Jan 6

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• Webpage to be up soon!

Approximate list of topics:

- Alg. sets, nullstellensatz, Noether normalization
- Affine alg. var.
- Proj. alg. var.
- Ringed spaces, abs. var.
- Schemes
- Degree & Hilbert polynomial
- Bernstein-Kushnirenko thm. & toric varieties
- Degree of a variety as volume
- Non-Sing. var.
- Étale maps & inverse function thm.
- Blow-up
- Curves
- Divisors, line bundles & RR thm.

Some text books:

- J. S. Milne Alg. Geo. (online notes)
- Hartshorne
- Mumford Red book of var. & schemes
  Complex proj. varieties
- Ravi Vakil Rising sea
- Karen Smith et. al. An invitation to alg. geo.
- Shafarevich Basic alg. geo. (I & II)
- Cutkosky (new) Intro. to alg. geo.
- Harris Alg. geo., a first course (lots of examples)
- Griffith & Harris Principles of alg. geo.
- Dummit & Foote <u>Chapter 15</u>

Intro

k field

$\bar{k} = k$ alg. closed

$k = \#$ field

$k = \mathbb{Q}$ arith. geo. / Diophantine geo. related to number theory

$k = \mathbb{R}$ real alg. geo. (lots of unsolved problems)
(some conn. with logic)

$k = \mathbb{Q}_p$ non-Arch. geo.

$k = \mathbb{F}_q$ finite field
(applications in Crypto, Control theory, ...)

k ring $\rightsquigarrow$ scheme theory

Also,
$k = F(t)$, $F((t))$, ...

Affine space $\mathbb{A}^n = k^n$

$k[x_1, \dots, x_n]$ $x = (x_1, \dots, x_n)$ short hand

$$V(f_1, \dots, f_r) = \{ x \in k^n \mid f_1(x) = \dots = f_r(x) = 0 \}$$

$S \subset k[x]$ any $\qquad V(S)$

Thm. $\forall S \subset k[x]$ $\exists$ finite set $f_1, \dots, f_r$ s.t.

$$V(S) = V(f_1, \dots, f_r).$$

I = ideal gen. by S

- $V(S) = V(I)$

$$I = \left\{ g_1 h_1 + \cdots + g_s h_s \;\middle|\; \begin{array}{l} \forall h_i \in S \\ \forall g_i \in k[x] \end{array} \right\}$$

__Thm__ (Hilbert basis thm) $I \subset k[x]$ f.g.

i.e. $k[x]$ Noetherian

- Examples: $X_f = \{ f(x,y) = 0 \} \subset \mathbb{A}^2$

(affine) plane alg. curve

__Def.__ X_f is "rational" if $\exists$ rat. functions

φ, ψ $\quad f(\varphi(t), \psi(t)) = 0 \quad \forall t$

not both const.

$\boxed{\underline{\text{Problem}}}$ Let $\deg f = 2 \Longrightarrow f$ rational.

Hint:

Fermat's last thm.

__Thm__ $n \geq 3$ char $k \nmid n$

Let $f = x^n + y^n - 1$ then X_f not rational

__proof__ $\varphi = \frac{p}{r}$, $\psi = \frac{q}{r}$ $\quad p^n + q^n - r^n = 0$

p, q, r rel. prime

$$\begin{cases} p' p^{n-1} + q' q^{n-1} - r' r^{n-1} = 0 \\ p \cdot p^{n-1} + q \cdot q^{n-1} - r \cdot r^{n-1} = 0 \end{cases}$$

$$\begin{bmatrix} p & q & r \\ p' & q' & r' \end{bmatrix} \begin{bmatrix} p^{n-1} \\ q^{n-1} \\ -r^{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (p^{n-1}, q^{n-1}, -r^{n-1}) \text{ proportional to } \overbrace{(qr'-rq', rp'-pr', pq'-qp')}^{\text{this is a sol.}}$$

$$p(qr'-rq') + q(rp'-pr') + r(pq'-qp') = 0$$

$$\cancel{pqr'} - \cancel{prq'} + \cancel{qrp'} - \cancel{qpr'} + \cancel{rpq'} - \cancel{rqp'}$$

$$p'(qr'-rq') + q'(rp'-pr') + r'(pq'-qp') = 0$$

$$\cancel{p'qr'} - \cancel{p'rq'} + \cancel{q'rp'} - \cancel{q'pr'} + \cancel{r'pq'} - \cancel{r'qp'}$$

$$p^{n-1} \mid qr'-rq' \qquad q^{n-1} \mid rp'-pr' \qquad r^{n-1} \mid pq'-qp'$$

$$\deg p = a \qquad \deg q = b \qquad \deg r = c \qquad a \geq b \geq c$$

$$(n-1)\,a \leq b + c - 1 \qquad b \leq a \;\&\; c \leq a \;\&\; n \geq 3$$

Contradiction 

Milne Chap. 2 Alg. sets

Zariski top.

