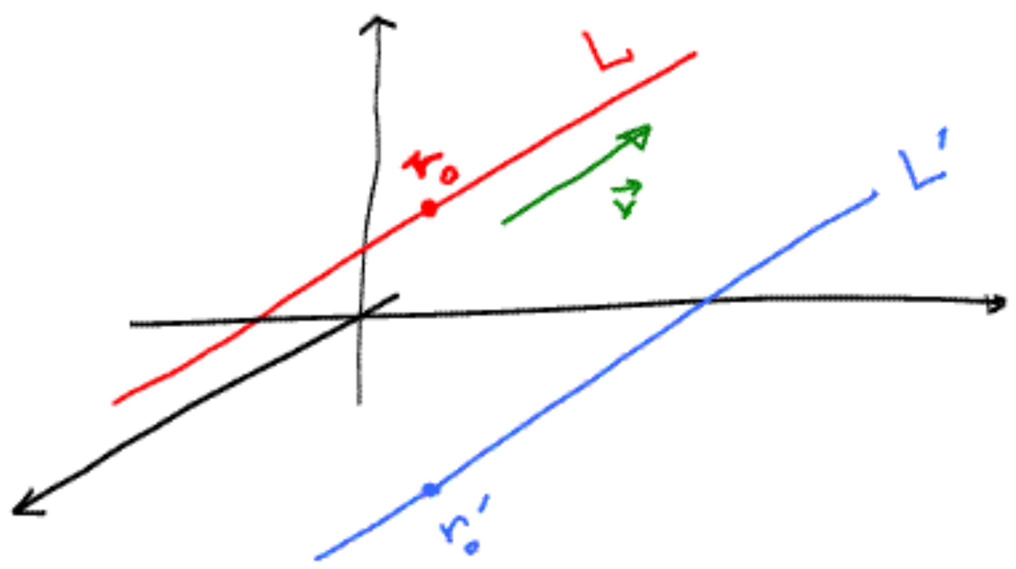
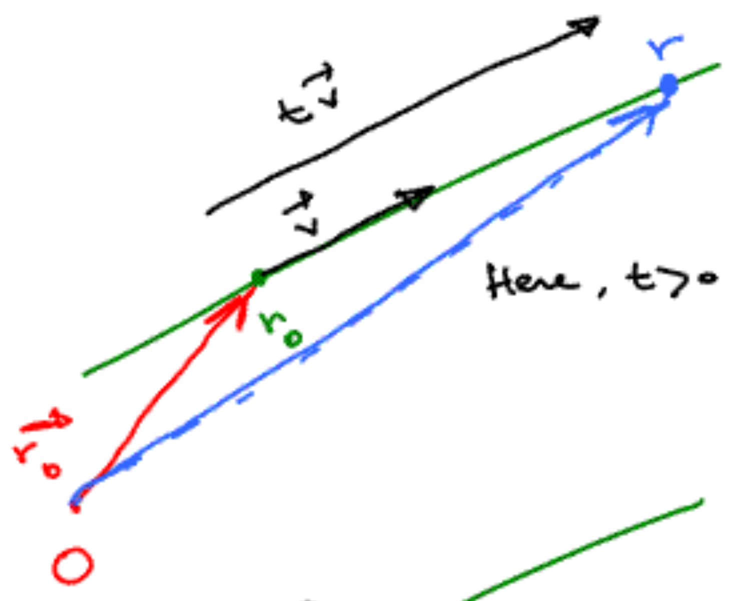
Calc 3, Session 4, 09/04/2014



Identify algebraically the points on the line L, which is parallel to the given vector $\vec{v}$ and passes through the point r_0.

$\vec{v} = \langle a,b,c \rangle$ $\quad$ $r_0 = (x_0, y_0, z_0)$ $\quad$ $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$

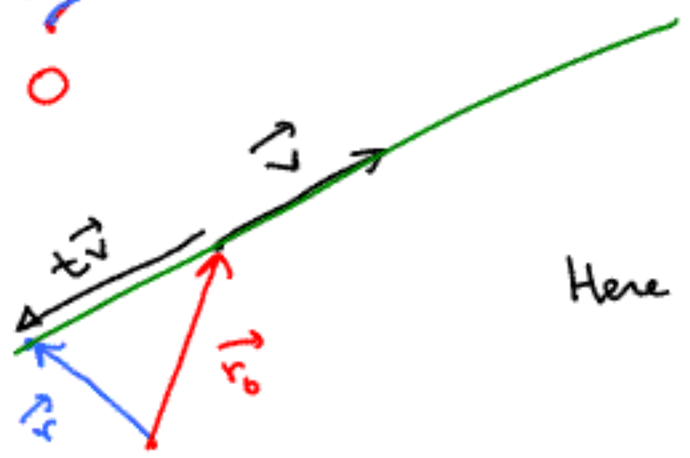
$r = (x, y, z)$ $\quad$ $\vec{r} = \langle x, y, z \rangle$



Here, $t>0$

It follows

$$\underline{\underline{\vec{r} = \vec{r_0} + t\vec{v}}}$$ where t is a number.



