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Exam 1

Fall 2011

Math 0240

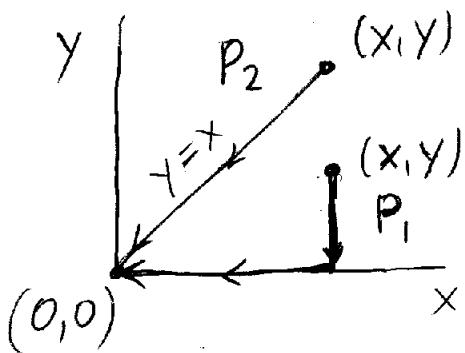
100 points total

Your name: Solutions

No calculators. Show all your work (no work = no credit). Explain every step. Write neatly.

1. [10 points] Find the limit, if it exists, or show that the limit does not exist:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^4 + y^6}$$

Along path P_1 :

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^4 + y^6} = \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} \frac{xy^3}{x^4 + y^6} \right) = \\ = \lim_{x \rightarrow 0} 0 = 0$$

Along path P_2 (choose (x, y) such that $y=x$)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^4 + y^6} = \lim_{x \rightarrow 0} \frac{x^4}{x^4 + x^6} = \lim_{x \rightarrow 0} \frac{x^4 \cdot 1}{x^4(1+x^2)} = \\ = \lim_{x \rightarrow 0} \frac{1}{1+x^2} = 1$$

The limits along paths P_1 and P_2 are different ($0 \neq 1$). Therefore the limit DNE.

2. [10 points] Show that the vector $\bar{v} = \text{orth}_{\bar{a}} \bar{b} = \bar{b} - \text{proj}_{\bar{a}} \bar{b}$ is orthogonal to \bar{a} .

Solution:

$$\bar{v} = \bar{b} - \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|^2} \bar{a}$$

$$\begin{aligned}\bar{v} \cdot \bar{a} &= \bar{b} \cdot \bar{a} - \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|^2} (\bar{a} \cdot \bar{a}) = \\&= \bar{b} \cdot \bar{a} - \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|^2} |\bar{a}|^2 = \bar{b} \cdot \bar{a} - \bar{a} \cdot \bar{b} = 0\end{aligned}$$

$$\bar{v} \cdot \bar{a} = 0 \Rightarrow \bar{v} \perp \bar{a}$$

3. [10 points] Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at the point $(1, 0)$ has the value 1.

$D_{\bar{u}} f(x, y) = \bar{\nabla} f(x, y) \cdot \bar{u}$, where
 $\bar{u} = \langle a, b \rangle$ is a unit vector that
needs to be found, $a^2 + b^2 = 1$.

$$\bar{\nabla} f = \langle 2x + y \cos(xy), x \cos(xy) \rangle$$

$$\bar{\nabla} f(1, 0) = \langle 2, 1 \rangle$$

$$\bar{\nabla} f(1, 0) \cdot \bar{u} = \langle 2, 1 \rangle \cdot \langle a, b \rangle = 2a + b = 1$$

