March 30 Zoom

Last time: irr.

$p \in C$ curve $\mathcal{O}_p$ local ring

$(\dim \mathcal{O}_p = \dim C = 1)$

- p non-sing. $\iff$ $\mathcal{O}_p$ is a DVR

$v_p(f)$ = order of vanishing of f at p.

f: rational function on C

- We stated a <u>Comm. alg. thm.</u> giving several equiv. cond. on local domain of dim 1 to be <u>regular</u>.

<u>Rem</u> Suppose X is an irr. var. & $\dim X > 1$. $\mathcal{O}_p$ local ring of $p \in X$.

Subvarieties $Y \subset X$ $p \in Y$ give ideals in $\mathcal{O}_p$.

$I(Y) \subset \mathcal{O}_p$.

Subvariety defined by m or its powers m^n are $\{p\}$. $V(m^n) =$

If $\dim X > 1$ we have subvar. $p \in Y \subset X$, $\{p\} \neq Y$, so $\exists$ many ideals that are NOT power of maximal ideal.

<u>Rem</u> $\mathfrak{a} \subset \mathcal{O}_p \rightsquigarrow V(\mathfrak{a})$ lives in a neigh. of $p \in X$. (germ of subvar.)

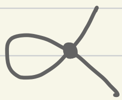
<u>Rem</u> If $\dim X = 1$ (curve) but $p \in X$ is a singular pt. i.e. $\mathcal{O}_p$ NOT a reg. local ring / DVR.

Then one may have ideals that are <u>not</u> power of max. ideal.

cusp

$\boxed{\text{Exercise}}$ ① In $X = V(y^2 - x^3) \subset \mathbb{A}^2$, $p = (0,0)$, find an ideal in $\mathcal{O}_p$ that is not power of max. ideal.

node



② Same question for $X = V(y^2 - x^2(x+1))$

...

$$\mathfrak{m} = (x,y) \begin{matrix} \neq (y) \\ \neq (x) \end{matrix} \subset \left(k[x,y] \big/ (y^2 - x^3) \right)_{\substack{\text{localized} \\ \text{at } p}}$$

Show (x) or (y) are not powers of max. ideal.

- $\underset{p}{\overset{\in}{Y}} \hookrightarrow X \implies$

$$\begin{tikzcd} k[X] \arrow[r] \arrow[d, dashed] & k[Y] \\ \mathcal{O}_{p,X} \arrow[r] & \mathcal{O}_{p,Y} \end{tikzcd}$$

<u>Aside</u> (Local ring of a subvar.)

$Y \subset X$ subvariety (suppose X is irr.)

$\mathcal{O}_{Y,X}$ = local ring of Y in X

= subring/subring of $k(X)$ field of rational functions

$$= \left\{ (f, U) \;\middle|\; \begin{matrix} f \in \mathcal{O}(U),\ U \subset X \\ U \cap Y \neq \emptyset \end{matrix} \right\} \Big/ \sim$$

$((f,U) \sim (g,V) \iff f = g \text{ on } U \cap V)$

One can show:

$Y \subset X$ affine, $\mathfrak{p} = I(Y)$ irr. subvariety

- $\mathcal{O}_{Y,X} \cong k[X]_{\mathfrak{p}}$
- $\dim \mathcal{O}_{Y,X} = \dim X - \dim Y$

Ex. $Y = X \rightsquigarrow \mathcal{O}_{Y,X} = k(X) =$ field of rat. functions

$\dim k(X) = 0.$

$\hookrightarrow$ Krull dim (only proper prime ideal is $\{0\}$)

Integral closure & normalization

X irr. affine $A = k[X] =$ coor. ring (domain)

$K = k(X) =$ field of rat. functions

$\overline{A} \subset K$ int. closure of A in $\overset{\text{field}}{K} \supset A$

$$\overline{A} = \left\{ f \in K \;\middle|\; \begin{array}{l} f \text{ satisfies an int. equ. over } A \\ f^m + a_{m-1} f^{m-1} + \cdots + a_0 = 0 \\ a_i \in A \end{array} \right\}$$

Example from number theory

K = number field

(i.e. finite ext. of $\mathbb{Q}$)

e.g. $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{2})$

....

$$\mathbb{Z} \subset \mathbb{Q} \rightsquigarrow \mathcal{O}_K = \underbrace{\text{int. closure of } \mathbb{Z} \text{ in } K}_{\text{algebraic integers}}$$

Famous thm. of Noether

Thm (Finiteness of int. closure)

Let A f.g. k-alg. domain, $K = \mathrm{Frac}(A)$

L/K finite ext.

Then int. closure of A in L is a finite module over A (in part. is a f.g. k-alg. itself).

(Unfortunately I skip the proof)

ref. Atiyah-Macdonald, Samuel-Zariski ...