Here, $t<0$

$\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ is the vector equation OR the parametric equation of the line L. Each point on the line corresponds to a real number t & vice-versa.

$\vec{r} = \langle x,y,z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, $\vec{v} = \langle a,b,c \rangle$

$\langle x,y,z \rangle = \langle x_0,y_0,z_0 \rangle + t\langle a,b,c \rangle$ - - - -

$= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$

$$\boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}}$$ Scalar system of parametric equations for the line L.

<u>Example</u> Find the parametric equation of the line L parallel to the vector $\langle 2,-1,3 \rangle$ and passing through the point $(1,0,2)$.

Find two other points lying on this line.

<u>ANSWER:</u>

$$\underbrace{\langle x,y,z \rangle}_{\vec{r}} = \underbrace{\langle 1,0,2 \rangle}_{\vec{r_0}} + t\,\underbrace{\langle 2,-1,3 \rangle}_{\vec{v}} \quad \text{OR} \quad \begin{cases} x = 1+2t \\ y = -t \\ z = 2+3t \end{cases}$$

For $t=1$, $\langle x,y,z \rangle = \langle 3,-1,5 \rangle$ $(3,-1,5)$ lies on the line.

For $t=-2$, $\langle x,y,z\rangle = \langle -3,2,-4\rangle$ $(-3,2,-4)$ lies on the line.

Question: Is the point $(9,7,-1)$ on the line L?

Answer: We check, is there a number t such that

$$\langle 9,7,-1\rangle = \langle 1,0,2\rangle + t\langle 2,-1,3\rangle$$

$$\Rightarrow \langle 9,7,-1\rangle - \langle 1,0,2\rangle = t\langle 2,-1,3\rangle$$

$$\langle 8,7,-3\rangle = t\langle 2,-1,3\rangle$$

$\Rightarrow$ No such t exists since $\langle 8,7,-3\rangle$ is not a multiple of $\langle 2,-1,3\rangle$.

Conclusion: The point $(9,7,-1)$ is NOT on the line.

Symmetric equations of the line.

Let L be the line with the scalar parametric equations

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

Assume that $a,b,c \neq 0$ (a,b,c are called direction numbers of the line. $\vec{v} = \langle a,b,c\rangle$)

Then $x = x_0 + ta \Rightarrow x - x_0 = ta \Rightarrow t = \dfrac{x-x_0}{a}$

$y = y_0 + tb \Rightarrow \quad \cdots \quad \Rightarrow t = \dfrac{y-y_0}{b}$

$z = z_0 + tc \Rightarrow \quad \cdots \quad \Rightarrow t = \dfrac{z-z_0}{c}$

Parametric equation is satisfied iff

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Example above $\langle a,b,c\rangle = \langle 2,-1,3\rangle$, $(x_0,y_0,z_0) = (1,0,2)$

Check the symmetric equations for $(9,7,-1)$:

$$\frac{9-1}{2} = \frac{7-0}{-1} = \frac{-1-2}{3} \quad ?$$

OBVIOUSLY NOT $\Rightarrow$ $(9,7,-1)$ is not on the line.