- $I \subset J \implies V(I) \supset V(J)$

- Prop.

(a) $V(0) = k^n \qquad V(k[x]) = \emptyset$.

(b) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.

(c) $V(\sum_i I_i) = \bigcap_i V(I_i)$

proof (a) ✓

(b) $IJ \subset I \cap J \subset I, J \implies V(IJ) \supset V(I \cap J) \supset V(I) \cup V(J)$.

If $a \notin V(I) \cup V(J) \implies \exists f \in I \quad \exists g \in J \quad \begin{matrix} f(a) \neq 0 \\ g(a) \neq 0 \end{matrix}$

$\implies (fg)(a) \neq 0 \implies a \notin V(IJ)$.

(c) ✓

- Cor. $\{V(I) \mid I \subset k[x]\}$ closed sets for a top. (Zariski top.)

Jan. 8

Über die vollen Invarianten systeme bottom of p. 320

Hilbert Nullstellensatz (1893)

Milne Chap 2 d & e

(Weak) Nullstellensatz : $I = k[x] \iff V(I) = \emptyset$

$P = (a_1, \dots, a_n) \xleftrightarrow{1-1} \phi : k[X] \longrightarrow k$

$\ker \phi \supset I$

Need to show $\exists\, \phi : k[x]/I \longrightarrow k$.

WLOG $I = \mathfrak{m}$ max. ideal

$\boxed{\text{\textbackslash mathfrak} \rightsquigarrow \text{Fraktur}}$

To complete the proof need to show $k[x]/\mathfrak{m} = k$.

<u>Lemma</u> $k[x]$ has ∞ many dist. monic irr. poly.

<u>proof</u> Euclid's proof.

<u>Zariski Lemma</u> $k \subset K$ fields (not nec. alg. closed)

If K f.g. k-alg. then K is alg. over k.

(i.e. K f.g. as a ring $\Longrightarrow$ K f.g. as a module).

Integrality (review)

$A \subset B$ $f \in B$ int. over A

Prop. $f \in B$ int. over $A \iff A[f]$ is f.g. A-module.

proof Cramer's rule.

Prop. Int. elements form a subring of B. $\rightsquigarrow$ int. closure $\overline{A}$

Prop. A int. domain $\rightsquigarrow F = \mathrm{Frac}(A)$.

f alg. over $F \Rightarrow \exists c \in A$ s.t. cf int. over A.

Prop. $k[t]$ int. closed

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proof of Zariski lemma by induction on # of gen. of K

(strong) Nullstellensatz

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

let $f \in I(V(\mathfrak{a}))$.

proof let $\mathfrak{a} = \langle g_1, \cdots, g_s \rangle$. Let $\tilde{\mathfrak{a}} = \langle g_1, \cdots, g_s, 1 - yf \rangle$

$\in k[x, y]$. Use weak nullstellensatz ...

Problem k infinite field $\Rightarrow$ $f \neq 0$ then $f: \mathbb{A}^n \longrightarrow k$ nonzero function. ($\in k[x_1, \dots x_n]$)

Jan 13 Milne chap. 2 e, f, g & Cutkosky 2.1

- Corr. between alg. sets & rad. ideals
- Galois Correspondence/connection

- $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{m}} \mathfrak{m}$
- pt $\xleftrightarrow{1-1}$ max ideals
- irr. variety/alg. set
- Decomposition into irr. varieties

- Def. of reg. function

Prop. ① $X_1 \subset X_2 \Longrightarrow I(X_1) \supset I(X_2)$.

② $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

$x = (x_1, \dots, x_n)$ $\qquad$ $a = (a_1, \dots, a_n)$

Prop. $\mathfrak{m} \subset k[x]$ max. $\iff$ $\mathfrak{m} = \overbrace{(x_1 - a_1, \dots, x_n - a_n)}^{\mathfrak{m}_a}$

proof ($\Rightarrow$) Let $a \in V(\mathfrak{m}) \Rightarrow \mathfrak{m} \subset \mathfrak{m}_a \Rightarrow \mathfrak{m} = \mathfrak{m}_a$.

