



(1)

13.6 parametric surfaces. S

- Area of surface (as an integral) $\iint_S 1 \, dS$
- Integrating a scalar function on a surface $\iint_S f \, dS$
- Integrating a vec. field \vec{F} on a surface.
("flux" of a vec. field on S) $\iint_S \vec{F} \cdot d\vec{n}$.

S Surface u, v parameters (u, v) in D
domain in \mathbb{R}^2

$$r(u, v) = (x(u, v), y(u, v), z(u, v)) \text{ in } S.$$

$f(x, y, z)$ scalar function

$$\iint_S f \, dS \stackrel{\text{def.}}{=} \iint_D f(x(u, v), y(u, v), z(u, v)) |r_u \times r_v| \, du \, dv$$

Recall: $r_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$ $r_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$

$$|r_u \times r_v| = \text{area of } r_v$$

$$dudv \quad (x, y, z) \quad \text{area} = |r_u \times r_v| dudv.$$

(2)

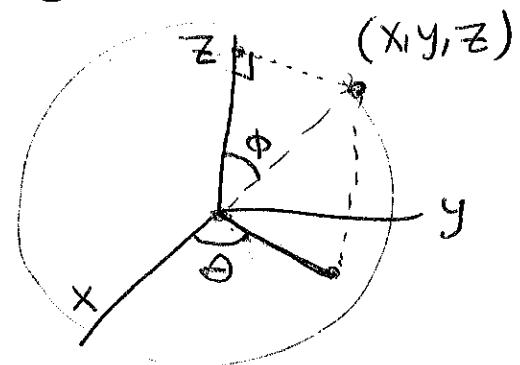
Ex. $x^2 + y^2 + z^2 = a^2$ $a = \text{radius}$

para. using sph. coor.

$$(\rho, \underbrace{\phi, \theta}_{\text{para.}})$$

$$\rho = a \quad \text{cont.}$$

$$\begin{cases} x = a \sin \phi \cos \theta \\ y = a \sin \phi \sin \theta \\ z = a \cos \phi \end{cases}$$

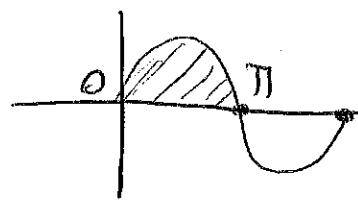


$$r_\theta = (-a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0)$$

$$r_\phi = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi)$$

$$r_\phi \times r_\theta = \begin{bmatrix} i & j & k \end{bmatrix}$$

$$= \begin{bmatrix} a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{bmatrix}$$



$$|r_\phi \times r_\theta| = a^2 \sin \phi$$

Area of sphere :

$$\iint_S 1 \, dS \stackrel{\text{def.}}{=} \iint_D a^2 \sin \phi \, d\phi d\theta$$

$$0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi$$

$$2\pi a^2 \int_0^\pi \sin \phi \, d\phi$$

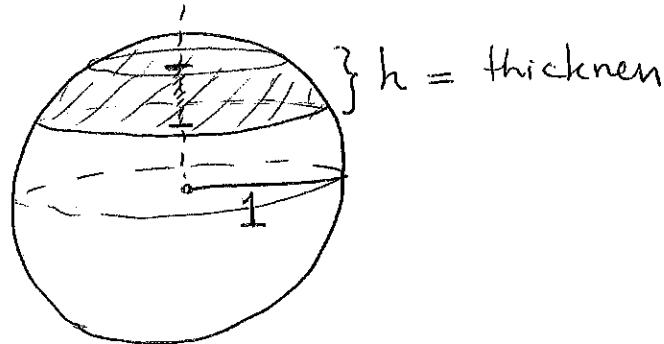
$$= 4\pi a^2 \quad \text{smiley face}$$

A problem of Archimedes

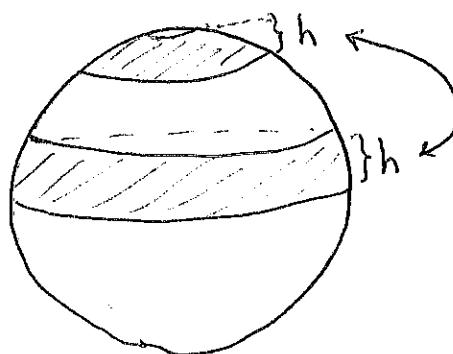
(3)

Exercise Use ~~the~~ formula for surface integral.

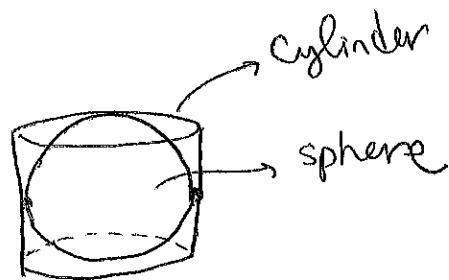
to ~~compute~~ Compute area of a stripe on sphere:



Archimedes: area of thin stripe only depends on h



Same area!



$$\frac{2}{3} \frac{\text{Vol/Cyl}}{\text{Area of Sphere}} = \frac{\text{Vol/Cyl}}{\text{Area of Sphere}}$$

Ex. Area of graph of a function $f(x,y) = z$.

