Sep 8 / 2017

Goal: Integrating $\dfrac{P(x)}{Q(x)}$ P, Q polynomials

simple Ex. $\displaystyle\int \frac{1}{x^2-1}\,dx \rightsquigarrow \boxed{\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} =}$

$$\boxed{= \frac{A}{x-1} + \frac{B}{x+1}}$$ A, B constants

Ex. $\displaystyle\int \frac{1}{x^2+1}\,dx \rightsquigarrow \big(\underbrace{\tan^{-1}(x)}_{\arctan(x)}\big)' = \frac{1}{x^2+1}$

$= \tan^{-1}(x)$

$$1 = (x^2-1)\left(\frac{A}{x-1} + \frac{B}{x+1}\right) = A(x+1) + B(x-1)$$

$$1 = \underbrace{(A+B)}_{0}x + \underbrace{(A-B)}_{1}$$

$\left.\begin{aligned} A+B &= 0 \\ A-B &= 1 \end{aligned}\right\}$ $A = \frac{1}{2}$, $B = -\frac{1}{2}$

$$\int \frac{1}{x^2-1}\,dx = \underbrace{\int \frac{(1/2)\,dx}{x-1}} - \int \frac{(\frac{1}{2})}{x+1}\,dx$$

$$= \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + C$$

Last time

$\displaystyle\int \frac{P(x)}{Q(x)}\,dx \longrightarrow$ If $\deg P \geq \deg Q$ then we do long division of poly.

Recall $P(x) = \underbrace{S(x)}_{\text{quotient}}\,Q(x) + \underbrace{R(x)}_{\text{remainder}}$ $\deg R < \deg Q$

$$\int \left(S(x) + \frac{R(x)}{Q(x)}\right) dx$$

#7 in 6.3

Ex. $\displaystyle\int \frac{x^4}{x-1}\,dx = \int (x^3+x^2+x+1)\,dx + \int \frac{1}{x-1}\,dx$

$x^4 = (x^3+x^2+x+1)(x-1) + 1$

$\displaystyle \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln(x-1) + C$

#9 6.2

Ex. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx \rightsquigarrow \frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$

A, B Constants.

Find A & B :

$5x+1 = (x-1)A + (2x+1)B$

$x=1$

$6 = 3B$

$\boxed{B=2}$

$x = -\frac{1}{2}$

$-\frac{5}{2}+1 = -\frac{3}{2}A$

$\frac{-3}{2} = \left(\frac{-3}{2}\right)A$

$\boxed{1=A}$ ☺

$\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx =$

$\displaystyle\int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$

$\frac{1}{2}\ln(2x+1) + 2\ln(x-1) + C$

General form of partial fractions

Case (1)

- $Q(x) = q_1(x) \cdots q_k(x)$

q_i's distinct linear poly.

Then P.F.T. Claims that for every $P_{(x)}$ $\deg P < \deg Q$

we can write:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{q_1(x)} + \cdots + \frac{A_k}{q_k(x)}$$

for some Constants $A_1, \ldots, A_k$.

<u>Case</u> 2 $\quad Q(x) = (q(x))^r \qquad q(x)$ linear poly.

Ex. $\quad Q(x) = (x+1)^3$

Then $\displaystyle \frac{P(x)}{Q(x)} = \frac{A_1}{q(x)} + \frac{A_2}{(q(x))^2} + \cdots + \frac{A_r}{(q(x))^r}$

for some Const. $A_1, \ldots, A_r$.

$\rightarrow$ #15 6.3

<u>Ex.</u> $\displaystyle \int \frac{2x+3}{(x+1)^2}\,dx \qquad \frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

$2x+3 = (x+1)A + B \rightsquigarrow x = -1 \quad B = 1$

$A = 2$

$\displaystyle \int = \int \frac{2}{x+1}dx + \int \frac{1}{(x+1)^2} = 2\ln(x+1) + -(x+1)^{-1} + C$

$\hookrightarrow$ Alternative way to solve for A & B:

$2x+3 = Ax + (A+B) \rightsquigarrow A = 2$

$A+B = 3 \rightsquigarrow B = 1.$

#19 6.3

Ex. $$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2+1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3).$$

.... $\rightsquigarrow$ $A = 10 \rightsquigarrow x = 3$

$B = -9$

$C = -5 \rightsquigarrow x = 2$

Case 3 If after factoring $Q(x)$ we have quadratic factors.

#22 6.3

Ex.

$$\frac{x^2-x+6}{x^3+3x} = \frac{x^2-x+6}{x(x^2+3)}$$

linear poly.

P.F.T. Claims:

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$x^2-x+6 = A(x^2+3) + (Bx+C)x.$$

$\rightsquigarrow x=0$ $\quad 6 = 3A \rightsquigarrow \boxed{A = 2}$

$$x^2-x+6 = 2x^2+6 + Bx^2 + Cx$$

$$1.x^2 - x + 6 = (2+B)x^2 + Cx + 6 \rightsquigarrow \boxed{C=-1} \quad \boxed{B=-1}$$

☺

Aside

Fundamental thm. of algebra

$Q(x)$ = product of linear or quadratic poly.

$\underbrace{Q}_{\text{poly. with real Coeff.}}$

poly. with
real Coeff.

over Complex numbers:

$Q(x)$ = product of linear polynomials.