Construct

- It is used to ^ resolution of sing. of curves.
- In higher dim. it is used to construct <u>normalization</u> of varieties.

<u>Recall res. of sing.</u>

X given variety, we want $\tilde{X}$ (a res. of sing. of X) & $\pi : \tilde{X} \longrightarrow X$ morphism

s.t. ① $\tilde{X}$ non-sing.

② π gives a birat. iso. i.e.

$\left(\exists\, \tilde{U} \subset \tilde{X} \text{ \& } U \subset X \text{ open} \right.$
$\left. \pi : \tilde{U} \longrightarrow U \text{ is iso}\underline{\text{ism}}. \right)$

<u>Big thm</u> (Hironaka, Fields medal)

char $k = 0$ then $\forall$ variety has a res. of sing.

- char $k = p > 0$ open problem.

X $\overset{\text{irr.}}{\text{affine}}$ $\qquad A = k[X]$

$\qquad\qquad\qquad K = k(X)$

let $\overline{A}$ = int. closure of A in K

(it is a f.g. k-alg.)

- $A \subset \overline{A}$ & $\overline{A}$ is a finite A-module. (Noether's finiteness of int. closure)

- Remember $\boxed{\text{f.g. } k\text{-domains} \xleftrightarrow{1-1} \text{irr. affine var.}}$

$A \qquad\qquad X$

$k[X] \longleftarrow X$

$A \longmapsto \operatorname{Spec}(A)$

$A = k[f_1, \dots, f_n] \qquad \operatorname{Spec}(A) := V(I) \subset \mathbb{A}^n$

I = ideal of rel. between the gen. f_i

$k[x_1 \cdots x_n] \longrightarrow A \qquad A \cong k[x_1 \cdots x_n]/I$

$x_i \longmapsto f_i$

$I = \ker$

<u>Def.</u>

- Normalization of $X := \operatorname{Spec}(\overline{A})$.

<u>Rem</u> If X not affine, cover X with finite number of affine neigh. & do normalization for each neigh. then <u>glue</u> these normalization together.

Rem $A \overset{i}{\hookrightarrow} \overline{A} \subset \boxed{\mathrm{Frac}(A) = \mathrm{Frac}(\overline{A}) = K}$

$\Downarrow$

$\pi = i^*: k[\tilde{X}] \longrightarrow k[X]$

$X = \mathrm{Spec}(A)$

$\tilde{X} = \mathrm{Spec}(\overline{A})$

finite map

i inj. $\Longrightarrow$ π is dominant.

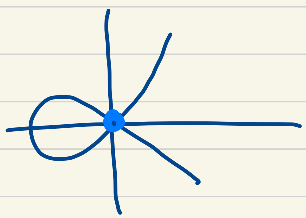
dominant + finite $\Longrightarrow$ surj.

So π is surj.

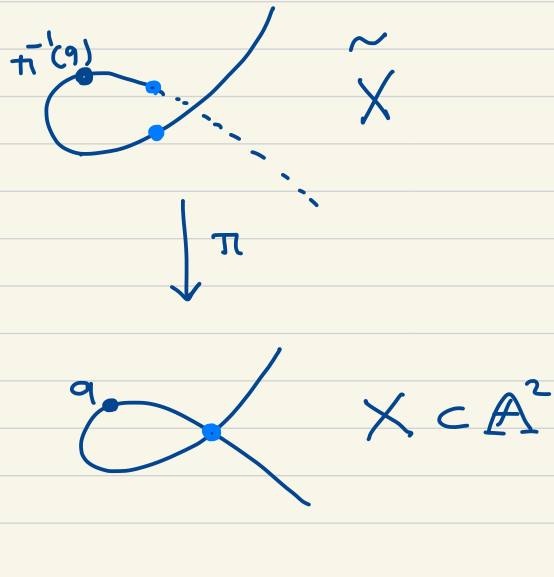
Exercise Show π is birational ism.

(because $k(X) = k(\tilde{X})$)

Ex.



$X = V(y^2 - x^2(x+1))$



Suppose

- X is an irr. affine curve.

One shows that localization commutes with taking int. closure.

Prop. $A \subseteq B$ rings & B integral over A.
If S is a multiplicative set in A (e.g. $S = A \setminus p$ where p is a prime ideal) then $S^{-1}B$ is integral over $S^{-1}A$.

(Proof of prop. can be found in Prop. 5.6. Atiyah–Macdonald)

Conclusion: If A is int. closed ($\overline{A} = A$) then all local rings $\mathcal{O}_p$ are also int. closed.

But $\dim X = 1$ then $\mathcal{O}_p$ int. closed means p non-sing. pt. (thm. from last time)

Cor. Normalization of an affine curve is non-sing. (at every point).