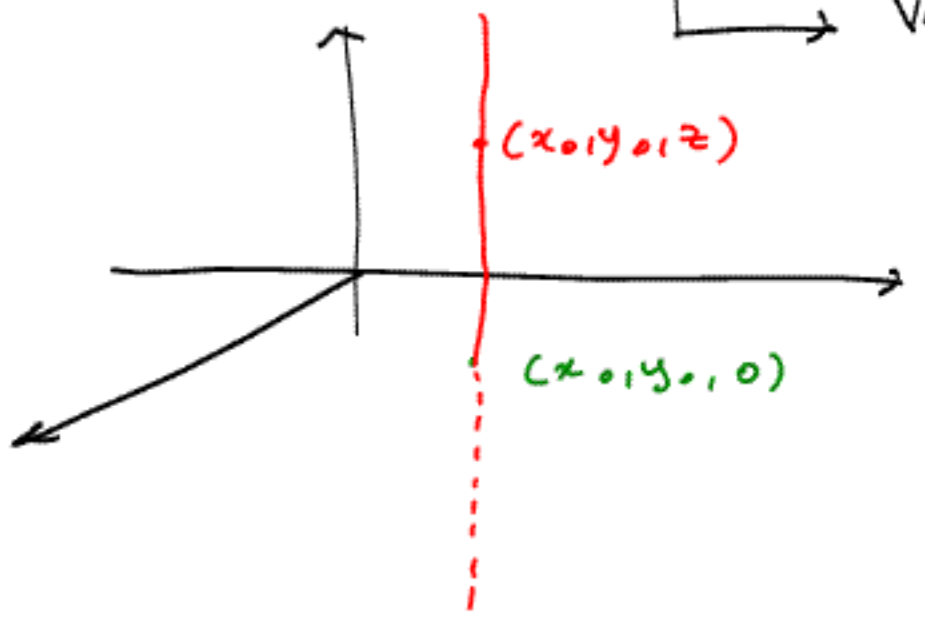
What if, say, $a = 0$, $b, c \neq 0$

$$\left.\begin{array}{l} x = x_0 + ta = x_0 \\ y = y_0 + tb \quad \rightarrow \quad t = \dfrac{y - y_0}{b} \\ z = z_0 + tc \quad \rightarrow \quad t = \dfrac{z - z_0}{c} \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x = x_0 \\ \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c} \end{array}}$$

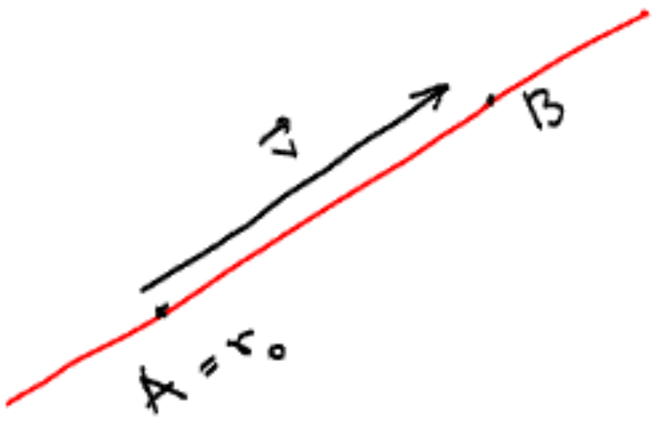
If, say, $a = b = 0$, $c \neq 0$

$$\begin{array}{l} x = x_0 \\ y = y_0 \\ t = \dfrac{z - z_0}{c} \end{array} \Rightarrow \boxed{\begin{array}{l} x = x_0 \\ y = y_0 \end{array}}$$

$\rightarrow$ Vertical line



Example: Find the equation of the line passing through the points $A(1,3,2)$ and $B(3,5,4)$.



$\vec{r_0} = \langle 1, 3, 2 \rangle$

$\vec{v} = \langle 3, 5, 4 \rangle - \langle 1, 3, 2 \rangle = \langle 2, 2, 2 \rangle$

$$\left\{\begin{array}{l} x = 1 + 2t \\ y = 3 + 2t \\ z = 2 + 2t \end{array}\right. \quad \text{or} \quad \boxed{\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-2}{2}}$$

$$\boxed{x - 1 = y - 3 = z - 2}$$

Find the coordinate of the intersection of the above line with the $\underbrace{x\text{-}z\text{ - plane}}_{\text{that is } y = 0}$.

$\left\{\begin{array}{l} y = 3 + 2t \\ y = 0 \end{array}\right. \Rightarrow 0 = 3 + 2t \Rightarrow$ the parameter for that point $= t = \frac{-3}{2}$

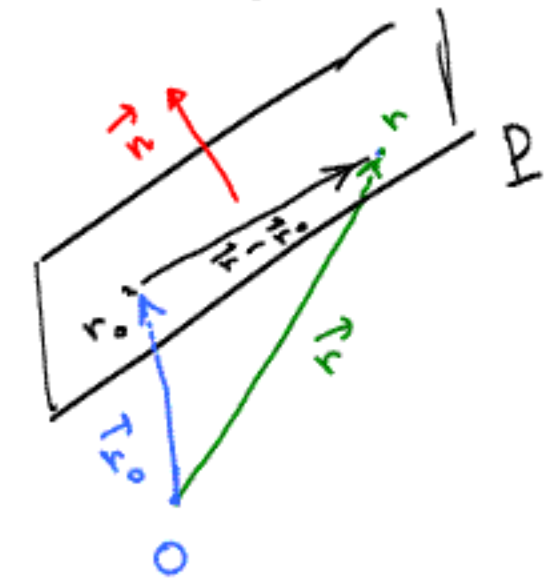
. Plug in for x & z : $\quad x = 1 + 2t = 1 + 2\left(\frac{-3}{2}\right) = 1 - 3 = -2$

$\quad z = 2 + 2t = 2 - 3 = -1$

ANSWER $= (-2, 0, -1)$

Equation of planes

A plane in 3d space is identified by a normal vector $\vec{n}$ and a point r_0 on the plane.



