Department of Mathematics

University of Pittsburgh

MATH 0230 (Calculus II)

Midterm 1, Fall 2017

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First Name: Aydin

TIME ALLOWED: 50 MINUTES. TOTAL MARKS: 50
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.
PLEASE READ THROUGH THE ENTIRE TEST BEFORE STARTING AND TAKE NOTE OF HOW MANY POINTS EACH QUESTION IS WORTH.
FOR FULL MARK YOU MUST PRESENT YOUR SOLUTION CLEARLY.

Question	Mark
1	/10
2	/10
3	/10
4	/10
5	/10
6	/1
TOTAL	/50 + 1

1.[10 points] Compute the following definite integral. Evaluate and simplify your answer:

$$\int_0^2 \frac{3x+1}{(x+2)^2} dx.$$

$$\frac{3x+1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

$$\Rightarrow 3x+1 = Ax + (2A+B) \Rightarrow A=3 \Rightarrow 6+B=1 \Rightarrow B=-5.$$

$$\int \frac{3x+1}{(x+2)^2} dx = 3\int \frac{1}{x+2} dx - 5\int \frac{1}{(x+2)^2} dx = 3\ln(x+2) - 5\left(\frac{-1}{x+2}\right) + C$$

$$\int_0^2 \frac{3x+1}{(x+2)^2} dx = \left[3\ln(x+2) + \frac{5}{x+2}\right]_0^2$$

$$= 3\ln(4) + \frac{5}{4} - 3\ln(2) - \frac{5}{2}.$$

$$= \boxed{3\ln(2) - \frac{5}{4}}.$$

2.[10 points] Compute the following integrals. Evaluate and simplify your answer.

(a) $\int_0^\infty xe^{-2x}dx$.

(b) $\int_0^1 \frac{1}{(1+x^2)^{3/2}}dx$. (Hint: trig substiution.)

a) Integration by parts: $u = x \quad dv = e^{-2x}dx$

$du = dx \quad v = -\frac{e^{-2x}}{2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int xe^{-2x}dx = -\frac{x}{2}e^{-2x} - \int -\frac{e^{-2x}}{2}dx = \frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$= \frac{e^{-2x}}{2}\left(-x - \frac{1}{2}\right)$$

$$\int_0^\infty xe^{-2x}dx = \lim_{x\to\infty} \underbrace{\cancel{\frac{e^{-2x}}{2}\left(-x-\frac{1}{2}\right)}}_{0} + \frac{1}{4} = \boxed{\frac{1}{4}}$$

(by L'Hospital)

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b) $x = \tan\theta \Rightarrow dx = \sec^2(\theta)d\theta \Rightarrow \int \frac{1}{(1+x^2)^{3/2}}dx = \int \frac{\sec^2\theta}{\sec^3\theta}d\theta =$

$1+x^2 = \sec^2(\theta)$

$$= \int \frac{1}{\sec\theta}d\theta = \int \cos\theta\, d\theta = \sin(\theta) + C$$

$x = 0 \Rightarrow \theta = \tan^{-1}(0) = 0$

$x = 1 \Rightarrow \theta = \tan^{-1}(1) = \pi/4$

$$\Rightarrow \int_0^1 \frac{1}{(1+x^2)^{3/2}}dx = \Big[\sin(\theta)\Big]_0^{\pi/4} = \sin(\pi/4) - \sin(0) = \boxed{\frac{\sqrt{2}}{2}}$$

3.[10 points] Set up (but do not evaluate) integrals that represents the following:

(a) [3 points] The length of the piece of the curve given by the graph of the function

$$y = \frac{e^x - e^{-x}}{2}$$

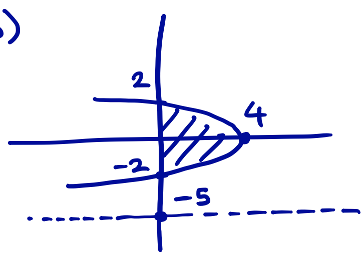
from $x = -1$ to $x = 1$.