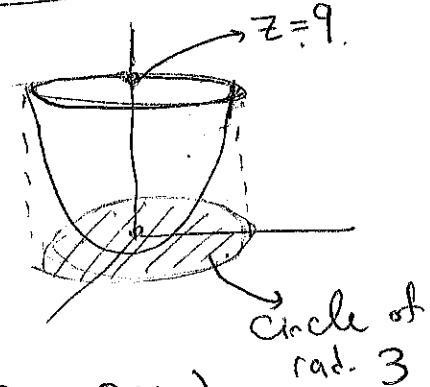
$$(u,v) \mapsto (u, v, f(u,v))$$

$$z = x^2 + y^2 \quad \text{paraboloid} \quad u^2 + v^2$$

$$\frac{z^2}{3} = 9$$

(u,v) inside circle of rad. 3

$$\begin{aligned} r_u &= (1 \ 0 \ 2u) \\ r_v &= (0 \ 1 \ 2v) \end{aligned}$$



(4)

$$r_1 \times r_2 = \det \begin{bmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{bmatrix}$$

$$|r_1 \times r_2| = \sqrt{(2u)^2 + (2v)^2 + 1}$$

(in general:

$$|r_1 \times r_2| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$\text{Area} = \iint_D \sqrt{4u^2 + 4v^2 + 1} \, du \, dv$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^3 \sqrt{4r^2 + 1} \, r \, dr \, d\theta$$

$$2\pi \left(\int_0^3 \sqrt{s} \, \frac{ds}{2} \right) = \dots$$

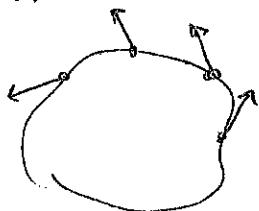
Can be easily done with

$$s = r^2 + 1$$

$$\frac{ds}{2} = r \, dr$$

Surface int. of vec. fields.

~~C~~ C curve in \mathbb{R}^2 & F vec. field in \mathbb{R}^2



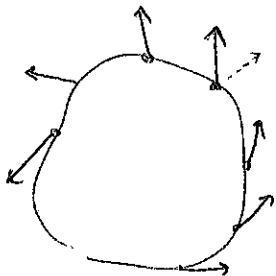
$$\int_C \vec{F} \cdot d\vec{r}$$

~~line int. of~~ line int. of \vec{F} along C
(work)

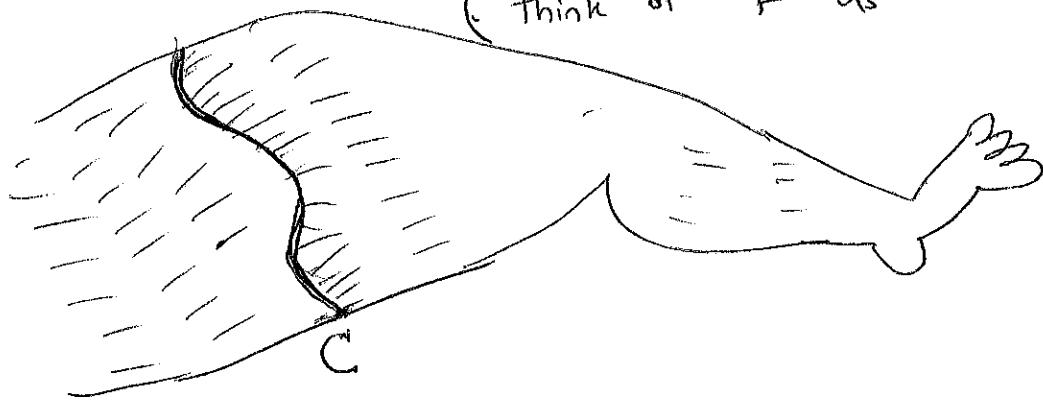
$$\int \vec{F} \cdot d\vec{n}$$

~~int. of~~ int. of \vec{F} in the normal direction to the curve C.
"flux" of \vec{F} on C.

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Flux of \vec{F} on C
measures how much \vec{F}
is going through the curve C
(think of \vec{F} as flow of a fluid)



In curve case :
 (x, y)

$$\vec{n} = \left(\frac{y'}{|r'(t)|}, \frac{-x'}{|r'(t)|} \right) \text{ unit normal}$$

$$\vec{\tau} = \left(\frac{x'}{|r'(t)|}, \frac{y'}{|r'(t)|} \right).$$

$$\text{Flux} = \int_C \vec{F} \cdot \vec{n} dS \stackrel{\text{def.}}{=} \int_a^b \vec{F} \cdot \vec{n} |r'(t)| dt.$$

Surface Case :

$$\iint_S \vec{F} \cdot d\vec{S} \stackrel{\text{def.}}{=} \underbrace{\iint_D}_{\text{Flux}} \vec{F} \cdot (\vec{r}_1 \times \vec{r}_2) dudv$$

$$\iint_D \vec{F} \cdot \underbrace{(\vec{r}_1 \times \vec{r}_2)}_{\text{vec.}} dudv$$