$\vec{r}-\vec{r_0}$ lies on the plane

$\Longleftrightarrow$

$\vec{r}-\vec{r_0}$ is orthogonal to $\vec{n}$

$\Longleftrightarrow$

Vector equation for the plane ———— $\boxed{(\vec{r}-\vec{r_0})\cdot\vec{n} = 0}$

$$(\vec{r}-\vec{r_0})\cdot\vec{n} = \vec{r}\cdot\vec{n} - \vec{r_0}\cdot\vec{n} = 0$$

$$\Longleftrightarrow \quad \boxed{\vec{r}\cdot\vec{n} = \vec{r_0}\cdot\vec{n}}$$

$\vec{r} = \langle x, y, z\rangle$, $\vec{r_0} = \langle x_0, y_0, z_0\rangle$, $\vec{n} = \langle a, b, c\rangle$ the normal vector

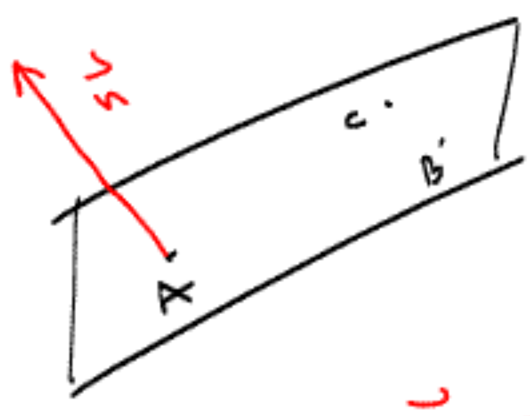
$$\boxed{\langle x, y, z\rangle \cdot \langle a, b, c\rangle = \langle x_0, y_0, z_0\rangle \cdot \langle a, b, c\rangle}$$

OR

$$\boxed{ax + by + cz = ax_0 + by_0 + cz_0}$$ ⟶ Scalar equation of the plane.

(Generic equation for a plane : $ax + by + cz = d$)

Example: Write the scalar & vector equations of the plane formed by the 3 points $(1,0,1)$, $(0,1,1)$ & $(1,1,0)$.
A B C



$$\vec{AB} = \langle 0,1,1\rangle - \langle 1,0,1\rangle = \langle -1, 1, 0\rangle$$

$$\vec{AC} = \langle 1,1,0\rangle - \langle 1,0,1\rangle = \langle 0, 1, -1\rangle$$

$$\vec{n} = \vec{AB} \times \vec{AC} = \langle -1, -1, -1\rangle$$

$$\vec{r_0} = (1,0,1), \quad d = \vec{n}\cdot\vec{r_0} = -1 - 0 - 1 = -2$$

Vector equation $\langle x, y, z\rangle \cdot \langle -1,-1,-1\rangle = \langle 1,0,1\rangle \cdot \langle -1,-1,-1\rangle$

$$\boxed{\langle x, y, z\rangle \cdot \langle -1,-1,-1\rangle = -2}$$

Scalar equation

All the same equation :

$$\boxed{-x-y-z=-2}$$

OR

$$\boxed{x+y+z=2}$$ → (This is for $\vec{n}=\langle 1,1,1\rangle$)

$$\boxed{x+y+z-2=0}$$

Example : The two planes with equations

$$2x-3y+z+1=0 \qquad P_1$$

and

$$x+y+z-2=0 \qquad P_2$$

are given.

(1) Find the angle between the two planes.

(2) Find the equation of the intersection line.

(1) We find the angle between the two normal vectors of the two planes, which would be the answer.

$\vec{n_1}$ normal to P_1 $=\langle 2,-3,1\rangle$

$\vec{n_2}$ " " P_2 $=\langle 1,1,1\rangle$

$$\cos\theta=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}=\frac{2-3+1}{\sqrt{14}\sqrt{3}}=\frac{0}{\sqrt{3}\sqrt{14}}=0.$$

So the angle $=90°$.

