Mach 27 Zoom

Last time: - Every irr. var. birat. to a hypersur.

- Nakayama's lemma
- Spanning set for $\mathfrak{m}/\mathfrak{m}^2 \Longleftrightarrow$ gen. for $\mathfrak{m}$
- Completion of a Noetherian ring & non-sing.

Milne Sec. 4.c

Quick example $X = V(F(x,y)) \subset \mathbb{A}^2$

$(0,0) = p \in X \qquad \mathfrak{m} = (x,y) \subset k[x,y]/(F(x,y))$

$\mathfrak{m}^2 = (x^2, xy, y^2)$

$$\mathfrak{m}/\mathfrak{m}^2 = (x,y)\Big/(x^2, xy, y^2, F(x,y)) \qquad F(0,0) = 0$$

$fx + gy \in \mathfrak{m} \rightsquigarrow$ reduce mod (x^2, xy, y^2)

$cx + dy \rightsquigarrow$ If $F_\ell =$ lin. part of $F = \square x + \square y$

$(0,0)$

Sing. $F_\ell = 0 \rightsquigarrow \mathfrak{m}/\mathfrak{m}^2 =$ All lin. poly $\boxed{\dim \mathfrak{m}/\mathfrak{m}^2 = 2}$

non-Sing $F_\ell \neq 0 \rightsquigarrow \mathfrak{m}/\mathfrak{m}^2 =$ All lin poly mod F_ℓ $\boxed{\dim \mathfrak{m}/\mathfrak{m}^2 = 1}$

Completion of a Noetherian ring

Many ref. e.g. Chap. 10 of Atiyah-Macdonald

Ex.

$\mathbb{Z} \supset (p)$ max ideal

$v_p(x) =$ order of p in x = how many times x div. by p

$v_p : \mathbb{Z} \setminus \{0\} \longrightarrow \mathbb{Z}_{\geq 0}$ $\quad v_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$

$v_p(0) := \infty$ $\quad v_p : \mathbb{Z} \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$

Ex. $k[t]$ (one variable) $\supset (t)$ max. ideal $\quad v_t : k(t) \longrightarrow \mathbb{Z} \cup \{\infty\}$

$v_t(f) =$ ord. of t in f = how many times f div. by t

$a = v_t(f)$ $\quad f = \square t^a + \square t^{a+1} + \cdots$

$$= t^a(\square + \square t + \cdots)$$

$\hookrightarrow$ order of vanishing of f at $\underline{0}$ = zero locus of (t).

$$\mathbb{Z} \xrightarrow{\text{Completion w.r.t. } (p)} \mathbb{Z}_p = p\text{-adic numbers}$$

$x = a_0 + a_1 p + \cdots + a_n p^n$ (x in base p)

$\mathbb{Z}_p = \{ a_0 + a_1 p + a_2 p^2 + \cdots \mid 0 \leq a_i < p \}$ all formal power series in p

$$k[t] \xrightarrow{\text{Completion w.r.t. } (t)} k[[t]]$$

don't care about conv.

$k[[t]]$ = ring of formal power series in t

$$= \{ a_0 + a_1 t + a_2 t^2 + \cdots \mid a_i \in k \}$$

These constructions generalize to arbitrary Noetherian ring A & ideal I

$I \rightsquigarrow$ a top. on A called I-adic top.

$\forall n > 0 \quad I^n$ = an open neighborhood of $0 \in A$

$x + I^n$ = - - - - - —— — - - $x \in A$

$x, y \in A \quad x$ & y are "close" if $x - y \in I^n$ for "large" n.

- Same way one constructs completion of a metric space, one defines $\hat{A}$ completion of A (w.r.t. I)

$$\hat{A} := \varprojlim A/I^n \qquad A/I \to A/I^2 \to A/I^3 \to \cdots$$

Atiyah-Mac. Chap 10

Cohen structure thm

A a <u>regular</u> local ring, $\mathfrak{m}$ max. ideal

$\dim A = r$.

then $\hat{A} \cong k[[x_1, \dots, x_r]]$.

k-alg.

$x_1, \dots, x_r \longleftrightarrow$ a set of gen. for $\mathfrak{m}$

Quick question: $k[[x_1, \dots, x_r]]$ local ring? yes

$\mathfrak{m} = (x_1, \dots, x_r) \rightsquigarrow$ If Const. term of f is nonzero then f unit (Newton's method)

Meaning: $p \in X$ $\qquad A = \mathcal{O}_p$ local ring

p is non-sing $\Longleftrightarrow \hat{\mathcal{O}}_p \cong$ a formal power series ring

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<u>Ex.</u> plane curve $X = \{ y^2 = x^3 + 1 \}$

$p = (0, 1)$

$\mathfrak{m}_p = (x, y-1)$ max. ideal in $\mathcal{O}_p = \left(\dfrac{k[x,y]}{(y^2 - x^3 - 1)} \right)_p$ localized

$\boxed{\text{Exercise}}$ Verify that $\mathfrak{m}_p = (x) \rightsquigarrow$ i.e. principal ideal.

