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Exam 2

Fall 2011 Math 0240

100 points total

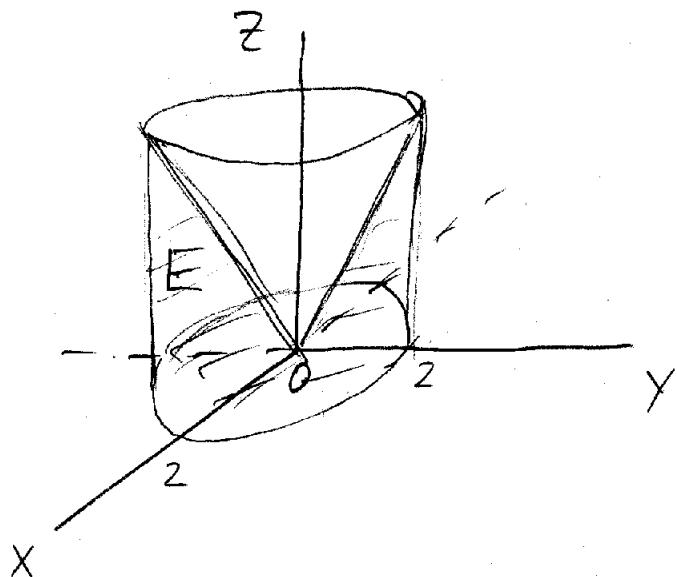
Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

1. The goal of this problem is to find the volume of the region E under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$.

a) [10 points] Define the region E in rectangular coordinates and sketch it.

$$E = \{(x, y, z) \mid -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}, 0 \leq z \leq \sqrt{x^2+y^2}\}$$



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b) [10 points] Define the region E in cylindrical coordinates.

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$E = \{(r, \theta, z) \mid 0 \leq r \leq 2, 0 \leq \theta < 2\pi, 0 \leq z \leq r\}$$

c) [10 points] Using appropriate coordinates represent the volume as an integral and evaluate it.

$$\begin{aligned} V &= \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^r r dz dr d\theta = \\ &= \int_0^{2\pi} d\theta \cdot \int_0^2 r z \Big|_0^r dr = 2\pi \cdot \int_0^2 r^2 dr = \\ &= 2\pi \cdot \frac{1}{3} r^3 \Big|_0^2 = \boxed{\frac{16}{3}\pi} \end{aligned}$$

2. Using Lagrange multipliers find the maximum and minimum values of the function $f(x, y) = 4xyz$ subject to the constraint $2x^2 + 3y^2 + z^2 = 6$.

a) [5 points] Write the Lagrange equations.

$$4yz = 4\lambda x \quad (1)$$

$$4xz = 6\lambda y \quad (2)$$

$$4xy = 2\lambda z \quad (3)$$

$$2x^2 + 3y^2 + z^2 = 6 \quad (4)$$

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b) [10 points] Find points where the function has possible extreme values.

Multiplying the first three eq's by x , y , and z correspondingly we obtain

$$4xyz = 4\lambda x^2 = 6\lambda y^2 = 2\lambda z^2 \Rightarrow 2x^2 = 3y^2 = z^2 \quad (\lambda \neq 0)$$

$$\text{Then } (4) \Rightarrow z^2 + z^2 + z^2 = 6 \Rightarrow z = \pm\sqrt{2}, x = \pm 1 \\ y = \pm\sqrt{\frac{2}{3}}$$

If $\lambda = 0$, then $yz = xz = xy = 0 \Rightarrow$ at least one of them x , y , or z is 0 $\Rightarrow f(x, y, z) = 0$

The points are $\boxed{(\pm 1, \pm \sqrt{\frac{2}{3}}, \pm \sqrt{2})} \quad (8 \text{ points})$

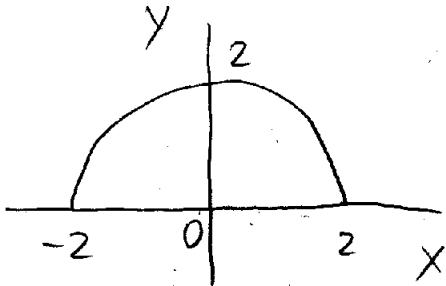
c) [5 points] Find extreme values.

