

10-10:50am

## Exam 1

Fall 2011 Math 0240

100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

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1. [10 points] Find the curvature of  $y = \cos 2x$ .

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}, \text{ where } f(x) = y = \cos 2x$$

$$f'(x) = -2\sin 2x, \quad |f''(x)| = |-4\cos 2x| = 4|\cos 2x|$$

(Note:  $\cos 2x$  can be negative)

$$\kappa(x) = \frac{4|\cos 2x|}{[1 + 4\sin^2 2x]^{3/2}}$$

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You also can try  $F(x) = \langle x, y \rangle = \langle x, \cos 2x \rangle$   
or  $F(t) = \langle t, \cos 2t \rangle$ , where  $t = x$ .

2. [10 points] Find the maximum rate of change of  $f(x, y) = \frac{y^2}{x}$  at the point  $(2, 4)$  and the direction in which it occurs.

$$\bar{\nabla} f = \langle f_x, f_y \rangle = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle$$

$$\bar{\nabla} f(2, 4) = \langle -4, 4 \rangle$$

max rate of change is

$$|\bar{\nabla} f(2, 4)| = 4\sqrt{2}$$

the direction is

$$\langle -4, 4 \rangle \quad (\text{or } \langle -1, 1 \rangle, \text{ or } \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle)$$

3. [15 points] Use implicit differentiation to find  $z_x$  and  $z_y$  if

$$yz = \ln(x + 2z)$$

Method 1. Differentiate both sides w.r.t.  $x$ :

$$yz_x = \frac{1+2z_x}{x+2z}, \quad (xy + 2yz - 2)z_x = 1, \quad z_x = \frac{1}{xy + 2yz - 2}$$

Dif-te w.r.t  $y$ :  $z + yz_y = \frac{2z_y}{x+2z},$

$$(xy + 2yz - 2)z_y = -z(x+2z), \quad z_y = -\frac{z(x+2z)}{xy + 2yz - 2}$$

Method 2. Denote  $F(x, y, z) = yz - \ln(x+2z).$

Then  $F_x = -\frac{1}{x+2z}, \quad F_y = z$

$$F_z = \frac{xy + 2yz - 2}{x+2z}, \quad \frac{1}{F_z} = \frac{x+2z}{xy + 2yz - 2}$$

$$z_x = -F_x \cdot \frac{1}{F_z} = -\frac{1}{xy + 2yz - 2}$$

$$z_y = -F_y \cdot \frac{1}{F_z} = -\frac{z(x+2z)}{xy + 2yz - 2} \quad (\text{as before})$$

4. [15 points] A gun is fired with angle of elevation  $30^\circ$ . What is the muzzle speed if the maximum height of the shell is 500 m?

$$\bar{r}(t) = \langle x(t), y(t) \rangle \Rightarrow \bar{v}(t) = \langle x'(t), y'(t) \rangle,$$

$$\bar{a}(t) = \langle x''(t), y''(t) \rangle = \langle 0, -g \rangle$$

$$\bar{v}_0 = \bar{v}(0) = \langle v_0 \cos 30^\circ, v_0 \sin 30^\circ \rangle = \left\langle \frac{\sqrt{3}}{2} v_0, \frac{1}{2} v_0 \right\rangle$$

where  $v_0$  is the muzzle speed that has to be found. ( $v_0 = |\bar{v}_0|$ )

$$\bar{F}_0 = \bar{F}(0) = \langle 0, 0 \rangle.$$

$$\text{Integrating } \bar{a}(t) : \quad \bar{v}(t) = \left\langle \frac{\sqrt{3}}{2} v_0, -gt + \frac{1}{2} v_0 \right\rangle$$

$$\text{Integrating } \bar{v}(t) : \quad \bar{r}(t) = \left\langle \frac{\sqrt{3}}{2} v_0 t, -\frac{g}{2} t^2 + \frac{1}{2} v_0 t \right\rangle$$

For y components of  $\bar{v}(t)$  and  $\bar{r}(t)$  at the maximum height we have

$$-gt + \frac{1}{2} v_0 = 0 \quad \Rightarrow \quad v_0 = 2gt$$

$$-\frac{g}{2} t^2 + \frac{1}{2} v_0 t = 500 \quad -\frac{g}{2} t^2 + \frac{1}{2} \cdot 2gt \cdot t = 500$$

$$\frac{g}{2} t^2 = 500 \quad t = \frac{10\sqrt{10}}{\sqrt{g}}$$

$$\text{Hence } v_0 = 2g \cdot \frac{10\sqrt{10}}{\sqrt{g}} = \boxed{20\sqrt{10}g \text{ m/s}} \quad (= \sqrt{4000g})$$

5. [10 points] Find an equation of the tangent plane to the surface  $z = 9x^2 + y^2 + 6x - 3y + 5$  at the point  $(1, 2, 18)$ .

