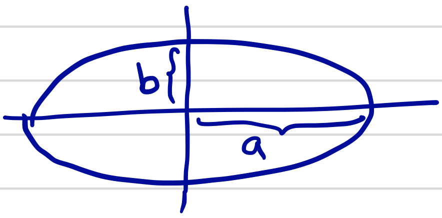
Sep. 6 / 2017

Rem $\int \tan(x)\,dx$

In a similar way one compute $\int \sec(x)\,dx$ using substitution $u = \sec(x) + \tan(x)$. ...

Last time: $\int \sqrt{a^2 - x^2}\,dx$ $\quad a^2 \rightsquigarrow$ some positive const.

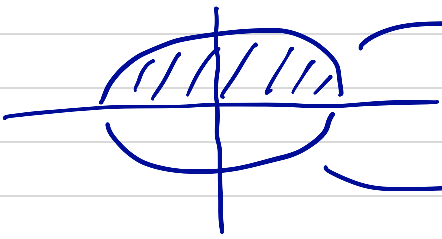


Area of ellipse

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1$$

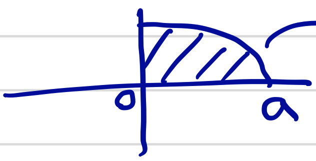
(x,y) lies on ellipse if $\rightsquigarrow \left(\frac{x}{a}, \frac{y}{b}\right)$ lies on unit circle

Area $= \pi a b$.



$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$y = -\frac{b}{a}\sqrt{a^2 - x^2}$$



$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Area} = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$\longrightarrow$ we use (inverse) trig sub.

$dx = a\cos\theta\, d\theta$

Integration fairy tells me to use $x = a\sin(\theta)$

new variable

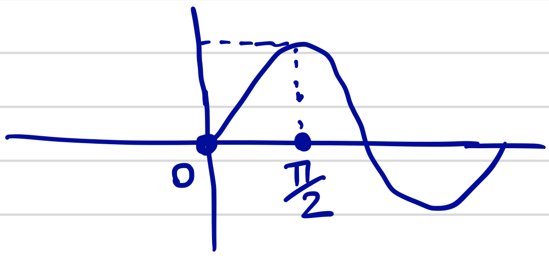
rewrite in terms of θ

$$\int_{x=0}^{a} \sqrt{a^2-x^2}\,dx = \int_{\theta=0}^{\pi/2} \left(\sqrt{a^2-a^2\sin^2\theta}\right) a\cos\theta \, d\theta .$$

$$0 \leq x \leq a \rightsquigarrow 0 \leq a\sin(\theta) \leq a$$

$$0 \leq \sin(\theta) \leq 1$$

$$0 \leq \theta \leq \pi/2$$



$$a\int_{0}^{\pi/2} \sqrt{a^2(1-\sin^2\theta)} \quad \cos\theta \quad d\theta$$

$$a^2 \int_{0}^{\pi/2} (\cos\theta)^2 d\theta = a^2 \int_{0}^{\pi/2} \frac{1}{2}\left(1+\cos 2\theta\right) d\theta$$

$$= \dots$$

$$\text{Area} = \frac{4b}{a} \cdot \text{integral} = \frac{4b}{a} \cdot \frac{a^2}{2}\left(\frac{\pi}{2}\right)$$

$$= \pi ab .$$

In general, we usually use the following substitutions:

$$\sqrt{a^2-x^2} \rightsquigarrow x = a\sin(\theta) \quad (\text{or } = a\cos\theta)$$

$$\dashrightarrow a\cos(\theta)$$

$-\infty < x < \infty \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

- $\sqrt{a^2+x^2} \rightsquigarrow x = a\tan(\theta)$

$$\sqrt{a^2 + a^2\tan^2(\theta)} = a\sqrt{1+\tan^2(\theta)}$$

$$= a\,|\sec(\theta)|$$

- $\sqrt{x^2-a^2} \rightsquigarrow x = a\sec(\theta)$

$$\sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1}$$

$$= a\sqrt{(\tan(\theta))^2}$$

$$= a\,|\tan(\theta)|$$

Ex. $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ (can be done without trig sub. also)

$x = \tan(\theta) \qquad dx = \sec^2(\theta)\,d\theta$

$$\int \frac{\tan^3(\theta)}{\cancel{\sec(\theta)}} \cdot \sec^{\cancel{2}}(\theta)\,d\theta = \int \frac{\sin^3\theta}{\cos^3\theta}\cdot\frac{1}{\cos\theta}\,d\theta$$

$$= \int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta = \int \frac{(1-\cos^2\theta)}{\cos^4\theta}\,\sin\theta\,d\theta$$

$u = \cos\theta$

$$u = \cos\theta$$
$$du = -\sin\theta\, d\theta$$

$$\ldots = \int \frac{1-\cos^2\theta}{\cos^4\theta} \underbrace{\sin\theta\, d\theta}_{-du}$$