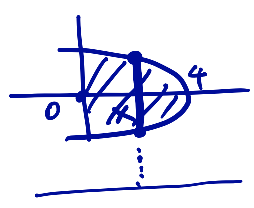
(b) [7 points] The volume of the solid obtained by revolving the region enclosed by the curves $x = 4 - y^2$ and the y-axis around the line $y = -5$.

a) $$\left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2}$$

$$\text{length} = \int_{-1}^{1} \sqrt{1 + \left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx$$

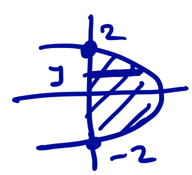
b)





washers

$$x = 4 - y^2$$
$$y = \pm\sqrt{4-x}$$



shells
cylindrical

washer method

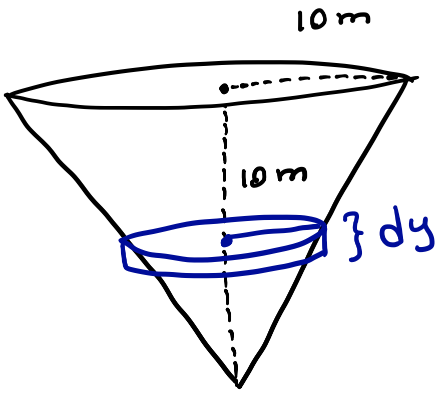
$$\text{vol.} = \pi \int_{0}^{4} \left(5 + \sqrt{4-x}\right)^2 - \left(5 - \sqrt{4-x}\right)^2 dx$$

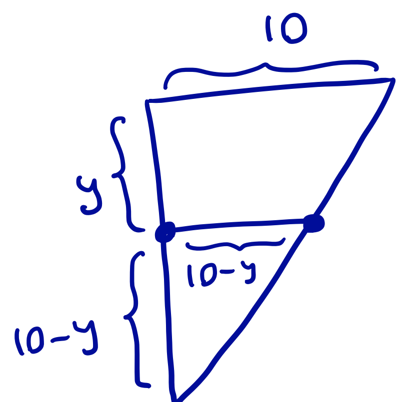
or

$$\text{vol.} = 2\pi \int_{-2}^{2} (y+5)(4-y^2)\,dy$$

cylindrical shells method

4.[10 points] Find the work done in pumping water out of a circular cone shaped pool which filled up completely. You do not need to simplify or compute any multiplication of constants like π. The water density is 1000 kilogram per meter cubed.





ρ = weight density of water = $1000\,\mathrm{kg} \cdot g/m^3$

$\rho \pi (10-y)^2 dy$ = weight of the layer at depth y

$$\text{work} = \int_0^{10} \rho\pi(10-y)^2 y\, dy = \rho\pi\int_0^{10}(100y - 20y^2 + y^3)\, dy$$

$$= \pi\rho \left[\frac{100y^2}{2} - \frac{20y^3}{3} + \frac{y^4}{4}\right]_0^{10} = \pi\rho\left(5000 - \frac{20{,}000}{3} + 2500\right)$$

5.[10 points] Consider the points $P = (1,1,1)$, $Q = (1,0,-1)$ and $R = (2,2,-1)$.

(a) Find the area of the triangle with vertices P, Q and R.

(b) Find the equation of the plane that passes through P, Q and R.

a)

Area of $\triangle PQR$ $= \frac{1}{2}$ Area of parallelogram $= \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$\vec{PQ} = (1,0,-1) - (1,1,1) = (0,-1,-2)$

$\vec{PR} = (2,2,-1) - (1,1,1) = (1,1,-2)$

$\vec{PQ} \times \vec{PR} = \det \begin{bmatrix} i & j & k \\ 0 & -1 & -2 \\ 1 & 1 & -2 \end{bmatrix} = 4\vec{i} - 2\vec{j} + \vec{k}$

$|\vec{PQ} \times \vec{PR}| = \sqrt{16+4+1} = \sqrt{21} \rightsquigarrow$ Area $= \frac{\sqrt{21}}{2}$.

b) $\vec{n} = (4,-2,1)$

$4x - 2y + z + d = 0$

$4 \cdot 1 - 2 \cdot 1 + 1 + d = 0 \Rightarrow d = -3$

$\boxed{4x - 2y + z - 3 = 0}$

6.[1 point] Draw a cartoon of yourself writing the test!

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