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Exam 2

Fall 2011 Math 0240

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

1. The goal of this problem is to evaluate the integral $\iiint_E z^2 dV$, where E is the portion of the upper half of the sphere $x^2 + y^2 + z^2 = R^2$ that is between the cones $x^2 + y^2 = 3z^2$ and $x^2 + y^2 = \frac{1}{3}z^2$.

(a) [10 points] Find the equation, in spherical coordinates of the cone $x^2 + y^2 = az^2$.

$$x = \rho \cos\theta \sin\varphi, \quad y = \rho \sin\theta \sin\varphi, \quad z = \rho \cos\varphi$$

$$x^2 + y^2 = \rho^2 \sin^2\varphi = a\rho^2 \cos^2\varphi = az^2$$

Assuming $\rho > 0$, $0 < \varphi < \frac{\pi}{2}$ we

get $\sin^2\varphi = a \cos^2\varphi$, $a = \tan^2\varphi$,

$$\boxed{\tan\varphi = \sqrt{a}}$$

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(b) [10 points] Use spherical coordinates to evaluate the integral above.

To find the bounds for φ we solve equations $\tan \varphi = \sqrt{a}$ for φ when $a = \frac{1}{3}$ and $a = 3$

$$\tan \varphi = \frac{1}{\sqrt{3}} \Rightarrow \varphi = \frac{\pi}{6}$$

$$\tan \varphi = \sqrt{3} \Rightarrow \varphi = \frac{\pi}{3}$$

Hence $E = \{(r, \theta, \varphi) \mid 0 \leq r \leq R, 0 \leq \theta < 2\pi, \frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}\}$

$$\text{Then } \iiint_E z^2 dV = \int_0^{2\pi} \int_0^R \int_{\pi/6}^{\pi/3} r^2 \cos^2 \varphi \cdot r^2 \sin \varphi d\varphi dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^R r^4 dr \int_{\pi/6}^{\pi/3} (-\cos^2 \varphi) d(\cos \varphi) = 2\pi \cdot \frac{R^5}{5} \cdot \left(-\frac{1}{3}\right) \cos^3 \varphi \Big|_{\pi/6}^{\pi/3}$$

$$= \frac{2}{5} \pi R^5 \left(-\frac{1}{3}\right) \left(\frac{1}{8} - \frac{3\sqrt{3}}{8}\right) = \frac{2}{15} \pi R^5 \frac{3\sqrt{3}-1}{8}$$

$$= \boxed{\frac{3\sqrt{3}-1}{60} \pi R^5}$$

See also example 7, page 739.

2. [10 points] Find the work done by the force field $\bar{F}(x, y) = 3x^2\bar{i} - 2xy\bar{j}$ in moving a particle along the upper half of the circle $\bar{r}(t) = 2 \cos t\bar{i} + 2 \sin t\bar{j}$, $0 \leq t \leq \pi$.

$$x = 2 \cos t, y = 2 \sin t,$$

$$\bar{F}(\bar{r}(t)) = 12 \cos^2 t \bar{i} - 8 \cos t \sin t \bar{j}$$

$$\bar{r}'(t) = -2 \sin t \bar{i} + 2 \cos t \bar{j}$$

$$\begin{aligned}\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) &= -24 \cos^2 t \sin t - 16 \cos^2 t \sin t = \\ &= -40 \cos^2 t \sin t\end{aligned}$$

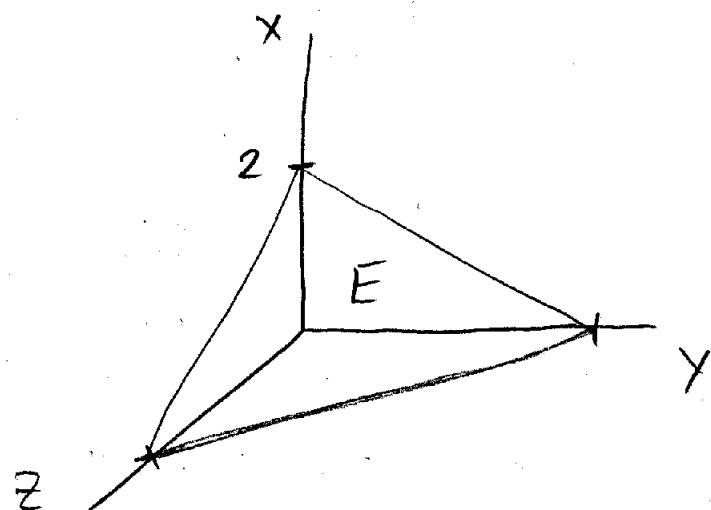
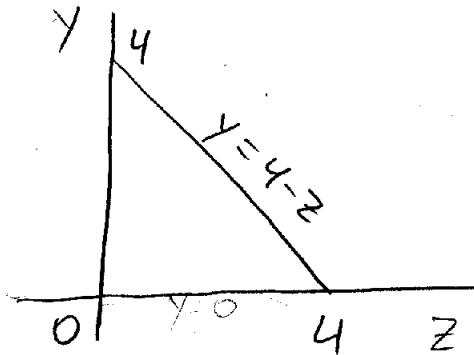
Therefore, the work done is

$$\begin{aligned}\int_C \bar{F} \cdot d\bar{r} &= \int_0^\pi \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) dt = \\ &= \int_0^\pi (-40 \cos^2 t \sin t) dt = 40 \int_0^\pi \cos^2 t d(\cos t) \\ &= 40 \left[\frac{\cos^3 t}{3} \right]_0^\pi = 40 \left(-\frac{1}{3} - \frac{1}{3} \right) = \boxed{-\frac{80}{3}}\end{aligned}$$

3. Evaluate the integral $\iiint_E z \, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 4$.

(a) [10 points] Define the region E in rectangular coordinates and sketch it.