Answer:

$y^2 = x^3 + 1$ $\qquad p = (0,1)$

$(y-1)(y+1) = x^3$ $\qquad$ $y+1$ is inv. in $\mathcal{O}_p$

$$y - 1 = \frac{x^3}{y+1} \in (x) \subset \mathcal{O}_p$$

agrees with Cohen st. th.

$\mathcal{O}_p \cong k[[x]]$

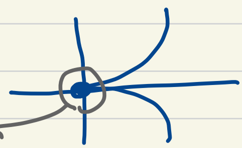
Calculus exercise

$$y = 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} + \cdots$$

y a power series in x

$y^2 = x^3 + 1 \rightsquigarrow$ solve y in terms of x

Ex. $y^2 = x^3 \rightsquigarrow$ sing. at $p = (0,0)$



Cusp

If you solve y in terms of x $\qquad y = x^{3/2}$

not a power series!

<u>Aside</u> $k((t))$ Laurent series in t

$$k((t)) = \{ f = \circledast t^{-m} + \circledast t^{-m+1} + \cdots \}$$

$$= k[[t]][t^{-1}].$$

Exercise: $k((t))$ is a field

<u>Thm</u> (Newton-Puiseux)

$$\overline{k((t))} = \{ \circledast t^{-m/k} + \text{higher terms} \}$$

Field of Puiseux series

<u>Local ring of pts of curves</u>

K field

<u>Def.</u> $v : K \setminus \{0\} \longrightarrow \mathbb{Z}$

(Discrete valuation) ⟿ gen. of notion of ord. of div. or vanishing

$\mathbb{Z}$ is discrete

a) $v(xy) = v(x) + v(y)$ $\quad \forall x, y \in K$, $x, y \neq 0$

b) $v(x+y) \geq \min\{v(x), v(y)\}$ $\quad v(0) := \infty$

Ex. of $A = \mathbb{Z}$ $\quad v = v_p$

$$x = p^a x' \qquad y = p^b y' \qquad \begin{array}{l} a \leq b \\ a = \min(a,b) \end{array}$$

$$x+y = p^a \big(\underbrace{x' + p^{b-a} y'}_{\text{If } a=b \text{ still maybe div. by } p} \big)$$

$$v_p(x+y) = \min(v(x), v(y)) \quad \text{if } a \neq b$$

$$\boxed{v_p(x+y) \geq \min(v(x), v(y)) \quad \text{if } a = b}$$

<u>Def.</u> $\boxed{\text{Valuation ring}}$ of v =

$$\mathcal{O}_v = \{ x \in K \mid v(x) \geq 0 \}$$

It is a local ring.

$$m_v = \text{max. ideal} = \{ x \in K \mid v(x) > 0 \}$$

x in $\mathcal{O}_v$ is a unit $\iff$ $v(x) = 0$

$$v(x^{-1}) = -v(x)$$

$K = \mathrm{Frac}(A)$

<u>Def.</u> Let A be a local domain.

A is a DVR (discrete val. ring)

if $\exists\, v : K \setminus \{0\} \longrightarrow \mathbb{Z}$ discrete val.

s.t. $A = \mathcal{O}_v = \{ x \in K \mid v(x) \geq 0 \}$

$\mathfrak{m} = \mathfrak{m}_v = \{ x \in K \mid v(x) > 0 \}$

Complete description of regular local rings in dim 1 (i.e. local rings of non-sing. pts on affine curves)

<u>Thm</u> (see for example Atiyah-Macdonald) Chap. 9 Prop. 9.2

Let A be a Noetherian local domain of dim 1. $\mathfrak{m}$ max ideal

TFAE

1. A is a <u>regular</u> local ring $\Leftrightarrow$ $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$
2. A is DVR
3. A is integrally closed (in its field of frac.)
4. $\mathfrak{m}$ is principal
5. $\forall\, \mathfrak{a} \subset A$ ideal $\mathfrak{a} = \mathfrak{m}^n$ $\exists\, n > 0$
6. $\exists\, x \in A$ $\forall\, \mathfrak{a} \subset A$ $\mathfrak{a} = (x)^n$ $\exists\, n > 0$

Suppose A is reg. local domain.

we define a val. v on A (& $K = \mathrm{Frac}(A)$)

Let $0 \neq x \in A$

all elements vanishing of ord. $\geq n$

Take $n > 0$ s.t. $x \in \mathfrak{m}^n$

$x \notin \mathfrak{m}^{n+1}$

(This n exists by Krull's intersec. thm)

$$\bigcap_{n>0} \mathfrak{m}^n = \{0\}$$

proof: $\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3 \supset \dots$

Let $M = \bigcap_{n>0} \mathfrak{m}^n \Rightarrow \mathfrak{m}M = M \Rightarrow M = \{0\}$

Then define $v(x) := n$.

One checks that v is a valuation.

Let $A = \mathcal{O}_p$ $p \in X$ $\dim X = 1$

Suppose A is a DVR

If $v(x) = 1$ then $\mathfrak{m} = (x)$ &

hence x is a <u>parameter</u> near $p \in X$