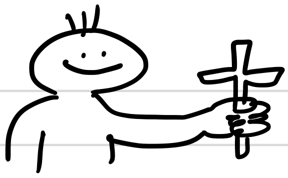
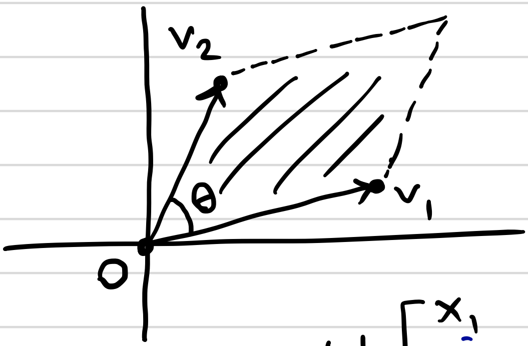
Oct. 4 / 2017



Cross product

2D



$v_1 = (x_1, y_1)$

$v_2 = (x_2, y_2)$

Area of parallelogram

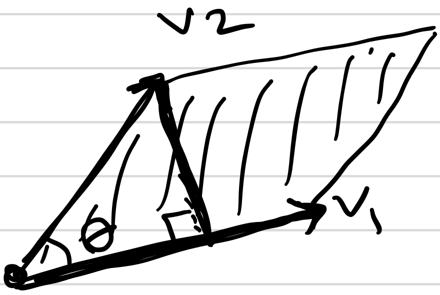
$$= \pm \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \end{matrix}$$

$$\det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = x_1 y_2 - y_1 x_2$$

$$\left(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc . \right)$$

$$\text{Area of parallogram} = \left| \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \right| = |x_1 y_2 - y_1 x_2|$$

$$= \underbrace{|v_1|}_{\text{base}} \underbrace{|v_2| \, |\sin(\theta)|}_{\text{height}}$$



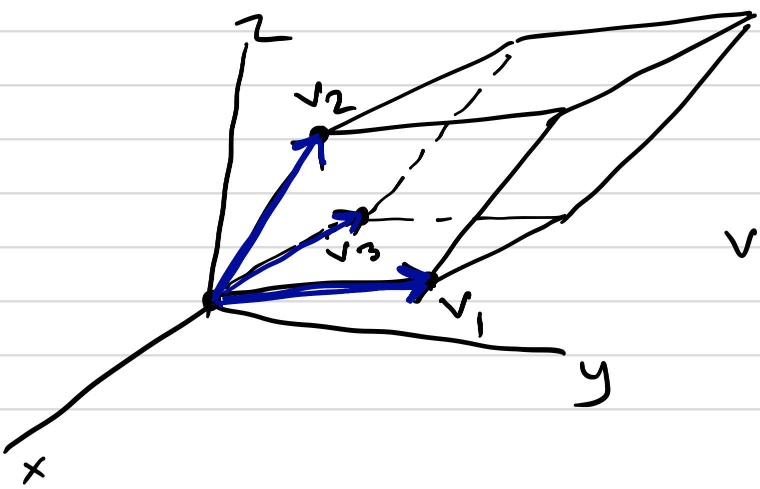
parallelepiped

$v_1 = (x_1, y_1, z_1)$

$v_2 = (x_2, y_2, z_2)$

$v_3 = (x_3, y_3, z_3)$

$$\text{volume} = \left| \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \right|$$



$$\det\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = x_1 \det\begin{bmatrix} y_2 & z_2 \\ y_3 & z_3 \end{bmatrix} - y_1 \det\begin{bmatrix} x_2 & z_2 \\ x_3 & z_3 \end{bmatrix} + z_1 \det\begin{bmatrix} x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}.$$

$$\left(\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right)$$

another notation

Ex.
$$\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \underbrace{1 \det\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{-1} - \underbrace{1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{1} + \underbrace{1 \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{1}$$

$$(-1) + (-1) + 1 = -1.$$

$$\text{Area} = |-1| = 1.$$

☺

<u>Cross product</u> V_1, V_2 3D vectors given.