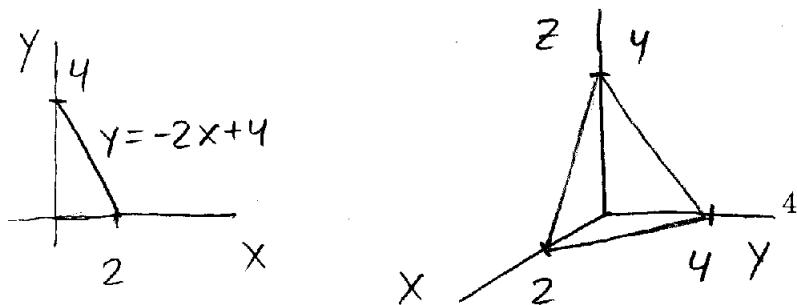
$$\text{In } yz\text{-plane} \quad x=0 \Rightarrow y+z=4 \\ y=4-z$$



$$E = \left\{ (x, y, z) \mid 0 \leq z \leq 4, 0 \leq y \leq 4-z, 0 \leq x \leq \frac{4-y-z}{2} \right\}$$

Another way:

$$E = \left\{ (x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq -2x+4, 0 \leq z \leq -2x-y+4 \right\}$$



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(b) [10 points] Represent the triple integral as an iterated integral and evaluate it.

$$\iiint_E z \, dV = \int_0^4 \int_0^{4-z} \int_0^{\frac{4-y-z}{2}} z \, dx \, dy \, dz =$$

$$= \int_0^4 z \int_0^{4-z} \frac{1}{2}(4-y-z) \, dy \, dz = \frac{1}{2} \int_0^4 z \int_0^{4-z} (4-y-z) \, dy \, dz$$

$$= \frac{1}{2} \int_0^4 \left[-\frac{1}{2} (4-y-z)^2 \right]_0^{4-z} \, dz = -\frac{1}{4} \int_0^4 \left[0 - (4-z)^2 \right] \, dz$$

$$= \frac{1}{4} \int_0^4 (4-z)^2 \, dz = \frac{1}{4} \left[-\frac{1}{3} (4-z)^3 \right]_0^4 = -\frac{1}{4 \cdot 3} [0 - 4^3] =$$

$$= \frac{4^3}{4 \cdot 3} = \frac{4^2}{3} = \boxed{\frac{16}{3}}$$

Another way: $\iiint_E z \, dV = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} z \, dz \, dy \, dx =$

$$= \frac{1}{2} \int_0^2 \int_0^{4-2x} (4-2x-y)^2 \, dy \, dx = \frac{1}{2} \int_0^2 \left[-\frac{1}{3} (4-2x-y)^3 \right]_0^{4-2x} \, dx$$

$$= -\frac{1}{6} \int_0^2 [0 - (4-2x)^3] \, dx = \frac{1}{6} \int_0^2 (4-2x)^3 \, dx =$$

$$= \frac{1}{6} \left[\frac{1}{4} (4-2x)^4 \left(-\frac{1}{2} \right) \right]_0^2 = -\frac{1}{6 \cdot 4 \cdot 2} [0 - 4^4] = \frac{4^4}{3 \cdot 4 \cdot 4} = \frac{16}{3}$$

4. Using Lagrange multipliers find the maximum and minimum values of the function $f(x, y) = 3x^2 + y^2$ on the circle $x^2 + y^2 = 1$.

(a) [10 points] Write the Lagrange equations.

$$6x = 2\lambda x$$

$$3x = \lambda x \quad (1)$$

$$2y = 2\lambda y$$

or

$$y(\lambda - 1) = 0 \quad (2)$$

$$x^2 + y^2 = 1$$

$$x^2 + y^2 = 1 \quad (3)$$

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(b) [10 points] Find points where the function has possible extreme values.

