

10-10:50am

Exam 2

Fall 2011 Math 0240

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

1. [10 points] Find the gradient vector field of $f(x, y, z) = \frac{y^2 \tan(x^3)}{z}$

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

$$f_x = \frac{y^2}{z} \cdot 3x^2 \sec(x^3)$$

$$f_y = 2y \frac{\tan(x^3)}{z}$$

$$f_z = -\frac{y^2 \tan(x^3)}{z^2}$$

$$\nabla f(x, y, z) = \left\langle \frac{3x^2y^2}{z} \sec(x^3), \frac{2y}{z} \tan(x^3), -\frac{y^2}{z^2} \tan(x^3) \right\rangle$$

2. The goal of this problem is to evaluate the integral $\iint_R \sin\left(\frac{x+2y}{x-2y}\right) dA$ where R is the triangle with vertices $(0, 0)$, $(2, -1)$, and $(4, 0)$.

a) [10 points] Consider the transformation $u = x - 2y$ and $v = x + 2y$. Solve these equations for x and y .

$$u+v=2x \Rightarrow x = \frac{u+v}{2}$$

$$y = \frac{v-x}{2}, \quad v-x = \frac{v-u}{2}, \quad y = \frac{v-u}{4}$$

b) [10 points] Find the Jacobian of the transformation.

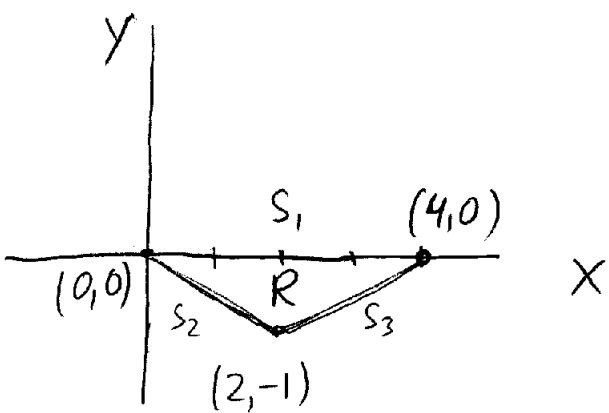
$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}, \quad x_u = x_v = \frac{1}{2}, \\ y_u = -\frac{1}{4}, \quad y_v = \frac{1}{4}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

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c) [10 points] Find the new region of integration (in the uv -plane) and sketch it.

$$T^{-1} : u = x - 2y, v = x + 2y$$



$$S_1 : y=0, 0 \leq x \leq 4$$

$$T^{-1}(S_1) : u=v=x$$

$$0 \leq u \leq 4, 0 \leq v \leq 4$$

$$S_2 : y = -\frac{x}{2}, 0 \leq x \leq 2$$

$$T^{-1}(S_2) : u = x + x = 2x \Rightarrow 0 \leq u \leq 4$$

$$v = 0$$

$$S_3 : y = \frac{x}{2} - 2, 2 \leq x \leq 4,$$

$$T^{-1}(S_3) : u = 4,$$

$$v = 2x - 4 \Rightarrow 0 \leq v \leq 4$$

Another way: Since the transformation is linear
it is enough to find

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images of vertices of

the triangle in xy -plane: $(0,0) \rightarrow (0,0)$,

3

$(2,-1) \rightarrow (4,0)$,

$(4,0) \rightarrow (4,4)$

d) [10 points] Evaluate the integral.

$$\iint_R \sin\left(\frac{x+2y}{x-2y}\right) dA = \iint_{R_{uv}} \sin\left(\frac{v}{u}\right) \cdot \frac{1}{4} dA =$$

$$R_{uv} = \{(u, v) \mid 0 \leq u \leq 4, 0 \leq v \leq u\}$$

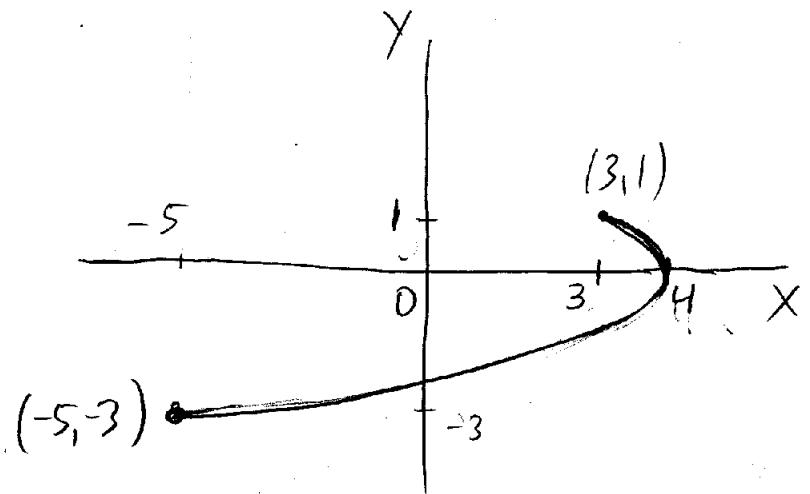
$$= \frac{1}{4} \int_0^4 \int_0^u \sin\left(\frac{v}{u}\right) dv du = \frac{1}{4} \int_0^4 -u \cos\left(\frac{v}{u}\right) \Big|_0^u du =$$

$$= \frac{1}{4} \int_0^4 -u(\cos 1 - 1) du = -\frac{\cos 1 - 1}{4} \int_0^4 u du =$$

$$\frac{1-\cos 1}{4} \cdot \frac{u^2}{2} \Big|_0^4 = 2(1-\cos 1)$$

see example 4, page 735

3. [10 points] Evaluate the integral $\int_C y^2 dx + 3x dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.



