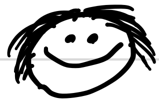
Sep 13 / 2017 

Improper integrals $\longrightarrow$ when x or y is infinite.

1. $\displaystyle\int_a^{\infty} f(x)\,dx$ or $\displaystyle\int_{-\infty}^{b} f(x)\,dx$ or $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

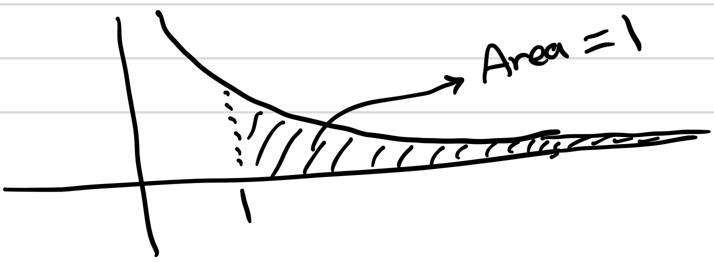
2. $\displaystyle\int_a^{b} f(x)\,dx$ but $\displaystyle\lim_{x\to a^+} f(x) = \pm\infty$ or $\displaystyle\lim_{x\to b^-} f(x) = \pm\infty$

- To compute improper integral: anti-derivative + Fundamental thm. of Calc. + take a limit.

- Comparison test.

we can find anti-derivative of $f(x)$ but
- - - - - - - - $g(x)$ such that $g(x) \geq f(x)$ &
$\int g(x)\,dx$ we can compute. we then use this to decide if $\displaystyle\int_a^{\infty} f(x)\,dx$ is finite or infinite.

Ex. $f(x) = \dfrac{1}{x^2} \longrightarrow \displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$



- Anti-derivative: $\displaystyle\int \frac{1}{x^2}\,dx = \frac{x^{-2+1}}{-2+1} + C$

$$= -\frac{1}{x} + C$$

- F.T.C. $\displaystyle\int_1^b \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^b = \left(-\frac{1}{b}\right) - (-1).$

$b > 1$

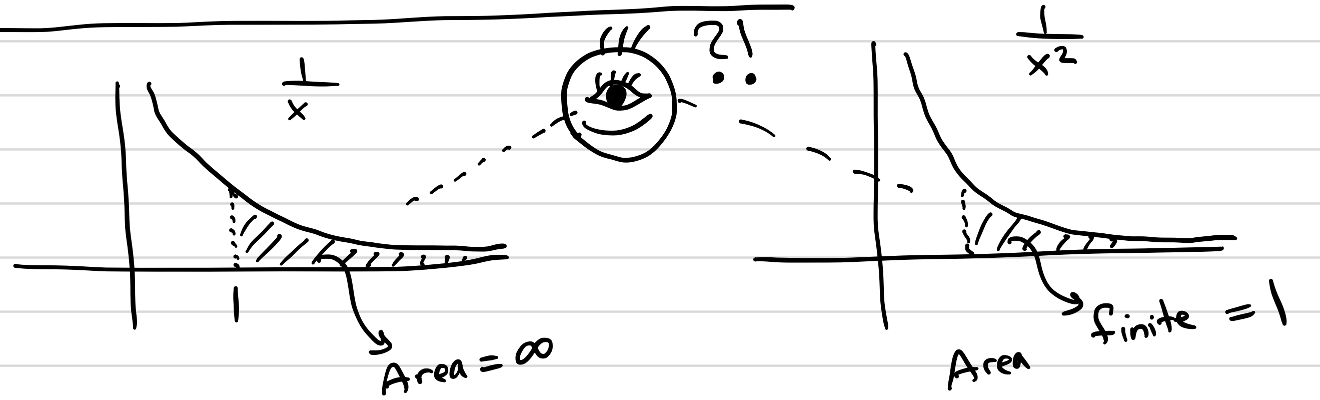
$$= 1 - \frac{1}{b}.$$

- Take limit as $b \longrightarrow +\infty$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} 1 - \frac{1}{b} = 1$$

<u>Ex.</u> $f(x) = \frac{1}{x}$ $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \underbrace{\left(\ln x \Big|_1^b\right)}_{\ln(b) - 0}$

$$= \lim_{b\to\infty} \ln(b) = +\infty$$



- Related to the series:

$1 + \frac{1}{2} + \frac{1}{3} + \cdots = +\infty$ $\qquad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

finite number

Aside: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$?!