($\Leftarrow$) Let $\underset{\text{max.}}{\mathfrak{m}'} \supset \mathfrak{m}_a \Longrightarrow V(\mathfrak{m}') \subset V(\mathfrak{m}_a)$

But $\underset{\exists b}{\wedge}$ $\mathfrak{m}' = \mathfrak{m}_b \Rightarrow b = a$.

Cor. (of Strong NS) ① $\mathfrak{a} \longmapsto V(\mathfrak{a})$

$X \longmapsto I(X)$

give a 1-1 corr. between alg. sets in $\mathbb{A}^n$ & rad. ideals in $k[x]$.

② $X \subset \mathbb{A}^n$ any set $\Rightarrow V(I(X)) = \overline{X}$ Zariski closure.

③ $\mathfrak{a} \subset k[x]$ any ideal $\Rightarrow I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$

restatement of SNS

proof ① Suppose X alg. set $= V(\mathfrak{a})$.

Then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ & $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a}) = X$.

$\Rightarrow V(I(X)) = X$.

② $X \subset Z = V(\mathfrak{a}) \Rightarrow \overbrace{V(I(X))}^{\overline{X}} \subset V(\underbrace{I(V(\mathfrak{a}))}_{\sqrt{\mathfrak{a}}}) = Z$.

Rem alg. sets = affine var. $\longleftrightarrow$ rad. ideals

(affine) schemes $\longleftrightarrow$ ideals

(abs.) var. $\rightsquigarrow$ covered by affine var.

scheme $\rightsquigarrow$ " " " schemes

(antitone Galois Conn.)

<u>Def.</u> A, B partially ordered sets.
$F : A \longrightarrow B$ $G : B \longrightarrow A$ order reversing maps

such that: $b \leq F(a) \iff a \leq G(b)$.

Then: $F(a)$ is largest b s.t. $a \leq G(b)$.
$G(b)$ .... - - - - -. a --- $b \leq F(a)$.

$GF : A \longrightarrow A$ & $FG : B \longrightarrow B$ ass. closure op.

monotone, idempotent, $a \leq GF(a)$ $\forall a \in A$
$b \leq FG(b)$ $\forall b \in B$.

- <u>Images of</u> GF & FG are in 1-1 Corr.

<u>Ex.</u>
Verify: $\mathfrak{a} \longmapsto V(\mathfrak{a})$ Galois Conn.:
$X \longmapsto I(X)$

$$Z \subseteq V(\mathfrak{a}) \iff \mathfrak{a} \subseteq I(Z)$$ (obvious from def.)

Ex. (Galois theory) L/K field ext. → Galois ext.? $A = \{\text{subfields } K \subseteq E \subseteq L\}$
$B =$ subgps of $\mathrm{Gal}(L/K)$
$E \longmapsto \mathrm{Gal}(L/E)$
$G \longmapsto \mathrm{Fix}(G)$

<u>Ex.</u> (alg. top.) Covering spaces & subgps of fundamental gp.

Rem irr $\Rightarrow$ Conn.
$\not\Leftarrow$

Def. $W \subset \mathbb{A}^n$ closed irr. if $W \neq Z_1 \cup Z_2$
$Z_1, Z_2 \neq W$ closed

Thm. $W \subset \mathbb{A}^n$ is irr. $\iff$ $I(W)$ prime.

proof $(\Rightarrow)$ $\mathfrak{a} = I(W)$. $fg \in \mathfrak{a} \Rightarrow W \subset V(fg) = V(f) \cup V(g)$.

$\Rightarrow W = V(f)$ or $W = V(g)$ $\Rightarrow$ $I(W) \supset I(V(f)) \supset (f)$
or
$I(W) \supset I(V(g)) \supset (g)$

$(\Leftarrow)$ Suppose $I(W) = \mathfrak{p}$ prime & $W = Z_1 \cup Z_2$.

$\Rightarrow I(W) = I(Z_1) \cap I(Z_2)$.

Now $I(Z_1) \not\subset I(Z_2)$ because otherwise $\underbrace{V(I(Z_1))}_{Z_1} \supset \overbrace{V(I(Z_2))}^{Z_2}$

So $\exists\, f_1 \in I(Z_1) \setminus I(Z_2)$ & $f_2 \in I(Z_2) \setminus I(Z_1)$.

