March 25 Zoom

Thm. 4.4 (Hartshorne)

$$\varphi : X \dashrightarrow Y \quad \xleftrightarrow{1-1} \quad F : k(Y) \hookrightarrow k(X)$$

dom. rat. maps — k-alg. homo$\underline{\text{ism}}$

<u>proof</u> $(\rightarrow)$ $\langle \varphi_U , U \rangle$ $\quad \varphi_U : U \xrightarrow{\text{morphism}} Y$

$\varphi_U(U)$ is dense in Y

$f \in k(Y) \Rightarrow f_V : V \xrightarrow{\text{reg.}} k$

$\langle f_V , V \rangle$

non-empty open

$$\begin{tikzcd} \varphi_U^{-1}(V) \arrow[r, "\varphi_U"'] \arrow[rr, bend right, "f \circ \varphi_U"'] & V \arrow[r, "f"'] & k \end{tikzcd}$$

$\varphi_U \circ f \in k(X) \Rightarrow$ we have a well-def. k-alg. homo$\underline{\text{ism}}$

$$k(Y) \longrightarrow k(X) \qquad f \longmapsto f \circ \varphi_U$$

$(\leftarrow)$ Let $F : k(Y) \hookrightarrow k(X)$

Y can be covered with affine open subsets.

WLOG Y affine variety.

$k[Y]$ Coor ring gen. by $y_1 , \dots , y_n$ as k-alg.

$F(y_1), \dots, F(y_n)$ rat. function on X.

$\exists\, U \subset X$ $\quad F(y_i)$ reg. functions on U, $\forall i$

$$F: k[Y] \hookrightarrow \underbrace{\mathcal{O}(U)}_{\text{ring of reg. function on } U} \Rightarrow \underset{\text{dominant morphism}}{\varphi: U \longrightarrow Y}$$

$\Rightarrow \varphi: X \dashrightarrow Y$ dom. rat. map.

Easy to check $\varphi \longmapsto F$ & $F \longmapsto \varphi$ are inv. of each other.

Recall

(Not necessary but for Convenience) ☺

Assume k char. 0 $\rightsquigarrow$ all ext. are $\underbrace{\text{separable}}$

no irr. poly. has repeated roots

<u>Thm</u> (Primitive element thm.)

L finite separable ext. of K & K infinite field $\Rightarrow \exists\, \alpha \in L$ st. $L = K(\alpha)$.

Moreover, if $L = K(\beta_1, \dots, \beta_n) \Rightarrow \alpha$ can be taken to be $\alpha = c_1\beta_1 + \dots + c_n\beta_n$, $c_i \in K$.

(proof relies on the lemma that a finite dim vec. space over K is NOT a union of finitely many proper subspaces)

<u>Prop.</u> X irr. var. of dim r $\Rightarrow X \underset{\text{bir.}}{\cong} H \subset \mathbb{A}^{r+1}$

$H = V(f)$ $\quad f$ irr. poly. in $k[x_1, \dots, x_{r+1}]$.

proof $K = k(X)$ field of rat. functions of X.

K/k f.g. field ext.

$\exists \underbrace{x_1, \ldots, x_r \in K \text{ tr. base}}_{\text{alg. ind. over } k}$, $K =$ finite ext. of $\underbrace{k(x_1 \ldots x_r)}_{\text{field of frac. poly. in } x_1 \ldots x_r}$

$K = k(x_1, \ldots, x_r)(y)$ $\quad \exists y \in K$ P.E.T.

$\exists f(\underbrace{X_1, \ldots, X_r, Y}_{\text{variables}})$ irr. over $k(x_1, \ldots, x_r)$

$f(x_1, \ldots, x_r, y) = 0$

let $H = V(f) \subset \mathbb{A}^{r+1} \rightsquigarrow k[H] = k[X_1, \ldots, X_r, Y] / \langle f \rangle$

$k(H) = k(x_1, \ldots, x_r, y) = K$.

$\underset{\text{as } k\text{-alg.}}{k(H) \cong k(X)} \implies H \underset{\text{bir.}}{\cong} X$. ☺

$p \in X \xrightarrow{} $ affine, irr. $\quad \mathfrak{m} =$ max. ideal in $\mathcal{O}_p =$ local ring of p.

$\underbrace{\dim_k \mathfrak{m}/\mathfrak{m}^2 = \overbrace{\dim \mathcal{O}_p}^{\dim k[X]}}_{\text{def. of reg. local ring}} \iff p$ non-sing pt.

$\dim_k \mathfrak{m}/\mathfrak{m}^2$ vs. min. # of gen. of $\mathfrak{m}$

(Atiyah–MacDonald Comm. alg. p. 121 Cor. 11.15)

Thm. A Noetherian local ring, $\mathfrak{m}$ max ideal

$\Longrightarrow \dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A.$

(Idea of proof: One shows by induction $\dim A =$ min. # of gen. for an $\mathfrak{m}$-primary ideal, & then one constructs an $\mathfrak{m}$-primary ideal with $\dim A$ many gen.)