$4 = 2^2$, $9 = 3^2$

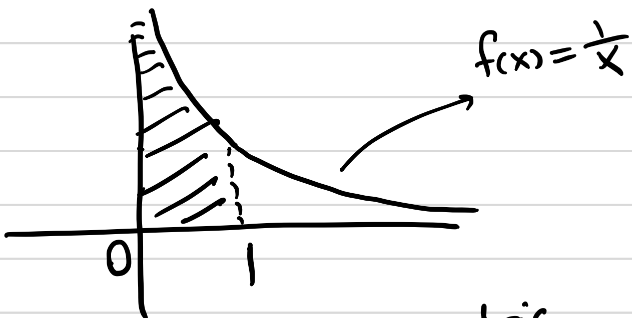
(discovered by Euler).

<u>Ex.</u> $\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ Decide if the improper integral $\boxed{\text{finite}}$ ✓ or infinite?

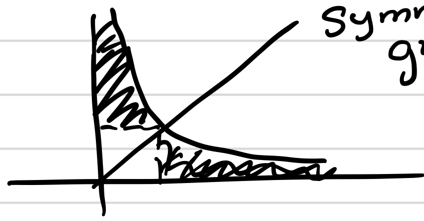
$$0 < \frac{\sin^2(x)}{x^2} \leq \frac{1}{x^2} \rightsquigarrow \underbrace{\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx}_{\text{finite}} \leq \int_1^\infty \frac{1}{x^2}\,dx = 1$$

Terminology: If $\int \ldots$ is finite we say it is "convergent". If $\int \ldots$ is infinite we say "divergent".

<u>Ex.</u> $\displaystyle\int_0^1 \frac{1}{x}\,dx \longrightarrow$



$\displaystyle\lim_{x\to 0^+} \frac{1}{x} = \infty$



symmetric graph

$$\lim_{a\to 0^+}\left(\ln(x)\Big|_a^1\right) = \lim_{a\to 0^+}\left(\ln(1) - \ln(a)\right) = +\infty$$

divergent

$$a = e^{\to -\infty},\ a \to 0^+ \qquad \lim_{a\to 0^+} \ln(a) = -\infty.$$

General fact

- $0 < p$ fixed.

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \nearrow \text{Convergent} & \text{if } 1 < p \\ \searrow \text{divergent} & 0 < p \leq 1 \end{cases}$$

Ex. $\displaystyle\int_0^3 \frac{1}{x^5}\,dx \rightsquigarrow \lim_{a\to 0^+} \left(\frac{x^{-4}}{-4} \Big|_a^3 \right) =$

$$= -\frac{1}{4 \cdot 3^4} - \left(-\frac{1}{4 \cdot a^4} \right)$$

$$= \frac{1}{4a^4} - \frac{1}{4 \cdot 3^4}$$

$\displaystyle\lim_{a\to 0} = +\infty$.

Ex. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$\int \frac{1}{\sqrt{x}}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2\sqrt{x}$$

$\displaystyle\lim_{a\to 0^+} (2 - 2\sqrt{a}) = 2$ ☺ convergent.

Ex. Is $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x}}\,dx$ Conv. or div. ?

$$= \int_2^\infty \frac{1}{\sqrt{u}}\,du$$ divergent

Ex. (You may remember $\int e^{-x^2}\,dx$ can not be computed).

Decide if $\int_1^{\infty} e^{-x^2}\,dx$ is div. or conv.

$$0 < \underbrace{e^{-x^2}}_{\frac{1}{e^{x^2}}} \leq \underbrace{e^{-x}}_{\frac{1}{e^{x}}}$$

$$x \geq 1 \Rightarrow x^2 \geq x$$

$$-x^2 \leq -x$$

$$e^{-x^2} \leq e^{-x}.$$

$$0 \leq \int_1^{\infty} e^{-x^2}\,dx \leq \int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty}$$

$$= 0 + \frac{1}{e}.$$