple)

Double integral (example is xy-coor. to polar coor. (r, θ))



$$\iint_S \text{?} \, du \, dv \quad \leftarrow \dots \quad \iint_R f(x,y) \frac{dA}{dx \, dy}$$

~~Ex.~~ $x = g(u,v) \quad y = h(u,v)$
 $(u,v) \xrightarrow{T} \left(\frac{g(u,v)}{x}, \frac{h(u,v)}{y} \right)$

Ex. $x = r \cos \theta \quad y = r \sin \theta \quad (r, \theta) \text{ new Coor.}$

Ex. $x = u+v \quad y = u-v \quad u, v \text{ new Coor.}$

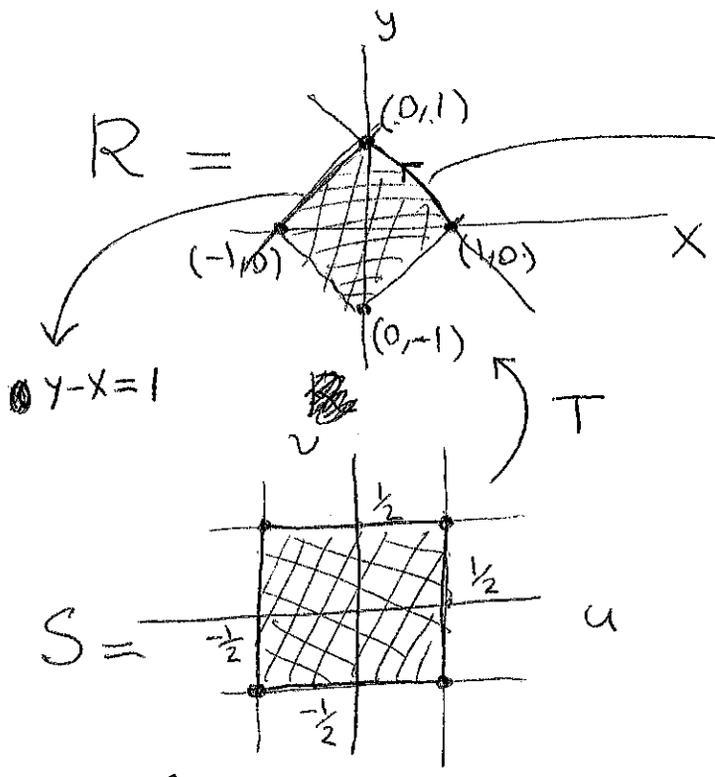
Solve for u & v :

$$\left. \begin{aligned} (u+v) + (u-v) &= x+y \\ 2u &= x+y \\ u &= \frac{x+y}{2} \end{aligned} \right\} (x,y) \xrightarrow{T^{-1}} \left(\frac{x+y}{2}, \frac{x-y}{2} \right)$$

Ex. (Cont'd).

$$\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}$$

$$\begin{aligned} u &= \frac{x+y}{2} \\ v &= \frac{x-y}{2} \end{aligned}$$



$$-x+1=y \quad (\text{or } x+y=1)$$

$$-(u+v)+1=u-v$$

$$1=2u \Rightarrow u=\frac{1}{2}$$

$$y-x=1$$

$$(u-v)-(u+v)=1$$

$$-2v=1 \Rightarrow v=-\frac{1}{2}$$

Thm (Change of var. formula) : in u, v

$$\underbrace{\iint_R f(x,y) dx dy}_{\text{given}} = \iint_S f(x(u,v), y(u,v)) \overbrace{J(u,v)}^{\text{Jacobian}} du dv$$

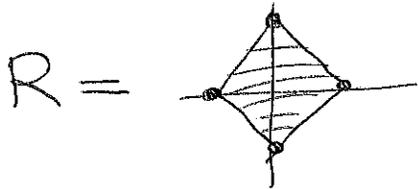
$$J(u,v) = \text{Jacobian of } T = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \boxed{\begin{matrix} \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \\ \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \end{matrix}}$$

Ex. (Cont'd)

$$f = x^2 y$$

(4)

$$\iint_R x^2 y \, dx dy = \iint_S (u+v)^2 (u-v) (-2) \, du dv$$



$$S = \left\{ \begin{array}{l} -\frac{1}{2} \leq u \leq \frac{1}{2} \\ -\frac{1}{2} \leq v \leq \frac{1}{2} \end{array} \right\}$$

$$J(u,v) = \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2$$

Rem If ~~Jacobian~~ Jacobian < 0 it means the transformation T reverse the "orientation".