$\Rightarrow f_1 f_2 \in I(W)^{=\mathfrak{p}}$ but $f_1, f_2 \notin \mathfrak{p}$ Contradiction.

Thm. $W \subset \mathbb{A}^n$ closed $\Rightarrow$ (uniquely) union of finite # of irr.

proof. By Noetherian property.

Def. Noetherian top. space : if desc. chain of closed sets stops.

Example $\overbrace{I(W \cap W')}^{\text{smallest rad. ideal containing } I(W) \,\&\, I(W')} = \operatorname{rad}(I(W) + I(W'))$

$W = V(x^2 - y) \qquad W' = V(x^2 + y)$

$I(W \cap W') = \operatorname{rad}(x^2, y) \underset{\substack{\text{assuming} \\ \text{char} \neq 2.}}{=} (x, y).$

- Milne 2(f):

Prop. $h \in \sqrt{\mathfrak{a}} \iff \underbrace{1 \in (\mathfrak{a}, 1 - yh)}_{\text{Gröbner basis theory}}.$

Zariski top:

Prop: ① pts are closed set.

② open cover has finite subcover (quasi-compact)

③ Not Hausdorff. $\rightsquigarrow$ Zariski top. on $\mathbb{A}^1$.

Example $\mathfrak{a} = (f)$

- (f) rad. $\iff$ f square free.
- (f) prime $\iff$ f irreducible

Rem Primary decomp

$X \subset \mathbb{A}^n$ alg. set

Rem: Some authors say affine var. = irr. alg. set

Def. $k[X] = k[x]/I(X)$ Coor. ring of X

Elements of $k[X]$ are regular functions on X.

- Zariski top. on $X \rightsquigarrow$ has same properties as on $\mathbb{A}^n$

Cutkosky

Example $Y = V(x^2 - yz, xz - x)$ is a union of 3 irr. Comp.

Example $Y = V(x^2+y^2+z^2, x^2-y^2-z^2+1)$ decomp. into irr. components.

Problem Zariski closure of graph of $y = e^x$.

Problem $X/\mathbb{C}$ irr. $\Rightarrow X$ Conn. in Euclidean top.

Not easy (ref. Shafarevich)

Jan. 15

- X irr. $\iff$ $I(X)$ prime
- $f : X \longrightarrow \mathbb{A}^1 = k$ reg.
- f regular at a pt p
- Zariski top. on X $\rightsquigarrow$ has similar properties.
- Finite & quasi-finite map
- $\varphi : X \longrightarrow Y \implies \varphi^* : k[Y] \longrightarrow k[X]$
  $F : k[Y] \to k[X] \implies F^* : X \longrightarrow Y$
- We have equiv. of categories.

Next time: - $\mathcal{O}(X) = k[X]$ - Finite & dominant is surj. $\rightsquigarrow$ going-up thm.

- Proof of Noether normalization

Jan. 22

Milne Chap. 2 Sec. k & l

- $\varphi : X \longrightarrow Y \iff F : k[Y] \longrightarrow k[X]$
- **Prop.** φ dom. $\iff$ φ^* inj.
- **Prop.** finite + dominant $\implies$ surj.
- Proj. of hypersur. in $\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^n$

- Going-up thm.

Ex.

φ dom. $\iff$ $a_m(T) = \text{Const.} \neq 0.$

- Noether normalization

↳ Complete the proof next time.

<u>Ex.</u> max. ideals in $k[X]$ $\xleftrightarrow{1-1}$ pts in X

prime ideals in $k[X]$ $\xleftrightarrow{1-1}$ irr. alg. subsets in X

(X irr. alg. set)

<u>Jan 27</u>

- Noether normalization $\rightsquigarrow$ Milne Sec. 2.l

Problem : Remark 2.47 in Milne

<u>Thm</u> tr deg = dim.

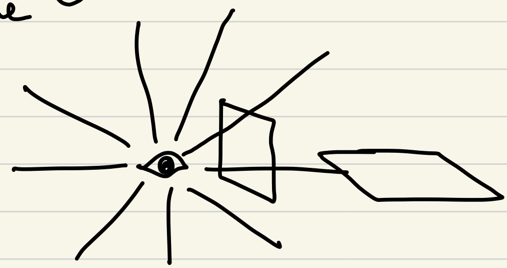
- dimension of an alg. set.

