March 22   Zoom

- We started with non-sing / sing. pt.
- Local notion

Today Hartshorne Chap I, Sec. 4

Recall: $Y \subset \mathbb{A}^n$ $I = I(Y) = (f_1, \ldots, f_t)$

$\dim Y = r$

$p \in Y$ is non-sing./smooth if

$$\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j}(p)\right] = n - r$$

$t \times n$ matrix

∇f row vec.

Tangent space $T_p Y = \left\{ v \in \mathbb{A}^n \;\middle|\; \left[\frac{\partial f}{\partial x_j}(p)\right] \cdot v = 0 \right\}$

$\forall f \in I$

$$= \left\{ v \in \mathbb{A}^n \;\middle|\; \left[\frac{\partial f_i}{\partial x_j}(p)\right] \cdot v = 0 \right\}$$

p non-sing $\iff$ rank $\overset{\text{Jac. matrix}}{J(p)} = n-r \underset{\text{rank-nullity thm.}}{\iff} \dim T_p Y = r.$

Last time: Comm. alg. desc. of non-sing.

<u>Thm</u> (Thm. 5.1 in Hartshorne I, Sec. 5)

(Zariski ?)

$p \in Y$ is non-sing. $\iff$ $\mathcal{O}_p = \mathcal{O}_{p,Y}$ is a <u>regular local ring</u>.

(with m = max ideal)

<u>Recall</u> R local ring is regular if

$$\underbrace{\dim_k m/m^2}_{\text{v.s. dim.}} = \underbrace{\dim R}_{\text{Krull dim.}} \qquad R/m = k \text{ res. field}$$

<u>Rem</u> We will see R regular $\iff$ m is gen. as an ideal by exactly $\dim R$ many elements.

<u>Rem/exercise</u> Y affine var., $R = k[Y]$

$p \in Y$, $\mathcal{O}_p$ = local ring $= R_m$, m = max. ideal of p in R

$n = mR_m$

One shows (Lemma 1.15 p.18 Milne)

$$k = R/m \cong R_m/mR_m \quad \& \quad m/m^2 \cong n/n^2 \quad \cdots \quad m^k/m^{k+1} \cong n^k/n^{k+1}$$

<u>Proof of Thm.</u> $Y \subset \mathbb{A}^n$

$p = (a_1, \dots, a_n) \in \mathbb{A}^n$ WLOG $p = (0, \dots, 0)$.

$\mathfrak{a}_p$ = max. ideal of p in $k[x_1 \cdots x_n] = (x_1, \dots, x_n)$

$$\theta : \mathfrak{a}_p \longrightarrow k^n$$

$$\theta(f) = \left(\frac{\partial f}{\partial x_1}(0), \dots, \frac{\partial f}{\partial x_n}(0) \right).$$

$$\theta' : \mathfrak{a}_p / \mathfrak{a}_p^2 \xrightarrow{\ \cong\ } k^n$$

$$\frac{\text{poly. without const. term}}{\text{poly. } \text{''} \quad \text{''} \ \& \text{ lin. terms}} \cong \text{all linear poly.}$$

$\mathfrak{b} = I(Y) \subset k[x_1, \dots, x_n]$ $\qquad J(p) = \left[\partial f_i / \partial x_j (p) \right]$

$\quad = (f_1, \dots, f_t)$

$$\boxed{\operatorname{rank}(J(p))} = \dim \theta(\mathfrak{b}) \subset k^n$$

$$= \boxed{\dim \left(\mathfrak{b} + \mathfrak{a}_p^2 \middle/ \mathfrak{a}_p^2 \right)} \subset \mathfrak{a}_p / \mathfrak{a}_p^2$$

$$\begin{tikzcd}
\mathfrak{a}_p / \mathfrak{a}_p^2 \arrow[r, "\cong"] & k^n \\
\mathfrak{b} \arrow[u] \arrow[ur, "\theta"] \arrow[r, hook] & \mathfrak{a}_p \arrow[u, "\theta"']
\end{tikzcd}$$

$$\mathfrak{m} = \text{max. ideal of } \mathcal{O}_p = \Big(\overbrace{k[x_1 \cdots x_n] / \mathfrak{b}}^{k[Y]} \Big)_{\mathfrak{a}'_p}$$

$$\boxed{\mathfrak{m} / \mathfrak{m}^2 \cong \mathfrak{a}'_p / \mathfrak{a}'^2_p \cong \mathfrak{a}_p \middle/ \mathfrak{b} + \mathfrak{a}_p^2}$$

$$\underbrace{\dim \mathfrak{a}_p \middle/ \mathfrak{b} + \mathfrak{a}_p^2} + \underbrace{\dim \mathfrak{b} + \mathfrak{a}_p^2 \middle/ \mathfrak{a}_p^2} = \dim \mathfrak{a}_p / \mathfrak{a}_p^2 = n$$

$$\boxed{\dim \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rank} J(p) = n.}$$

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rank} J(p) = n \quad \rightsquigarrow \text{always}$$

$$\boxed{\operatorname{rank} J(p) = n - r} \iff \boxed{\dim_k \mathfrak{m}/\mathfrak{m}^2 = r}$$

p non-sing. $\qquad\qquad$ $\mathcal{O}_p$ reg. local ring

☺

Rem ① One knows that $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$ $\rightsquigarrow$ local ring

(next time).