Max occurs when the product of xyz is positive:

$$f(1, \sqrt{\frac{2}{3}}, \sqrt{2}) = 4 \cdot 1 \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \sqrt{2} = \boxed{\frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}}$$

$$\text{Min: } f(-1, \sqrt{\frac{2}{3}}, \sqrt{2}) = \boxed{-\frac{8}{\sqrt{3}} = -\frac{8\sqrt{3}}{3}}$$

3. [10 points] Evaluate the integral $\int_C (2 + x^2y) ds$, where C is the upper half of the circle $x^2 + y^2 = 4$.



$$C: \begin{aligned} x &= 2\cos t \\ y &= 2\sin t \end{aligned}$$

$$0 \leq t \leq \pi$$

$$\begin{aligned} ds &= \sqrt{(-2\sin t)^2 + (2\cos t)^2} dt = \\ &= 2dt \end{aligned}$$

$$\begin{aligned} \int_C (2 + x^2y) ds &= \int_0^\pi (2 + 8\cos^2 t \sin t)(2dt) = 4 \int_0^\pi dt = \\ &= 4\pi - 16 \int_0^\pi \cos^2 t d(\cos t) = 4\pi - 16 \frac{\cos^3 t}{3} \Big|_0^\pi = \\ &= 4\pi - 16 \left[-\frac{1}{3} - \frac{1}{3} \right] = \boxed{4\pi + \frac{32}{3}} \end{aligned}$$

4. [10 points] Find the gradient vector field of $f(x, y, z) = \sin(x^2/z) \cos(y^2/z)$

$$f_x = \left(\frac{2x}{z}\right) \cos\left(\frac{x^2}{z}\right) \cos\left(\frac{y^2}{z}\right)$$

$$f_y = \sin\left(\frac{x^2}{z}\right) \cdot \left(\frac{2y}{z}\right) \cdot \left(-\sin\left(\frac{y^2}{z}\right)\right)$$

$$f_z = \cos\left(\frac{x^2}{z}\right) \cdot \left(-\frac{x^2}{z^2}\right) \cos\left(\frac{y^2}{z}\right) + \sin\left(\frac{x^2}{z}\right) \cdot \left(-\sin\left(\frac{y^2}{z}\right)\right) \left(-\frac{y^2}{z^2}\right)$$

$$\nabla f(x, y, z) = \frac{2x}{z} \cos\left(\frac{x^2}{z}\right) \cos\left(\frac{y^2}{z}\right) \vec{i}$$

$$- \frac{2y}{z} \sin\left(\frac{x^2}{z}\right) \sin\left(\frac{y^2}{z}\right) \vec{j}$$

$$+ \left[-\frac{x^2}{z^2} \cos\left(\frac{x^2}{z}\right) \cos\left(\frac{y^2}{z}\right) + \frac{y^2}{z^2} \sin\left(\frac{x^2}{z}\right) \sin\left(\frac{y^2}{z}\right) \right] \vec{k}$$

5. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is known that the area enclosed by this ellipse is $ab\pi$.

a) [10 points] Set up a double integral in x and y variables to find the area enclosed by this ellipse.

$$A = \iint_R dA$$

$$-a \leq x \leq a$$

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2} (a^2 - x^2)$$

$$-\frac{b}{a} \sqrt{a^2 - x^2} \leq y \leq \frac{b}{a} \sqrt{a^2 - x^2} \quad (b, a > 0)$$

$$R = \{(x, y) \mid -a \leq x \leq a, -\frac{b}{a} \sqrt{a^2 - x^2} \leq y \leq \frac{b}{a} \sqrt{a^2 - x^2}\}$$