- dim. of a hypersurface & Krull Hauptidealsatz

↳ where in Milne?

Jan 29

Proj. varieties $\rightsquigarrow$ Hartshorne
Milne Chap. 6



- Proj. space
- Alg. sets in proj. space
- Proj. Nullstellen satz
- Corr. between homog. rad. ideals & proj. alg. sets

Feb. 5 (Feb. 3 canceled)

- Homog. Coor. ring & homog. ideal
- Affine var. $\longleftrightarrow$ f.g. alg.
- proj. var. $\longleftrightarrow$ graded f.g. alg. gen. in deg 1
- Closure of an affine var. & homogenization of an ideal.
- Quasi-affine & quasi-proj. var.

- Visualizing $\mathbb{P}^n$ by a simplex & moment map

Problem dim. of a hypersurface & Krull Hauptideal satz

Example of rational normal curve. (?)
Example of pt at ∞ on an elliptic curve.

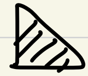
Problem: If I homog. $\Rightarrow \sqrt{I}$ homog.

- Proof that I homog. $\Leftrightarrow$ ($\forall f = \sum f_i$ then $f \in I \Leftrightarrow \forall f_i \in I$).

Plan of next several lectures:

- Sheaf of reg. functions
- Cat. of quasi-proj. var.
- $\exists$ basis of affine neighb.
- Equiv. of cat. of affine var. & f.g. alg.
- Abs. alg. var. $\rightsquigarrow$ sheaf, ringed space, Spec of a ring, Proj of a graded ring

Weil for Riemann Hyp. for function fields

Problem Contin. Map from $\mathbb{CP}^2$ to  $\Delta =$ which is fiberation over the interior. Δ°

$$\Delta = \{(x,y,z) \mid x+y+z=1\} \subset \mathbb{R}^3$$

Problem: Prop. 6.7 in Milne

Problem*: Show rational normal curve (or twisted cubic?) is NOT a complete intersection.

- Example that $X \longmapsto \mathbb{C}[X]_h$ is not 1-1

  X with 2 different embeddings can have 2 non-iso. coor. rings.
- Veronese embedding

Feb. 10

- Def. of complete intersec. I(X) can be gen. by codim X many elements.

- Chap 6 Milne

- Morphisms of proj. var.
- Aut $\mathbb{P}^n$ = PGL(n+1) $\rightsquigarrow$ HW 2 Aut($\mathbb{P}^1$)
  Thm Cross ratio
- Veronese embedding $\rightsquigarrow$ Image closed subvar.
- Segre embedding

Problem Plücker embedding of Gr(2,4)

- Segre embedding. $\rightsquigarrow$ Thm Image closed subvar.

- Plücker embedding $\rightsquigarrow$ Determinantal formula giving Plücker rel. (check Harris) Karen Smith et. al.
  ↳ Example of Gr(2,4)

Problem

- Category of quasi-affine & quasi-proj. var.

- Sheaf of reg. functions $\rightsquigarrow$ Structure sheaf $\mathcal{O}_X(U)$
- Morphisms

Ultra important: X does not have alg. para./charts unlike manifolds.

Rem An open subset can be iso. to an affine alg. set.

Problem $(k^*)^n$ is an alg. gp.
$(k^*)^n$ is an affine alg. var.

- If time permits:

<u>Thm</u> X affine $\Longrightarrow \mathcal{O}_X(X) =$ rest. of poly.

<u>Thm</u> X proj. $\Longrightarrow \mathcal{O}_X(X) = k$.

<u>Thm</u> Equiv. of cat. of f.g. k-alg ...

- Goal: Develope cat. of (abs) alg. var. / schemes
  these of def.
  & extend thms from complex manifolds to them.
  Prominent example: Riemann-Roch thm.
  Hirzbruch
- After def of abs. var. introduce Proj Construction.

<u>Feb. 12</u>

Plücker rel. for $Gr(2,4)$

- Segre embedding
- Grassmannian $Gr(k,n)$
- Plücker embedding & Plücker relations
- Defined quasi-affine & quasi-proj. var.

Next time, reg. functions etc.

Plücker embedding

$$\text{span}\langle v_1, \dots, v_k\rangle \longmapsto v_1 \wedge \cdots \wedge v_k$$

HW3

- $(k^*)^n$ affine var. & alg. gp.