It follows from proof of thm. that

$$\operatorname{rank} J(p) \leq n - r.$$

② From proof one also sees that

$$\mathfrak{m}/\mathfrak{m}^2 \cong \text{null space of } J(p) =: T_p Y.$$

alg. def. of dual tangent space $T_p^* Y$

Thm (Thm. 5.3 in Hartshorne I, Sec. 5)

Y_{sing} = set of all sing. pts of Y

is a proper closed set.

(or $Y_{non\text{-}sing} \subset Y$ open & non-empty).

proof WLOG Y affine irr. variety.

$\forall p \in Y \qquad r = \dim Y$

$\operatorname{rank} J(p) \leq n - r$

$Y_{sing} = \{ p \in Y \mid \operatorname{rank} J(p) < n-r \}$

= alg. set defined by ideal gen. by all $(n-r)\times(n-r)$ $\underbrace{\text{sub-det.}}_{\text{minors}}$ of $J(p)$.

(Linear alg. : $A_{t\times n}$ matrix, $\operatorname{rank} A < k \iff$ all $k\times k$ minor vanish)

So Y_{sing} closed ✓ ☺

Remains to show $Y_{sing} \neq Y$.

We use the following fact: (to be proven)

<u>Thm.</u> $\forall$ irr. variety Y is $\underbrace{\text{birationally iso.}}_{\text{birational}}$ to a hypersurface $H \subset \mathbb{A}^{r+1}$.

($X \cong_{bir.} Y \iff \exists U \subset X$ open $\exists V \subset Y$ open $U \cong V$ as varieties)

($X \cong_{bir.} Y \iff k(X) \cong k(Y)$ as k-alg. fields over k)

→ Follows from Field theory statement "Primitive Element" Thm.

L/K finite ext. char $=0$ (or separable ext.) $\Longrightarrow L = K(\alpha)$ $\exists \alpha \in L$

WLOG we can assume $Y = H \subset \mathbb{A}^{r+1}$ defined by $f(x_1, \dots, x_{r+1}) = 0$ irr. poly.

$$Y_{sing} = \left\{ p \in \mathbb{A} \;\middle|\; \begin{matrix} f(p) = 0 \\ \nabla f(p) = 0 \end{matrix} \right\}$$

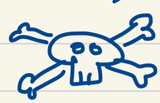
If $Y = Y_{sing.}$ $\Longrightarrow$ $\dfrac{\partial f}{\partial x_j} \in (f) = I(Y)$ $\forall j$

But $\deg \dfrac{\partial f}{\partial x_j} = \deg f - 1$ so $\dfrac{\partial f}{\partial x_j} \equiv 0$.

- If char $k = 0$, not possible (f is not const.)
- If char $k = p > 0$ $\dfrac{\partial}{\partial x}(x^p) = p\,x^{p-1} = 0$.

$$\frac{\partial f}{\partial x_j} = 0 \quad \forall j = 1 \dots r+1 \Longrightarrow f = g(x_1^p, \dots, x_{r+1}^p)$$
$$= g(x_1, \dots, x_{r+1})^p \rightsquigarrow$$

f not irr.



- We (= mathematicians) dislike singular points! because we can not do diff. geo./calculus.

Y variety $\rightsquigarrow$ You can partition it into smooth subvar.

$$Y_0 \supset Y_1 \supset Y_2 \supset \dots \supset Y_k \qquad k \le \dim Y$$

$Y_0 = Y_{non\text{-}sing}$ $\qquad Y_i := (Y_{i-1})_{sing.}$

Review & Completion of material about $\overset{\text{iso.}}{\text{birat.}}$

Hartshorne I. Sec. 4

- Recall X, Y affine varieties

$$\underset{\text{morphism}}{\varphi : X \longrightarrow Y} \quad \xleftrightarrow{1-1} \quad \underset{k\text{-alg. homo}\underline{\text{ism}}}{F : k[Y] \longrightarrow k[X]}$$

(Moreover, Y affine & X any var. :

$$\underset{\text{morphism}}{\varphi : X \longrightarrow Y} \quad \xleftrightarrow{1-1} \quad \underset{k\text{-alg. homo}\underline{\text{ism}}}{F : k[Y] \longrightarrow \mathcal{O}(X)}$$
)

<u>Thm</u> (Thm 4.4 in Hartshorne)

X, Y irr. var.

$$\varphi : X \dashrightarrow Y \quad \xleftrightarrow{1-1} \quad \underset{k\text{-alg. homo}\underline{\text{ism}}}{F : k(Y) \hookrightarrow k(X)}$$

dominant rat. map

$\Updownarrow$

$\operatorname{Im} \varphi$ dense in Y

Next time we finish this & thm. about

$\forall\, Y \underset{\text{bir.}}{\cong}$ hypersurface.