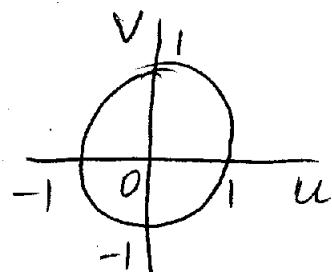
$$A = \int_{-a}^a \int_{-\frac{b}{a} \sqrt{a^2 - x^2}}^{\frac{b}{a} \sqrt{a^2 - x^2}} dy dx$$

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b) [10 points] Now transform the above integral via the transformation $x = au$, $y = bv$ (that is, find the Jacobian, integral, and the new region of integration). Sketch the new region in uv -plane.

$$u = \frac{x}{a}, \quad v = \frac{y}{b}$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = u^2 + v^2 = 1 \quad (\text{the unit circle})$$



$$x_u = a, \quad x_v = 0, \quad y_u = 0, \quad y_v = b$$

$$J = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$A = \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab \, dv \, du$$

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c) [10 points] Evaluate the integral to find the area enclosed by the ellipse.

$$\int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dv du = \pi \cdot 1^2 = \pi$$

(the integral is the area of the unit circle)

Hence $A = ab \cdot \pi$ or $A = \pi ab$

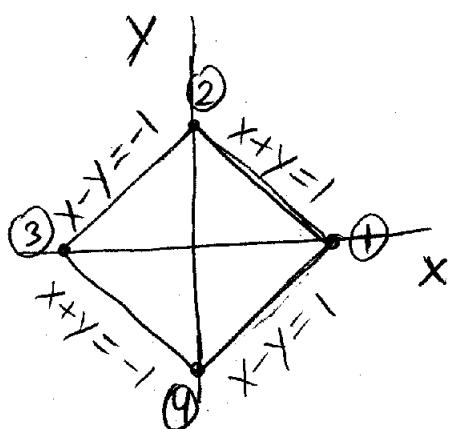
Another way (by using polar coord's)

$$A = ab \int_0^{2\pi} \int_0^1 r dr d\theta = ab \cdot 2\pi \cdot \frac{1}{2} = \pi ab$$

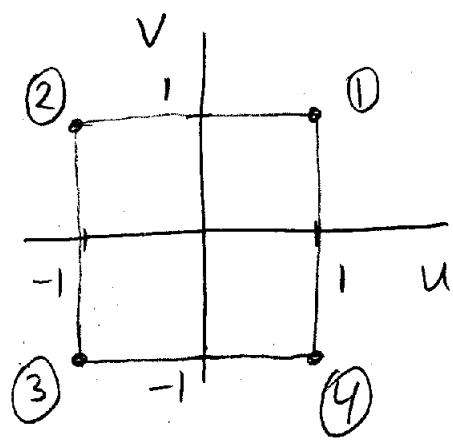
bonus problem [7 points extra] Evaluate the integral $\iint_R e^{x+y} dA$ by making an appropriate change of variables, where R is the region given by inequality $|x| + |y| \leq 1$.

$$\text{C.O.V. } u = x - y, v = x + y$$

$$\text{Then } x = \frac{u+v}{2}, y = \frac{-u+v}{2} \quad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$



the transformation is linear
It transforms a line segment
into a line segment. It is
enough to find images
of the points ①, ②, ③, and ④
under T^{-1} : $u = x - y, v = x + y$



← The image is square S

$$S = \{(u, v) \mid -1 \leq u \leq 1, -1 \leq v \leq 1\}$$

$$\iint_R e^{x+y} dA = \iint_S e^v \cdot \frac{1}{2} dv du = \int_{-1}^1 du \cdot \frac{1}{2} \int_{-1}^1 e^v dv =$$

$$= 2 \cdot \frac{1}{2} \cdot e^v \Big|_{-1}^1 = [e - e^{-1}]$$