  $GL(n)$ " " " " "

- Plücker rel. for $Gr(2,4)$

- Def. of quasi-affine & quasi-proj. ⇝ obviously have quasi-affine basis of neighborhoods
  - will see have basis of affine neighborhoods
- Def of reg. function (on quasi-proj.)
- Def. of $\mathcal{O}_X(U)$
- Def. of morphism (of q-proj. varieties) & iso<u>ism</u>
- Def of sheaf → in general
  → of k-alg. (from Milne)
- Stalk of a sheaf germ → Ex. Conv. power series

<u>Ex. & non-ex.</u>

- Const. functions
- Bounded contin. functions
- Contin. functions
- Diff. / holo.
- Reg. functions

- Ringed space

Feb. 17 Milne Chap 3 Sec. a–c

- Def. of reg. function
- sheaf of reg. functions
- Thm $k[X]_h \xrightarrow{\cong} \mathcal{O}_X(D(h))$
- Thm $k[X]_{m_p} \xrightarrow{\cong} \mathcal{O}_{X,p}$

Feb 19

- X top. space

Functor $U \longmapsto \mathcal{F}(U) \rightsquigarrow$ usually some kind of functions on U

$\mathcal{C}$ cat. of ab. gps, v.s., rings, alg. modules, alg.

$\boxed{\text{• Def. } p \in X \quad \mathcal{F}_p = \text{stalk of } \mathcal{F} \text{ at } p}$

$V \subset U \longmapsto \operatorname{res}_{V,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

$$\left.\begin{array}{l} ① \quad \operatorname{res}_{U,U} = \text{id. on } \mathcal{F}(U) \\ ② \quad W \subseteq V \subseteq U \Rightarrow \operatorname{res}_{W,V} \circ \operatorname{res}_{V,U} = \operatorname{res}_{W,U} \end{array}\right\} \text{presheaf}$$

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③ (Locality) $\{U_i\}$ open cover of U $\quad s, t \in \mathcal{F}(U)$ agree on the U_i $\Rightarrow s = t$

④ (gluing) $s_i \in \mathcal{F}(U_i)$ & $s_i = s_j$ on $U_i \cap U_j \Rightarrow \exists s \in \mathcal{F}(U)$ $s|_{U_i} = s_i$

Ex. ④ fails $\rightsquigarrow$ bounded functions on $\mathbb{R}$.

Thm (non-trivial but easy) $\mathcal{F}$ presheaf $\rightsquigarrow \mathcal{F}^+$ sheaf with universal property

Rem $(X, \mathcal{F}) \rightsquigarrow H^i(X, \mathcal{F})$ Čech coh.

- Morphism of sheaves : Natural transformation

$\varphi : \mathcal{G} \longrightarrow \mathcal{F}$ $\quad \varphi(U) : \mathcal{G}(U) \longrightarrow \mathcal{F}(U)$.
$\forall U$

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ordnung in German?

- $U \longmapsto \mathcal{O}_X(U)$ **structure sheaf** of X $\quad \Gamma(U, \mathcal{O}_X)$

Rem X proj. then $\mathcal{O}_X(X) = k$ so need reg. functions on smaller open sets.

Def. $f : X \longrightarrow Y$ Contin. map $\quad \mathcal{F}$ sheaf on X

$f_*\mathcal{F}$ direct image sheaf on Y

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)) \quad \forall V \subset Y \text{ open}$$

$f^{-1}\mathcal{G}$ inv. image sheaf on X $\quad \mathcal{G}$ sheaf on Y

$$U \longmapsto \varinjlim_{f(U) \subset V} \mathcal{G}(V)$$ $\rightsquigarrow$ Take sheaf ass. to this presheaf

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- **Def** Ringed space $\quad$ top. space + sheaf of alg.
- **Def.** Morphism of ringed spaces.
  Iso ism

$A \longmapsto \underset{}{\overset{\max}{\mathrm{Spec}}}(A)$ or $\mathrm{Spm}(A)$ A any ring

- $X \subset \mathbb{A}^n$ affine var.

$(X, \mathcal{O}_X)$ can be purely described using $A = k[X]$

$p \in X \longleftrightarrow$ max. ideals $m_p \subset A$

$D(f) = \{ m \mid f \notin m \}$

$f \in A_h$, $m \in D(h)$, $f(m) :=$ image of f in $A_h / mA_h \cong k.$ canonically iso.