So we have the system

$$2a + b = 1$$

$$b = 1 - 2a$$

$$a^2 + b^2 = 1$$

$$a^2 + (1-2a)^2 = a^2 + 1 - 4a + 4a^2 =$$

$$5a^2 - 4a = 0, a(5a-4) = 0$$

$$a = 0, b = 1$$

$$a = \frac{4}{5}, b = -\frac{3}{5}$$

Answer: $\langle 0, 1 \rangle$ and

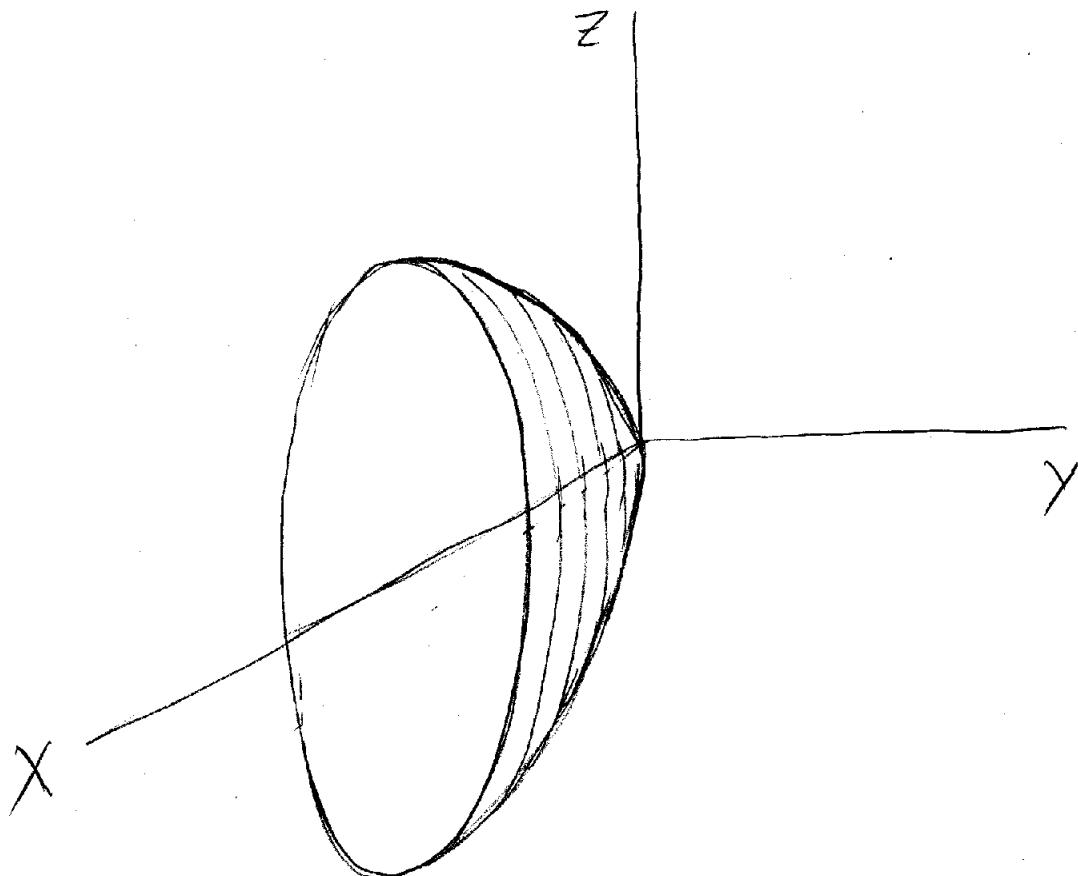
$$\left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

³ are the direction vectors.

4. [10 points] Reduce the equation $9x - y^2 - 9z^2 = 0$ to one of the standard forms, classify the surface and sketch it.

$$x = \frac{y^2}{3^2} + z^2 \text{ elliptic paraboloid}$$

(Note: $x \geq 0$)



5. (a) [10 points] Find the unit tangent and unit normal vectors $\bar{T}(t)$ and $\bar{N}(t)$ to the curve

$$\bar{r}(t) = t^2 \bar{i} + (\sin t - t \cos t) \bar{j} + (\cos t + t \sin t) \bar{k}, \quad t > 0$$

$$\bar{T}(t) = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|} \quad \bar{r}'(t) = \langle 2t, \cos t - \cos t + t \sin t, -\sin t + \sin t + t \cos t \rangle \\ = t \langle 2, \sin t, \cos t \rangle$$

$$\|\bar{r}'(t)\| = t \sqrt{4+1} = t \sqrt{5} \quad (\text{not } |t|, \text{ since } t > 0)$$

$$\bar{T}(t) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \sin t, \frac{1}{\sqrt{5}} \cos t \right\rangle$$

$$\bar{N}(t) = \frac{\bar{T}'(t)}{\|\bar{T}'(t)\|} \quad \bar{T}'(t) = \left\langle 0, \frac{1}{\sqrt{5}} \cos t, -\frac{1}{\sqrt{5}} \sin t \right\rangle \\ = \frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle$$

$$|\bar{T}'(t)| = \frac{1}{\sqrt{5}} \Rightarrow \frac{1}{\|\bar{T}'(t)\|} = \sqrt{5}$$

$$\bar{N}(t) = \langle 0, \cos t, -\sin t \rangle$$

(b) [10 points] Find its curvature.