Set $y = t$, then $x = 4 - t^2$, $-3 \leq t \leq 1$.

$$dy = dt, \quad dx = -2t dt, \quad y^2 dx = t^2(-2t dt) = -2t^3 dt$$

$$3x dy = 3(4 - t^2) dt$$

$$\int_C y^2 dx + 3x dy = \int_{-3}^1 [-2t^3 + 3(4 - t^2)] dt =$$

$$= \int_{-3}^1 (-2t^3 - 3t^2 + 12) dt = \left[-\frac{1}{2}t^4 - t^3 + 12t \right]_{-3}^1 =$$

$$= \left[-\frac{1}{2} - 1 + 12 \right] - \left[-\frac{81}{2} + 27 - 36 \right] = \boxed{60}$$

4. Using Lagrange multipliers find the maximum and minimum values of the function $f(x, y) = 3xy^2$ subject to the constraint $2x^2 + y^2 = 6$.

a) [10 points] Write the Lagrange equations.

$$3y^2 = 4\lambda x \quad (1)$$

$$6xy = 2\lambda y \quad (2)$$

$$2x^2 + y^2 = 6 \quad (3)$$

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b) [10 points] Find points where the function has possible extreme values.

Assume $y \neq 0$. (2) $\Rightarrow 3x = \lambda$,

$$(1) \Rightarrow 3y^2 = 12x^2, y^2 = 4x^2 = (2x)^2, y = \pm 2x$$

$$(3) \Rightarrow 2x^2 + 4x^2 = 6 \Rightarrow x = \pm 1, y = \pm 2$$

If $y = 0$, then (1) \Rightarrow either $\lambda = 0$ or $x = 0$
 $x = 0$: (3) $\Rightarrow 0 = 6$, a contradiction $\Rightarrow x \neq 0$

$$\lambda = 0: (3) \Rightarrow x^2 = 3, x = \pm \sqrt{3}$$

points are $(\pm 1, \pm 2)$ (4 points) and $(\pm \sqrt{3}, 0)$
(2 points)

c) [10 points] Find extreme values.

Max attains when $x > 0$:

$$f(1, \pm 2) = \boxed{12} \text{ (maximum value)}$$

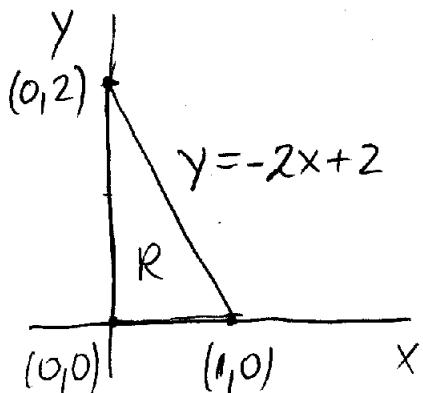
Min attains when $x < 0$:

$$f(-1, \pm 2) = \boxed{-12} \text{ (minimum value)}$$

(Note $f(\pm \sqrt{3}, 0) = 0$, no max/min)

\bar{y} set up, do not evaluate

5. [10 points] Find the y -coordinate of the center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.



$$R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq -2x + 2\}$$

$$\begin{aligned} m &= \int_0^1 \int_0^{-2x+2} (1+3x+y) dy dx = \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_0^{-2x+2} dx \\ &= \int_0^1 (-2x+2 - 6x^2 + 6x + 2x^2 - 4x + 2) dx \\ &= \int_0^1 (-4x^2 + 4) dx = \left[-\frac{4}{3}x^3 + 4x \right]_0^1 = \frac{8}{3} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m} \int_0^1 \int_0^{-2x+2} (y + 3xy + y^2) dy dx = \frac{3}{8} \int_0^1 \left[\frac{y^2}{2} + \frac{3xy^2}{2} \right. \\ &\quad \left. + \frac{y^3}{3} \right]_0^{-2x+2} dx = \frac{3}{8} \int_0^1 (2x^2 - 4x + 2 + 6x^3 - 12x^2 + 6x \\ &\quad - \frac{8}{3}x^3 + 8x^2 - 8x + \frac{8}{3}) dx = \frac{3}{8} \int_0^1 \left(\frac{10}{3}x^3 - 2x^2 - 6x + \frac{14}{3} \right) dx \\ &= \frac{3}{8} \left[\frac{5}{6}x^4 - \frac{2}{3}x^3 - 3x^2 + \frac{14}{3}x \right]_0^1 = \frac{11}{16} \end{aligned}$$

bonus problem [7 points extra] Find the region E for which the triple integral
 $\iiint_E (1 - x^2 - y^2 - z^2) dV$ is a maximum.

The function $1 - x^2 - y^2 - z^2$ is negative outside the unit sphere $x^2 + y^2 + z^2 = 1$ and positive inside it. Hence the integral attains its maximum when E is the unit sphere

$$x^2 + y^2 + z^2 = 1$$