$\mathcal{O}_X(D(h)) := A_h$

- Suppose X irr. then $\mathcal{O}_X(U)$ & $\mathcal{O}_p \subset k(X)$.

$\mathcal{O}_X(U) = \bigcap_{p \in U} \mathcal{O}_p$ $\quad \mathcal{O}_p = \{ \frac{g}{h} \in A \mid h(p) \neq 0 \}$

$= A_{m_p}$

Problem $\mathcal{O}_p$ int. domain iff $p \in$ unique irr. component.

Ex. $U = \mathbb{A}^2 \setminus 0$.

$\mathcal{O}(U) = k[x, y]$.

Rem Gelfand-Kolmogorov : $A =$ ring of real valued functions on X compact

max. ideals of $A \overset{1-1}{\longleftrightarrow}$ pts in X.

Thm $X \longmapsto k[X]$

equiv. of cat. between $\underbrace{\text{f.g. reduced alg.}/k}_{\text{affine } k\text{-alg.}}$ (domain)

& affine alg. var. (irr.)

$A \rightsquigarrow \mathrm{Spm}(A)$ (Contravariant) equiv.

$(X, \mathcal{O}_X) \rightsquigarrow k[X] = \mathcal{O}_X(X)$ is a quasi-inverse

Def. Affine alg. var. $\rightsquigarrow$ X quasi-proj. $\cong$ alg set in $\mathbb{A}^n$

Prop. $D(f) \subset \mathbb{A}^n$ affine alg.

Ex. $GL(n)$ & $(k^*)^n$

Cor. Every quasi-proj. var. has basis of affine open sets.

$\hookrightarrow$ For all Local problems can assume X affine.

Next lecture(s):

- Reg. functions on proj. var. $\rightsquigarrow$ proof from CLS toric var.?
- Abs. var. obtained by gluing
- Proj. Construction (as abs. var.)
- Def. of scheme

- smoothness & tangent space (from Milne & Hartshorne)
- Hilbert poly. (from Hartshorne & Harris)
- Deg. of a var. (from Harris)
- Toric var. & BKK thm.

Feb. 24

Coming up:
- Tangent space & Smoothness
- Abs. alg. var.
- Separatedness axiom
- Abs. alg. var. by patching
- Scheme
- Proj. & Spec

- Rat. map
- Birat. map
- Birat. iso., res. of sing., birat. geo., blowup, normalization

Example of $X = V(y^2 = x^2(x+1))$

$\frac{y}{x}$ is int. over $k[X]$

- Field of rat. functions $k(X)$ for irr. q-proj. X

Problem X affine irr. $\Rightarrow k(X) =$ field of frac. of $k[X]$

Problem All quadrics in $\mathbb{P}^n$ (smooth) are isomorphic.

Feb. 26

- Conics & quadrics
- Deg. 3 curves in plane $\rightsquigarrow$ elliptic curves gp. law

<u>Thm.</u> Every $\underbrace{\text{proj. alg. gp.}}_{\text{abelian variety}}$ is commutative.

- Spm = max Spectrum A f.g. k-alg. & domain
- structure sheaf on Spm(A)
- <u>Thm</u> Cat. of k-domains equiv. to Cat. of irr. affine varieties.

<u>Thm</u> X var., Y affine var. then:

$$\mathrm{Hom}(X,Y) \xleftrightarrow{1-1} \mathrm{Hom}(k[Y], \overbrace{\mathcal{O}_X(X)}^{\Gamma(X,\mathcal{O}_X)})$$

Next time: X proj. then $\Gamma(X, \mathcal{O}_X) = k$.

March 2

(positively)
S graded ring
$\mathfrak{p} \subset S$ homog. prime ideal
$S_{(\mathfrak{p})}$ = subring of deg. $\underline{0}$ elements in local. of S w.r.t. $\mathfrak{p}$.

Thm. 3.4 (Hartshorne I.3)

$X \subset \mathbb{P}^n$ irr. proj. var.

$k[X]$ homog. coor. ring $= k[x_0, \dots, x_n]/I(X)$.

(a) $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X) = k$.

(b) $\forall p \in X$, let $m_p \subseteq k[X]$ ideal gen. by $f \in k[X]$ s.t. $f(p) = 0$. Then $\mathcal{O}_{p,X} = k[X]_{(m_p)}$.