(2) For finding a point r_0 on the line, solve the system

$$\begin{cases}2x-3y+z+1=0\\ x+y+z-2=0\end{cases}$$

Let $z=2$ to get
$$\begin{cases}2x-3y=-3\\ x+y=0 \Rightarrow y=-x\end{cases}$$

(2 equations, 3 unknowns) implies one free unknown

$$-2y-3y=-3 \Rightarrow y=\frac{3}{5}$$

$$x=-\frac{3}{5}$$

$$r_0=\left(-\frac{3}{5},\frac{3}{5},2\right)$$

[If, say we chose any other value for z at the beginning, we'd obtain other points on the line]

To find a vector $\vec{v}$ in the direction of the line, we observe that the intersection line L lies on both planes P_1 & P_2

Therefore, the direction $\vec{v}$ must be parallel to both planes, and hence orthogonal to both directions $\vec{n}_1$ and $\vec{n}_2$.

Therefore we can choose $\boxed{\vec{v} = \vec{n}_1 \times \vec{n}_2}$.

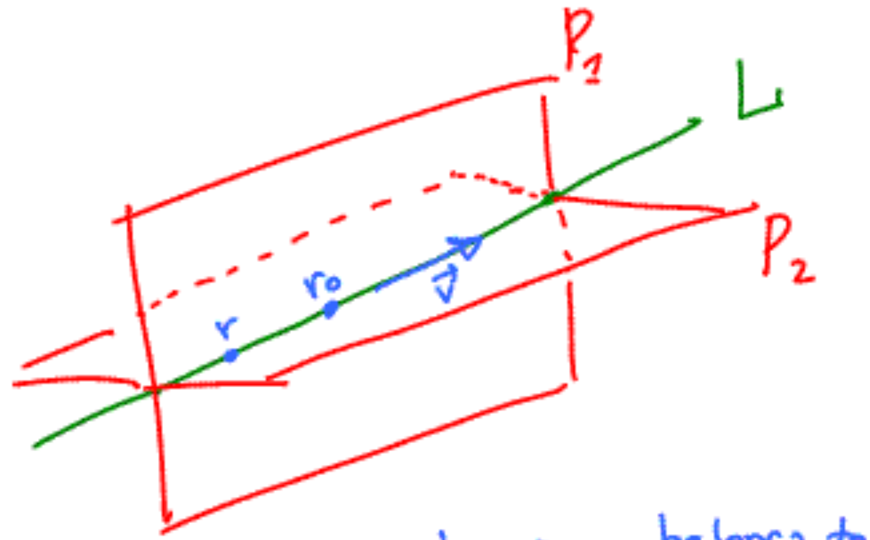
$$\boxed{\vec{v} = \langle 2, -3, 1\rangle \times \langle 1, 1, 1\rangle = \langle -4, -1, 5\rangle}$$

Sym. equations of line passing through $r_0 = \left(-\frac{3}{5}, \frac{3}{5}, 2\right)$ and parallel to $\vec{v} = \langle -4, -1, 5\rangle$ is

$$\frac{x + \frac{3}{5}}{-4} = \frac{y - \frac{3}{5}}{-1} = \frac{z-2}{5}$$

which simplifies to (through algebraic simplification)

$$\boxed{\frac{5x+3}{4} = 5y - 3 = z - 2}$$



Parametric equation of the same line are

$$(L)\begin{cases} x = -\frac{3}{5} - 4t \\ y = \frac{3}{5} - t \\ z = 2 + 5t \end{cases}$$

r belongs to L $\Leftrightarrow$ r belongs to P_1 and P_2

$r = (x, y, z)$

Test: check whether for all t, the solution r of the parametric equations (L) satisfy both $\begin{cases} 2x - 3y + z + 1 = 0 & (P_1) \\ x + y + z - 2 = 0 & (P_2) \end{cases}$?

$$2\left(-\frac{3}{5} - 4t\right) - 3\left(\frac{3}{5} - t\right) + (2 + 5t) + 1 = \left(-\frac{6}{5} - \frac{9}{5} + 2\right) - 8t + 3t + 5t + 1 = 0 \quad \checkmark \text{ ok}$$

$$\left(-\tfrac{3}{5} - 4t\right) + \left(\tfrac{3}{5} - t\right) + (2 + 5t) - 2 = \left(-\tfrac{3}{5} + \tfrac{3}{5} + 2\right) + (-4t - t + 5t) - 2 = 0 \quad \checkmark \text{ ok}$$