$$(2) \Rightarrow y = 0 \text{ or } \lambda = 1$$

Case $\lambda = 1$: (1) $\Rightarrow 3x = x \Rightarrow x = 0$

$$(3) \Rightarrow y^2 = 1, y = \pm 1, \text{ Points: } (0, \pm 1)$$

Case $y = 0$: (3) $\Rightarrow x^2 = 1, x = \pm 1$

Points : $(\pm 1, 0)$

Answer: 4 points $(0, \pm 1), (\pm 1, 0)$

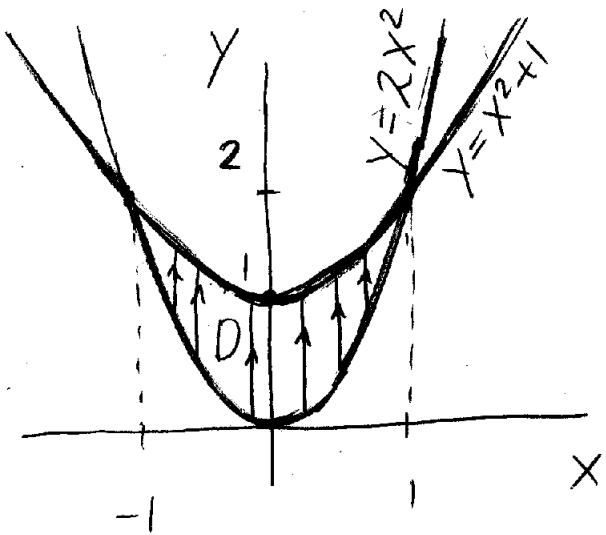
(c) [10 points] Find extreme values.

$$f(0, \pm 1) = 1 \quad \text{minimum value}$$

$$f(\pm 1, 0) = 3 \quad \text{maximum value}$$

5. The goal of this problem is to evaluate the integral $\iint_D (3x + y) dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = x^2 + 1$.

(a) [10 points] Define the region D and sketch it. Describe how you filled it in.



$$2x^2 = x^2 + 1$$

$$x^2 = 1$$

$$x = \pm 1$$

two parabolas intersect at the points $(-1, 2)$ and $(1, 2)$

D is filled in by using vertical lines between $y = 2x^2$ and $y = x^2 + 1$

$$D = \{(x, y) \mid -1 \leq x \leq 1, 2x^2 \leq y \leq x^2 + 1\}$$

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(b) [10 points] Evaluate the given integral as an iterated integral.

$$\iint_D (3x+y) dA = \int_{-1}^1 \int_{2x^2}^{x^2+1} (3x+y) dy dx$$

$$\begin{aligned}
 &= \int_{-1}^1 \left[3xy + \frac{y^2}{2} \right]_{2x^2}^{x^2+1} dx = \int_{-1}^1 [3x(x^2+1-2x^2) \\
 &\quad + \frac{1}{2}((x^2+1)^2 - (2x^2)^2)] dx = \int_{-1}^1 [3x^3 + 3x - 6x^3 \\
 &\quad + \frac{1}{2}(x^4 + 2x^2 + 1 - 4x^4)] dx = \int_{-1}^1 [-\frac{3}{2}x^4 - 3x^3 + x^2 \\
 &\quad + 3x + \frac{1}{2}] dx = \left[-\frac{3}{10}x^5 - \frac{3}{4}x^4 + \frac{1}{3}x^3 + \frac{3}{2}x^2 \right. \\
 &\quad \left. + 3x + \frac{1}{2} \right]_{-1}^1 = \left(-\frac{3}{10} - \frac{3}{4} + \frac{1}{3} + \frac{3}{2} + \frac{1}{2} \right) - \left(\frac{3}{10} - \frac{3}{4} \right. \\
 &\quad \left. - \frac{1}{3} + \frac{3}{2} - \frac{1}{2} \right) = -\frac{6}{10} + \frac{2}{3} + 1 = \frac{5}{3} - \frac{3}{5} = \boxed{\frac{16}{15}}
 \end{aligned}$$

bonus problem [7 points extra] Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw \quad (u, v, w > 0)$$

$$\text{Surface: } u+v+w=1, \quad w=0 \Rightarrow u+v=1 \\ v=1-u$$

$$E = \{(u, v, w) \mid 0 \leq u \leq 1, 0 \leq v \leq 1-u, 0 \leq w \leq 1-u-v\}$$

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw dw dv du = 4 \int_0^1 u \int_0^{1-u} v(v+u-1)^2 dv du$$

$$= 4 \int_0^1 u \int_0^{1-u} (v^3 + 2(u-1)v^2 + (u-1)^2 v) dv du =$$

$$= 4 \int_0^1 u \left[\frac{1}{4} v^4 + \frac{2}{3} (u-1) v^3 + \frac{(u-1)^2}{2} v^2 \right]_0^{1-u} du$$

$$= 4 \int_0^1 u \left[\frac{(u-1)^4}{4} - \frac{2}{3} (u-1)^4 + \frac{1}{2} (u-1)^4 \right] du =$$

$$= 4 \int_0^1 u \cdot \frac{(u-1)^4}{12} du = \frac{1}{3} \int_0^1 (u-1+1)(u-1)^4 du =$$

$$= \frac{1}{3} \int_0^1 [(u-1)^5 + (u-1)^4] du = \frac{1}{3} \left[\frac{(u-1)^6}{6} + \frac{(u-1)^5}{5} \right]_0^1 \\ = \frac{1}{3} \left[0 - \frac{1}{6} + 0 - \left(-\frac{1}{5} \right) \right] = \frac{1}{3} \cdot \frac{1}{30} = \frac{1}{90}$$