$$V_1 \times V_2 = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \rightsquigarrow \text{3D vector}$$

means

$$= \underbrace{\det\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}}_{x\text{-Coor.}} \vec{i} + \underbrace{-\det\begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix}}_{y\text{-Coor.}} \vec{j} + \underbrace{\det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}}_{z\text{-Coor.}} \vec{k}$$

One can show the following :

(1)

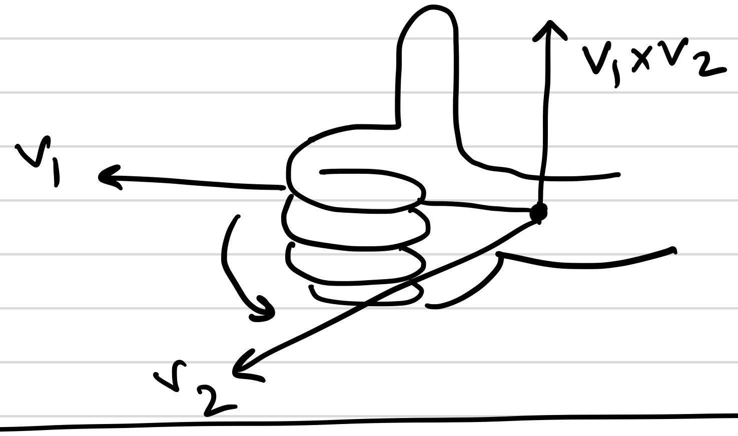
$$\underbrace{|V_1 \times V_2|}_{\text{length of } V_1\times V_2 \text{ vector}} = \underbrace{|V_1|\,|V_2|\,|\sin(\theta)|}_{\text{area of parallelogram made by } V_1 \,\&\, V_2}$$

angle between V_1 & V_2

(2) $V_1 \times V_2$ is orthogonal to both V_1 & V_2.

(to show this we show $(V_1 \times V_2)\cdot V_1 = 0$ & $(V_1 \times V_2)\cdot V_2 = 0$)

(3) The direction of $V_1 \times V_2$ is given by "righthand rule".



Some algebraic properities :

- $V_1 \times (V_2 + V_3) = (V_1 \times V_2) + (V_1 \times V_3)$
- $(cV_1) \times V_2 = c\,(V_1 \times V_2)$
- $V_1 \times V_2 = -(V_2 \times V_1)$

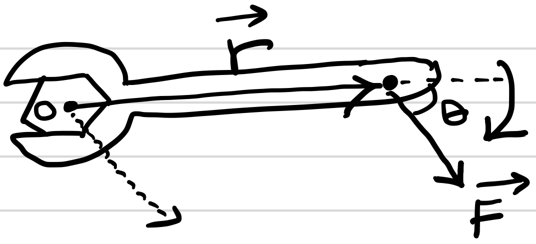
- $V_1 \times V_2 = 0 \iff V_1$ & V_2 are parallel.

An application in physics

Torque (and not turk!) $\rightsquigarrow$ Cross product

(work $\rightsquigarrow$ dot product)

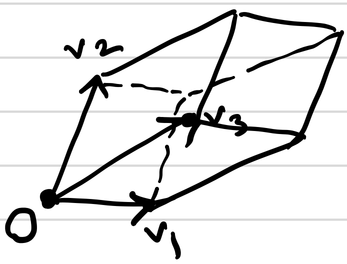
Greek tau   torque



$$\tau = \vec{r} \times \vec{F}$$

$$|\tau| = |r| \underbrace{|F|\ |\sin\theta|}_{\text{orth. comp. of force}}$$

Computing volume (of a parallelepiped) using $\times$ & $\cdot$ products



$$v_1 \cdot (v_2 \times v_3) = \det \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{bmatrix}$$

$$\text{Volume} = |v_1 \cdot (v_2 \times v_3)| = \left| \det \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{bmatrix} \right|.$$