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\sqrt{5}}{t\sqrt{5}} = \frac{1}{\sqrt{5} \cdot \sqrt{5} \cdot t} = \boxed{\frac{1}{5t}}$$

6. [10 points] Find the linear approximation of the function

$f(x, y, z) = \sqrt{x^2 + y^2 + 3z^2}$ at the point $(3, 2, 2)$ and use it to approximate the number $\sqrt{(3.01)^2 + (2.02)^2 + 3(1.98)^2}$.

$$f(x, y, z) \approx f(3, 2, 2) + f_x(3, 2, 2)(x-3) + f_y(3, 2, 2)(y-2) + f_z(3, 2, 2)(z-2) \quad (\text{linear approximation})$$

$$f_x = \frac{x}{f}, f_y = \frac{y}{f}, f_z = \frac{3z}{f}$$

$$f(3, 2, 2) = \sqrt{9+4+12} = 5$$

$$f_x(3, 2, 2) = \frac{3}{5}, f_y(3, 2, 2) = \frac{2}{5}, f_z(3, 2, 2) = \frac{6}{5}$$

$$\boxed{f(x, y, z) \approx 5 + \frac{3}{5}(x-3) + \frac{2}{5}(y-2) + \frac{6}{5}(z-2)}$$

$$\text{or } \boxed{f(x, y, z) \approx \frac{1}{5}(3x + 2y + 6z)} \quad (= L(x, y, z))$$

$$\sqrt{(3.01)^2 + (2.02)^2 + 3(1.98)^2} = f(3.01, 2.02, 1.98)$$

$$\begin{aligned} &\approx L(3.01, 2.02, 1.98) = 5 + \frac{3}{5} \cdot 0.01 + \frac{2}{5} \cdot 0.02 + \\ &+ \frac{6}{5} \cdot (-0.02) = 5 - \frac{0.05}{5} = 5 - 0.01 = \boxed{4.99} \end{aligned}$$

[You do not need to use $L(x, y, z)$]

7. (a) [10 points] Find a nonzero vector orthogonal to the plane through the points $P(2, 1, 5)$, $Q(-1, 3, 4)$ and $R(3, 0, 6)$.

$$\bar{PQ} = \langle -3, 2, -1 \rangle, \quad \bar{PR} = \langle 1, -1, 1 \rangle$$

$$\text{The vector } \bar{a} = \bar{PQ} \times \bar{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} + 2\hat{j} + \hat{k}$$

(or $\bar{a} = \langle 1, 2, 1 \rangle$) is orthogonal to the plane.

(In general, any vector $c\bar{a} = \langle c, 2c, c \rangle$, where c is a constant, is \perp to the plane).

- (b) [10 points] Find the area of the triangle PQR .

$$\text{Area of } \triangle PQR = \frac{1}{2} |\bar{a}| = \frac{1}{2} \sqrt{1+4+1} = \frac{\sqrt{6}}{2}$$

bonus problem [7 points extra] Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = (1 + xy)(x + y)$.

$$f = x + y + x^2y + xy^2$$

$$\text{CP's: } f_x = 1 + 2xy + y^2 = 0, f_y = 1 + x^2 + 2xy = 0$$

$$f_x - f_y = y^2 - x^2 = 0 \Leftrightarrow y = -x \quad \text{or} \\ y = x$$

$$y = -x : f_x = 1 - 2x^2 + x^2 = 1 - x^2 = 0 \Leftrightarrow x = \pm 1$$

$$y = x : f_x = 1 + 2x^2 + x^2 = 1 + 3x^2 = 0, \text{ no solutions}$$

Two CP's: $(1, -1)$ and $(-1, 1)$ another way:

$$f_{xx} = 2y, f_{yy} = 2x, f_{xy} = 2x + 2y$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = -4(x^2 + xy + y^2)$$

$$\begin{cases} f_{xy} = 0 \\ D = -4x^2 \\ \text{when } y = -x \end{cases}$$

$$D(1, -1) = -4(1 - 1 + 1) = -4 < 0 \quad \text{saddle pt.}$$

$$D(-1, 1) = -4(1 - 1 + 1) = -4 < 0 \quad \text{saddle pt.}$$

The function has no loc max or min values. It has two saddle points $(1, -1)$ and $(-1, 1)$.