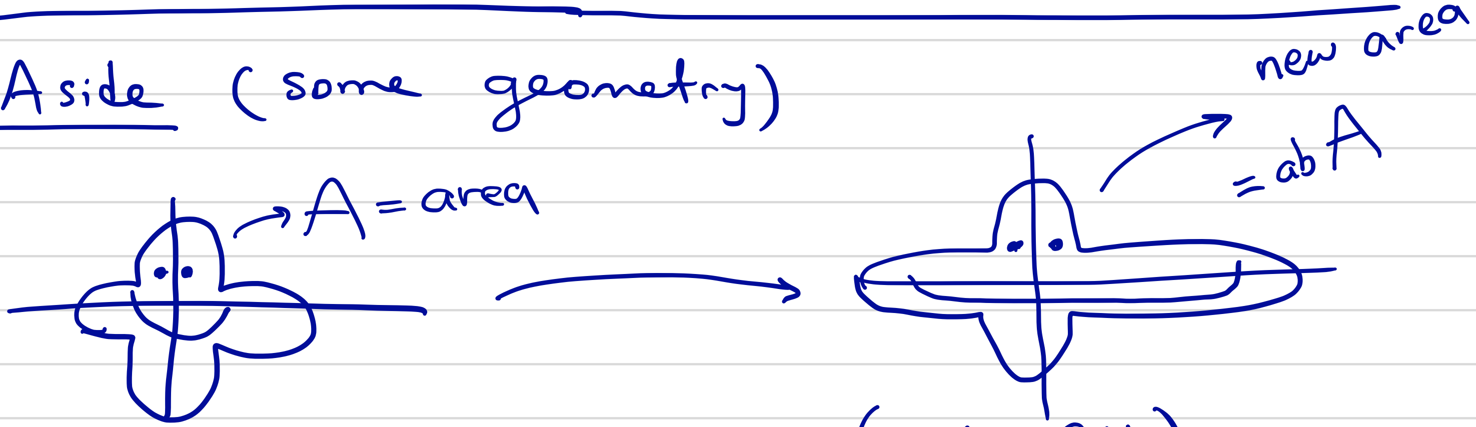
$$= -\int \frac{1-u^2}{u^4}\, du = -\left(\frac{u^{-3}}{-3} - \frac{u^{-1}}{-1}\right)$$

$$= \frac{u^{-3}}{3} - u^{-1}.$$

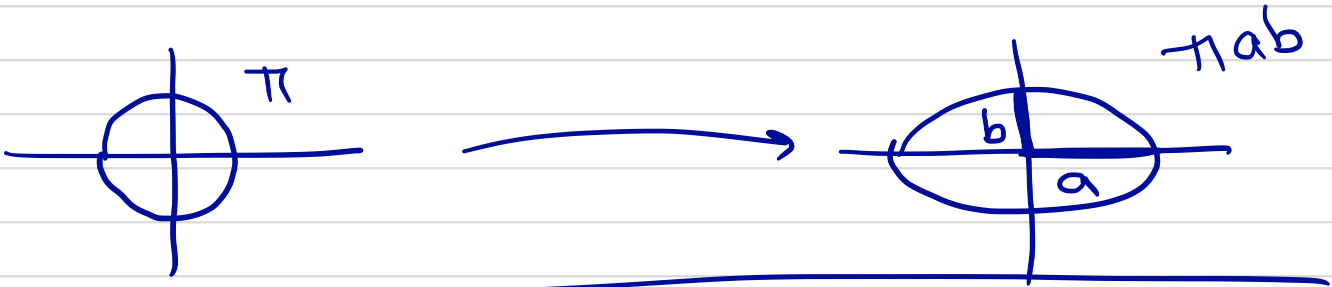
$$= \frac{(\cos\theta)^{-3}}{3} - \frac{1}{\cos\theta}$$

(rewrite back in terms of $x = \tan(\theta)$ ...)

Aside (some geometry)



$(x, y) \longmapsto (ax, ay)$



6.3 Partial fractions

- Some alg. which helps to find integral of

$\frac{\text{poly.}}{\text{poly.}}$ i.e. $\frac{P(x)}{Q(x)}$.

Ex. $\int (x^2+1)\,dx \rightsquigarrow$  piece of cake

$\frac{x^3}{3} + x + C$

Ex. $\int \frac{x^2+1}{x-3}\,dx \rightsquigarrow$ do long division of poly.

$(x^2+1) \div (x-3)$

$$\begin{array}{r|l} x^2+1 & x-3 \\ \cline{2-2} x^2-3x & x+3 \\ \hline 3x+1 & \\ 3x-9 & \\ \hline 10 & \end{array}$$

$\rightsquigarrow$ $x^2+1 = (x-3)(x+3) + 10$

$x^2+1 = x^2-9+10$

$$\int \frac{(x-3)(x+3)+10}{(x-3)}\,dx = \underbrace{\int (x+3)\,dx}_{\frac{x^2}{2}+3x} + \int \frac{10}{x-3}\,dx$$

$\frac{x^2}{2} + 3x + 10\ln(x-3) + C$

. Next time $\int \frac{P(x)}{Q(x)}\,dx$ $\deg P < \deg Q$.