(c) $K(X)$ = Field of rat. functions on X $= k[X]_{(0)}$.

proof $U_i \subset \mathbb{P}^n$ $x_i \neq 0$, $X_i := X \cap U_i$ affine var.

$$\varphi_i : U_i \longrightarrow \mathbb{A}^n$$

$$\varphi_i^* : k[y_1, \dots, y_n] \xrightarrow{\cong} k[x_0, \dots, x_n]_{(x_i)} \qquad y_j \longmapsto x_j/x_i$$

$$k[X_i] \xrightarrow{\cong} k[X]_{(x_i)} \qquad I(X_i) \longrightarrow I(X)\, k[x_0 \cdots x_n]_{(x_i)}$$

(b) $p \in X_i \implies \mathcal{O}_{p,X} = \mathcal{O}_{p,X_i} \cong k[X_i]_{m'_p}$ $\quad$ m'_p max. ideal of $k[X_i]$ corr. to p.

$$\varphi_i^*(m'_p) = m_p \cdot k[X]_{(x_i)}.$$

(c) $K(X) = K(X_i)$ = quotient field of $k[X_i]$

$$\varphi_i^* : K(X_i) \xrightarrow{\cong} k[X]_{(0)}.$$

(a) Let $f \in \Gamma(X, \mathcal{O}_X)$.

$\forall i$ f reg. on $X_i \implies f \in k[X_i] = k[X]_{(x_i)}$

$\Rightarrow$ f can be written as $g_i / x_i^{N_i}$, $g_i \in k[X]$

homog. of deg. N_i

$\mathcal{O}(X)$, $K(X)$ & $k[X]$ $\subset$ L = quotient field of $k[X]$.

$x_i^{N_i} f \in k[X]_{N_i}$, $\forall i$

$N \geq \sum N_i \Rightarrow k[X]_N$ spanned as k-vec- space by

monomials of deg. N in $x_0, \ldots, x_n$.

$\Rightarrow$ $k[X]_N f \subseteq k[X]_N \Rightarrow k[X]_N f^q \subset k[X]_N \quad \forall q>0$

$\Rightarrow$ $x_0^N f^q \in k[X] \quad \forall q>0 \Rightarrow \underbrace{k[X][f]}_{k[X]\text{-module}} \subset x_0^{-N} k[X]$.

Hence f.g. $k[X]$-mod. $\Rightarrow$ f int. over $k[X]$.

$\Rightarrow \exists\, m>0 \quad a_0, \ldots, a_{m-1} \in k[X] \qquad f^m + a_{m-1} f^{m-1} + \cdots + a_0 = 0$

f has degree $\underline{0}$ $\Rightarrow$ replace the a_i by their homog. comp. of deg $\underline{0}$.

$\Rightarrow$ still valid equ.. But $k[X]_0 = k \Rightarrow a_i \in k$

$\Rightarrow$ f alg. over k $\Rightarrow$ $f \in k$ since $k = \bar{k}$.

Problem Let $x \in k[\tilde{X}]$ be homog. deg. 1 .

Show that $\Gamma(D(f), \mathcal{O}_X) = k[\tilde{X}]_{(f)} \rightsquigarrow$

degree $\underline{0}$ part of localization ...

March 4

- Started with tangent space/cone last time.

- Intuitive def. of tangent space.
- " " " " cone.
- M manifold $\rightsquigarrow$ TM tangent bundle.
- Recall: Rank thm. from

- $X = V(f_1, \dots, f_t)$ is a submanifold of $\mathbb{R}^n$ of dim d

  rank $\left(\underbrace{\partial f_i / \partial x_j}_{Jac(f_1,\dots,f_t)}\right)$ is $n-d$ at $\forall x \in X$.

- Def (Tangent space) $\rightsquigarrow$ depends only on a generating set of ideal.
- Def. (Sing./non-Sing. pt)
- Def. (Regular local ring) $\rightsquigarrow$ Prop. $\dim_k m/m^2 \geq \dim A$

- Ex. of curves $\rightsquigarrow$ From Milne.

(Milne)

- **Lemma** $\quad \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\cong} \mathfrak{m}A_{\mathfrak{m}} / \mathfrak{m}^2 A_{\mathfrak{m}}$

- Later: $\mathfrak{m}/\mathfrak{m}^2$ v. s. basis gen. $\mathfrak{m}$ as ideal
  (Nakayama's lemma)

- Completion & Cohen st. thm.

Next time:
- $\forall X$ var. X birat. to a hypersur.
- $\dim \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim A$ (Atiyah-McDonald)