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13.5 curl & divergence

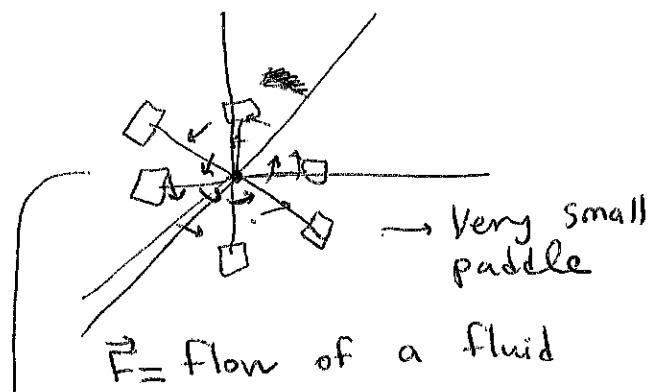
\vec{F} vec. field in 3D (or \mathbb{R}^3)

$$\vec{F}(x, y, z) = P\vec{i} + Q\vec{j} + R\vec{k} = (P, Q, R)$$

\vec{F} \rightarrow curl \vec{F} vec. field in \mathbb{R}^3 .
 \vec{F} \rightarrow div \vec{F} scalar function \mathbb{R}^3

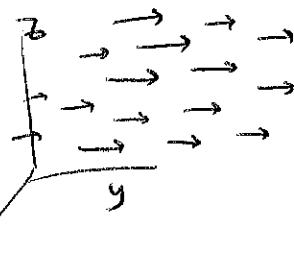
curl $\vec{F}_{(x, y, z)}$ (roughly speaking) how much vec. field \vec{F} curls around x, y, z axes.

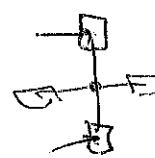
(or the direction where \vec{F} has max. curl).



Pic. P. 822. (Sec. 13.8)

→ curl \vec{F} measures how fast the paddle rotates & in which direction has max. rotation ...

Ex.  $\vec{F} = (0, 1, 0)$ const. vect.
 $\text{curl } \vec{F} = 0$



• Important in physics (electromagnetism ...) ②

\vec{F} : flow of a fluid
(velocity)

$\text{div}(\vec{F}) =$ How much fluid is ^{Created} born/dies at given point (x, y, z) .



has positive $\text{div}(\vec{F})$

$$\text{Ex. } \vec{F} = (0, 1, 0)$$

Cont. vec. field



has negative $\text{div}(\vec{F})$

$$\text{div}(\vec{F}) \leq 0$$



Def. $\text{div}(\vec{F})(x, y, z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

$$\vec{F} = (P, Q, R)$$

$$\begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{bmatrix}$$

div.
sum
of
diagonal.

$$\text{Curl}(\vec{F})(x, y, z) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

Rem Suppose $\vec{F}(x, y) = (P(x, y), Q(x, y), R)$

$$\text{curl } \vec{F} = (0 \ i + 0 \ j + \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{no } z} \ k)$$

exactly what
appeared in
Green's thm.

Thm. ~~Assume~~ F defined on a domain D in \mathbb{R}^3 .
(Suppose par. der.
of f are contin.)

Suppose $F = \nabla f$. Then:

$$\text{curl}(F) = \text{curl}(\nabla f) = 0.$$

Proof $P = \frac{\partial f}{\partial x}$ $Q = \frac{\partial f}{\partial y}$ $R = \frac{\partial f}{\partial z}$

$$\left(\cancel{\frac{\partial f}{\partial y \partial z}} - \cancel{\frac{\partial f}{\partial z \partial y}} \right), \quad \left(\cancel{\frac{\partial f}{\partial x \partial z}} - \cancel{\frac{\partial f}{\partial z \partial x}} \right), \quad \left(\cancel{\frac{\partial f}{\partial x \partial y}} - \cancel{\frac{\partial f}{\partial y \partial x}} \right).$$

Another way to see this:

$$\text{curl}(\nabla f) = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = 0 \text{ because}$$

$$\det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} = 0$$

Same rows

Thm. F is defined on a domain D which is
simply connected (no holes in D !)

Thm 2. Curl F = 0 then $F = \nabla f$ for some scalar function f .

(simply connected domain \rightsquigarrow more precise definition started "topology").

Similar identity for div :

Thm. F has second par. der. on some domain D .

$$\text{div}(\text{curl } \vec{F}) = 0.$$

proof. Easy calculation using def. of $\text{curl } \vec{F}$.

$$\begin{aligned}\text{div}(\vec{F}) &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (\overset{\circ}{P}, Q, R) \\ &\quad \text{"formal" vector} \quad \nabla\end{aligned}$$

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

(For any two vec. A & B $\underbrace{A \cdot (A \times B) = 0}_{\substack{\text{volume of a} \\ \text{flat box}}}$)

$$\text{curl}(\nabla) = 0$$

$$\text{div}(\text{curl}) = 0$$

$$\nabla \leftrightarrow \text{curl } \vec{B} \leftrightarrow \text{div } \vec{B}.$$

$$\begin{array}{c} f \xrightarrow{\text{scalar}} \nabla f \xrightarrow{\text{vec.}} \\ \vec{F} \xrightarrow{\text{vec.}} \text{curl } \vec{F} \xrightarrow{\text{vec.}} \\ \vec{F} \xrightarrow{\text{vec.}} \text{div}(\vec{F}) \xrightarrow{\text{Scalar}} \end{array}$$

(S)

Two terminologies : \vec{F} vec. field

$\operatorname{curl} \vec{F} = 0 \rightsquigarrow$ irrotational

$\operatorname{div}(\vec{F}) = 0 \rightsquigarrow$ incompressible.

(no flow is
created or disappeared)

One more terminology :

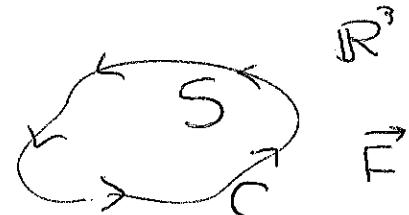
$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$\xrightarrow{\text{Laplacian of } f}$

$$\text{Laplace} \quad \frac{\partial f}{\partial x \partial x} + \frac{\partial f}{\partial y \partial y} + \frac{\partial f}{\partial z \partial z}$$

French mathematician

~~Stokes~~ Stokes thm.



$$\int_C \vec{F} \cdot d\vec{r} = \boxed{\iint_S \operatorname{curl} \vec{F}}$$

\rightarrow we will
make sense
hex + time.