Let  $f(x, y) = 9x^2 + y^2 + 6x - 3y + 5 \quad (= z)$

Then the tan. plane equation  
at  $(1, 2, 18)$  is

$$z = 18 + f_x(1, 2)(x-1) + f_y(1, 2)(y-2)$$

$$f_x = 18x + 6, \quad f_x(1, 2) = 24$$

$$f_y = 2y - 3, \quad f_y(1, 2) = 1$$

tan. plane: 
$$\boxed{z = 18 + 24(x-1) + y-2}$$

or 
$$\boxed{24x + y - z - 8 = 0}$$

Another way: Let  $F(x, y, z) = 9x^2 + y^2 + 6x - 3y + 5 - z$

$$F_x = 18x + 6, \quad F_x(1, 2, 18) = 24, \quad F_y = 2y - 3, \quad F_y(1, 2, 18) = 1, \\ F_z = -1. \quad \text{Tan plane: } F_x(1, 2, 18)(x-1) + F_y(1, 2, 18)(y-2) + \\ + F_z(1, 2, 18)(z-18) = 0$$

$$\text{or } 24(x-1) + 1(y-2) - 1(z-18) = 0$$

$$\text{or } 24x + y - z - 8 = 0 \quad (\text{as before})$$

6. [20 points] Find the local maximum and minimum values and saddle point(s) of the function  $f(x, y) = x^3y + 12x^2 - 8y$ .

CPs:  $f_x = 3x^2y + 24x = 0 \Rightarrow 12y + 48 = 0, y = -4$   
 $f_y = x^3 - 8 = 0 \Rightarrow x = 2$

CP:  $(2, -4)$

$$f_{xx} = 6xy + 24, f_{xx}(2, -4) = -24 < 0$$

$$f_{yy} = 0, f_{xy} = 3x^2, f_{xy}(2, -4) = 12.$$

$$D = 0 - 12^2 = -12^2 < 0$$

Hence  $(2, -4)$  is a saddle point.

There is no loc. max or min. values.

7. (a) [10 points] For the given points  $P(2, 0, -1)$ ,  $Q(4, 1, 0)$ ,  $R(3, -1, 1)$  and  $S(2, -2, 2)$  find the area of the triangle  $PRS$ .

$$\bar{b} = \overline{PR} = \langle 1, -1, 2 \rangle, \quad \bar{c} = \overline{PS} = \langle 0, -2, 3 \rangle$$

$$\text{Area of } \triangle PRS = \frac{1}{2} |\bar{a} \times \bar{b}|$$

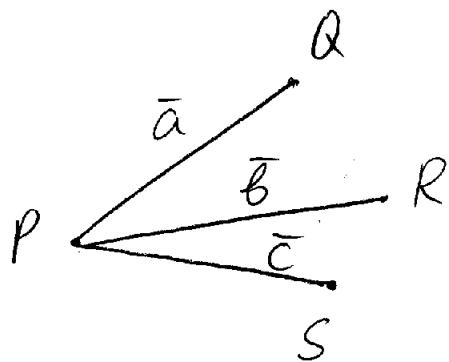
$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = \bar{i} - 3\bar{j} - 2\bar{k} = \langle 1, -3, -2 \rangle$$

$$|\bar{b} \times \bar{c}| = \sqrt{1+9+4} = \sqrt{14}$$

$$\text{Area of } \triangle PRS = \boxed{\frac{\sqrt{14}}{2}}$$

(b) [10 points] Find the volume of the parallelepiped with adjacent edges  $P, Q, R$ , and  $S$ .

$$\text{Let } \bar{a} = \bar{PQ} = \langle 2, 1, 1 \rangle$$



$$\bar{b} = \bar{PR} \quad (\text{as before})$$

$$\bar{c} = \bar{PS}$$

$$\text{Then } V = |\bar{a} \cdot (\bar{b} \times \bar{c})| = |\bar{a} \cdot \langle 1, -3, -2 \rangle|$$

$$= |\langle 2, 1, 1 \rangle \cdot \langle 1, -3, -2 \rangle| = |2 - 3 - 2| = |-3| = 3$$

$$\boxed{V = 3}$$

$$\text{Another way: } \bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} =$$

$$= -6 + 0 - 2 - 0 - 3 - (-8) = -8 - 3 + 8 = -3$$

$$V = |-3| = 3$$

bonus problem [7 points extra] Determine whether the lines

$$L_1: x = 1 + 2t, \quad y = 3t, \quad z = 2 - t$$

$$L_2: x = -1 + s, \quad y = 4 + s, \quad z = 1 + 3s$$

are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$L_1 \neq L_2$  because vectors  $\langle 2, 3, -1 \rangle$  and  $\langle 1, 1, 3 \rangle$  are not  $\parallel$ .

If  $L_1$  and  $L_2$  had a point of intersection it would give:

$$\begin{aligned} x = 1 + 2t &= -1 + s &\Rightarrow s = 2t + 2 \\ y = 3t &= 4 + s &\Rightarrow s = 3t - 4 \\ z = 2 - t &= 1 + 3s \end{aligned} \quad \left. \begin{array}{l} \Rightarrow 2t + 2 = 3t - 4 \\ t = 6, s = 14 \end{array} \right\}$$

$$\text{But } 2 - 6 \neq 1 + 3 \cdot 14 \quad (-4 \neq 43)$$

Hence there is no a point of intersection

$\Rightarrow L_1$  and  $L_2$  are skew lines.