Department of Mathematics

University of Pittsburgh

MATH 0230 (Calculus II)

Midterm 2, Fall 2017

Instructor: Kiumars Kaveh

Last Name: Kaveh **Student Number:** 1

First Name: Aydin

TIME ALLOWED: 50 MINUTES. TOTAL MARKS: 50
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.
PLEASE READ THROUGH THE ENTIRE TEST BEFORE STARTING
AND TAKE NOTE OF HOW MANY POINTS EACH QUESTION IS WORTH.
FOR FULL MARK YOU MUST PRESENT YOUR SOLUTION CLEARLY.

Question	Mark
1	/10
2	/10
3	/8
4	/12
5	/10
6	/1
TOTAL	/50 + 1

1.[10 points] Consider the curve in the plane given parametrically by:

$$x(t) = \sin(t), \quad y(t) = \tan(t).$$

(a) [6 points] Find the (parametric) equation of tangent line to the curve at the point $t = \pi/3$.

(b) [4 points] Setup an integral that represents the length of this curve from $t = 0$ to $t = \pi$.

(a) $x'(t) = \cos(t) \qquad y'(t) = \sec^2(t)$

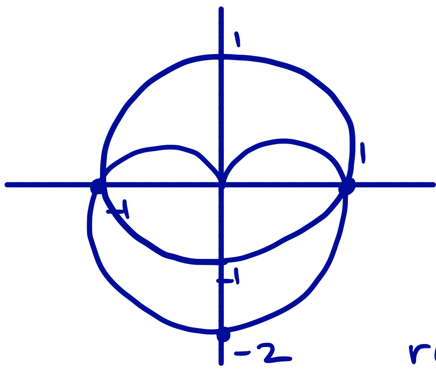
$x'(\pi/3) = \frac{1}{2} \qquad y'(\pi/3) = 4$

$x(\pi/3) = \sqrt{3}/2 \qquad y(\pi/3) = \sqrt{3}$.

Equ. of tangent line at $t = \pi/3$: $\begin{cases} x(t) = \frac{1}{2}t + \frac{\sqrt{3}}{2} \\ y(t) = 4t + \sqrt{3}. \end{cases}$

(b) length $= \int_0^{\pi} \sqrt{\cos^2(t) + \sec^4(t)} \, dt.$

2.[10 points] Find the area of the region that lies inside the polar curve $r = 1 - \sin(\theta)$ and outside the polar curve $r = 1$.



Area =

$$\int_{\pi}^{2\pi} \frac{1}{2}(r(t))^2 \, dt - \left(\text{half of the area inside the circle } r=1\right)$$

$$r(t) = 1 - \sin(\theta)$$

$$\int_{\pi}^{2\pi} \frac{1}{2}(1-\sin(\theta))^2 \, d\theta = \int_{\pi}^{2\pi} \frac{1}{2} - \sin(\theta) + \frac{\sin^2(\theta)}{2} \, d\theta =$$

$$= \frac{1}{2}(2\pi - \pi) + \cos(\theta)\Big|_{\pi}^{2\pi} + \frac{1}{4}\left(x - \frac{\sin(2x)}{2}\right)\Big|_{\pi}^{2\pi}$$

$$= \frac{\pi}{2} + 2 + \frac{\pi}{4} = 2 + \frac{3\pi}{4}.$$

$$\text{Area} = \left(2 + \frac{3\pi}{4}\right) - \frac{\pi}{2} = 2 + \frac{\pi}{4}.$$

$\sin(2n) \leq 1$

because $2+\sin(2n) \leq 3$

Note $0 \leq 2+\sin(2n)$. for all n

3.[12 points]

(a) [3 points] Show that the series $\sum_{n=1}^{\infty} \frac{2+\sin(2n)}{n^2}$ is absolutely convergent.

$\sum_n \left|\frac{2+\sin(2n)}{n^2}\right| \leq \sum_n \frac{3}{n^2}$ but $\sum_n \frac{1}{n^2}$ is conv.

so by Comparison test $\sum_n \frac{2+\sin(2n)}{n^2}$ is abs. conv.

(b) [3 points] Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent. In the case of convergence, determine if it is also absolutely convergent.

$0 < \frac{1}{\sqrt{n}}$ & decreasing seq. so by (Leibnitz) alt. series test $\sum_n (-1)^n/\sqrt{n}$ is conv. But $\sum_n \frac{1}{\sqrt{n}}$ is not conv (for example by integral test). So it is not abs. conv.

(c) [3 points] Find the sum of the series $\sum_{n=2}^{\infty} \frac{2^n}{3^n}$.

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3.$$

$$\sum_{n=2}^{\infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^2 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{4}{9} \cdot 3 = \frac{4}{3}.$$

(d) [3 points] Find the sum of the series $\sum_{n=3}^{\infty} 2/n(n-2)$.

$$\frac{2}{n(n-2)} = \frac{1}{n-2} - \frac{1}{n} \quad \text{(partial fraction)}$$

$$\sum_{n=3}^{\infty} \frac{2}{n(n-2)} = \sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}.$$

4.[8 points]

(a) [4 points] Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n2^n}$.

(b) [4 points] Find the power series representation of the function $\frac{1+x^2}{1+3x}$.

(a) Root test: $\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{n2^n}\right|} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}\cdot 2}$

$= \frac{|x|}{2}$. By root test $\frac{|x|}{2}<1$ series (abs.) conv.

& $\frac{|x|}{2}>1$ series div. So $|x|<2$ conv. $|x|>2$ div.

$\Rightarrow$ radius of conv. $= 2$.

(b) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n$$

$$\frac{1+x^2}{1+3x} = \sum_{n=0}^{\infty} (1+x^2)(-3x)^n$$

$$= \sum_{n=0}^{\infty} (-3)^n x^{n+2} + \sum_{n=0}^{\infty} (-3x)^n$$

$$= \sum_{n=2}^{\infty} (-3)^{n-2} x^n + \sum_{n=0}^{\infty} (-3x)^n = 1+(-3)x + \sum_{n=2}^{\infty} \left((-3)^{n-2}+(-3)^n\right)x^n.$$

5.[10 points] Consider the function $f(x) = 1/(1-x)^2$.

(a) [4 points] Find the Taylor polynomial $T_2(x)$ at $x = 0$ of $f(x)$.

(b) [4 points] Find the Taylor series $T(x)$ of $f(x)$.

(c) [2 points] Find the radius of convergence of the power series $T(x)$.

$$f'(x) = \left((1-x)^{-2}\right)' = 2(1-x)^{-3}$$

$$f''(x) = (2\times 3)(1-x)^{-4}$$

$$\vdots$$

$$f^{(n)} = (n+1)!\,(1-x)^{-(n+2)}$$

$$f^{(n)}(0) = (n+1)!$$

(a) $$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$= 1 + 2x + 3x^2$$

(b) $$T(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)x^n$$

(c) By root test radius of convergence is 1.

6.[1 point] Draw a cartoon of yourself eating a whole turkey!

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