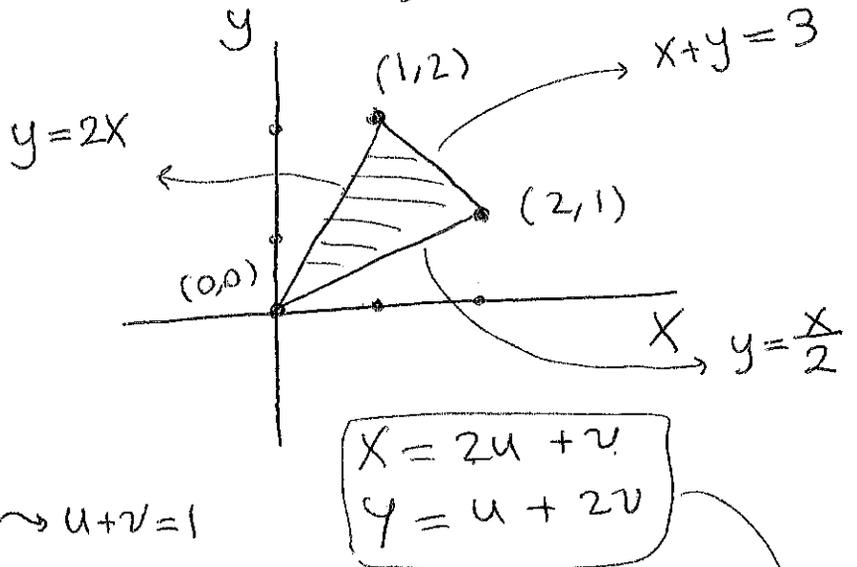
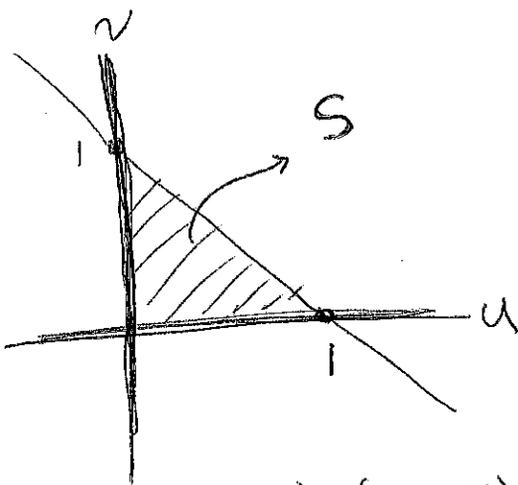
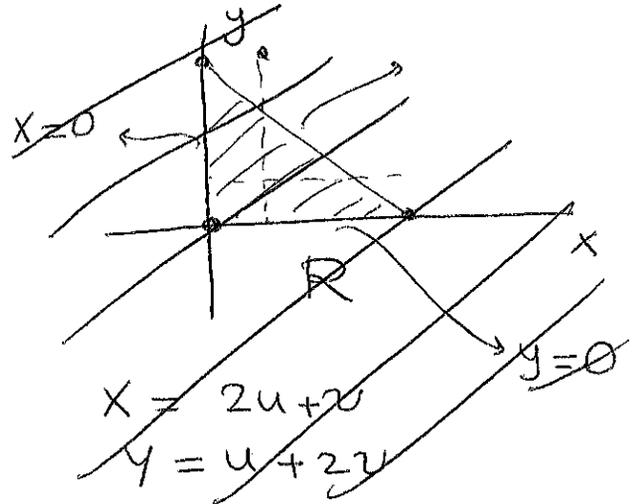
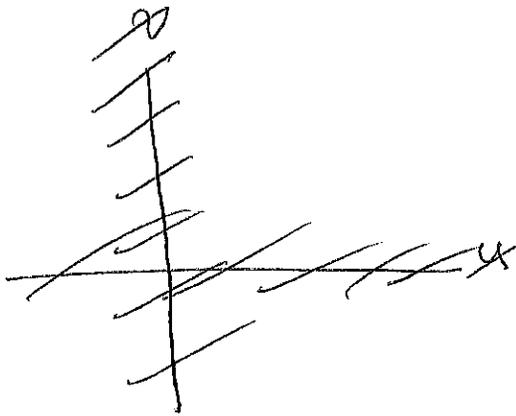
- Most of the time we deal with transformations which do not change the orientation.



In 1-variable

$$\int_a^b f(x) dx \rightarrow \int_b^a f(x) dx = -\int_a^b f(x) dx$$

Ex. R triangular region with vertices (0,0), (2,1), (1,2).



$$(2u+v) + (u+2v) = 3 \Rightarrow u+v=1$$

$$(u+2v) = 2(2u+v)$$

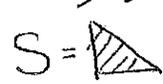
$$0 = 3u \Rightarrow u=0$$

$$(u+2v) = \frac{1}{2}(2u+v)$$

$$\rightarrow 2u+4v = 2u+v \Rightarrow v=0$$

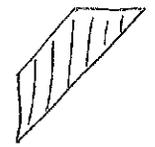
$$J = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 4 - 1 = 3$$

$$\iint_R 1 \, dx \, dy = \iint_S 1 \cdot 3 \, du \, dv$$



Ex. $\iint_R e^{x+y/x-y} dx dy$

R trapezoidal region vertices (1,0), (2,0), (0,-2), (0,-1).



$$\left. \begin{matrix} u = x+y \\ v = x-y \end{matrix} \right\} \xrightarrow{\text{Solve for } x \text{ and } y} \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$$

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

$$\iint_S e^{u/v} \left(\frac{-1}{2}\right) du dv.$$

find!
in terms of u,v