Milne p.14 Cor. 1.4

Prop. A local ring, $\mathfrak{m}$ max. ideal.

$\mathfrak{m} = (a_1, \ldots, a_r) \Longleftrightarrow \bar{a}_1, \ldots, \bar{a}_r$ in $\mathfrak{m}/\mathfrak{m}^2$ span $\mathfrak{m}/\mathfrak{m}^2$ as k-vec. space.

famous but obvious lemma!

Lemma (Nakayama lemma) A local ring ...

M f.g. A-module

(a) $M = \mathfrak{m}M \Longrightarrow M = \{0\}$.

(b) If $M = N + \mathfrak{m}M$ for some sub-mod. $N \subset M$ then $M = N$.

proof (a) $\Rightarrow$ (b) Apply (a) to M/N.

(a) Let $\{e_1, \ldots, e_r\}$ minimal set of A-mod. gen. of M

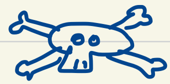
$M = \mathfrak{m}M \Longrightarrow e_1 = c_1 e_1 + \cdots + c_r e_r \quad c_i \in \mathfrak{m}.$

$$\underbrace{e_1}_{\text{in } M} = \underbrace{c_1 e_1 + \cdots + c_r e_r}_{\text{in } \mathfrak{m}M} \qquad c_i \in \mathfrak{m}$$

$$0 = (1-c_1)e_1 + \cdots + c_r e_r$$

$1-c_1$ is a unit $\Longrightarrow$ e_1 is a comb. of $e_2, \ldots, e_r$

Contradiction with minimality of r.



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proof of prop.

$(\Rightarrow)$ $\quad \mathfrak{m} = (a_1, \ldots, a_r) \Longrightarrow \bar{a}_1, \ldots, \bar{a}_r$ span $\mathfrak{m}/\mathfrak{m}^2$.

$\forall c \in \mathfrak{m}$ $\quad c = \sum_i c_i a_i \qquad c_i \in A$

$$\bar{c} = \sum_i \bar{c_i}\, \bar{a_i} \qquad \bar{c}, \bar{a_i} \in \mathfrak{m}/\mathfrak{m}^2$$

$$\bar{c_i} \in A/\mathfrak{m} = k.$$

$(\Leftarrow)$ $\quad \bar{a}_1, \ldots, \bar{a}_r$ span $\mathfrak{m}/\mathfrak{m}^2 \Longrightarrow \underbrace{\mathfrak{m}}_{M} = \underbrace{(a_1, \ldots, a_r)}_{N} + \underbrace{\mathfrak{m}^2}_{\mathfrak{m}M}$

Take $N = (a_1, \ldots, a_r)$ $\quad M = \mathfrak{m}$

$\Longrightarrow M = N$ 

Rem This prop. spanning set / basis for $\mathfrak{m}/\mathfrak{m}^2 \cong T_p^*X$

$\Updownarrow$

(minimal) gen. for $\mathfrak{m}$

is analogue of the following in diff. geo :

$p \in X =$ manifold $/\mathbb{R}$ or $/\mathbb{C}$ $\quad \dim X = r$

$f_1, \dots, f_r$ diff. functions at p with $f_1(p) = \dots = f_r(p) = 0$

$\underbrace{df_1(p), \dots, df_r(p)}_{\text{vec. in } T_p^*X}$ are lin. ind. $\iff$ $x \longmapsto (f_1(x), \dots, f_r(x))$ local chart / local coor. at p.

(Just Corollary of Inv. F. T.)

Let $\mathcal{O}_p$ reg. local ring.

Def. $a_1, \dots, a_r \in A = \mathcal{O}_p$ are a <u>local system of para.</u>

if $\mathfrak{m} = (a_1, \dots, a_r)$.

Next time : we consider 1-dim. local rings.

- A is 1-dim. reg. local ring $\iff$ A is a discrete valuation ring ($\mathfrak{m}$ is principal ideal)

I-Sec. 4 Hartshorne

Completion of a ring ($\rightsquigarrow$ many books in Comm. alg. e.g. Atiyah-Mac.)

$$(A, m) \rightsquigarrow (\hat{A}, \hat{m})$$

top. on A $\quad$ m-adic top. $\rightsquigarrow$ gen. of p-adic top. on $\mathbb{Q}$ or $\mathbb{Z}$

m^i, $i = 0, 1, \ldots$ define a top. on A (basis of neighborhoods of $\underline{0}$)

$$\hat{A} = \varprojlim A/m^i$$

Ex. $A = k[x_1, \ldots, x_r]$, $m = (x_1, \ldots, x_r)$ $\Longrightarrow$ $\hat{A} = \overbrace{k[[x_1, \ldots, x_r]]}^{\text{formal power series}}$, $\hat{m} = (x_1, \ldots, x_r)$

<u>Thm</u> (Cohen St. thm.) $p \in X$ irr. var.

$A = \mathcal{O}_p$

- p is non-sing. / A is reg. local ring

$$\Longleftrightarrow \hat{A} \underset{\text{as } k\text{-alg.}}{\cong} k[[x_1, \ldots, x_r]] \qquad r = \dim A$$