Oct. 30 / 2017

Happy Dracula

- Quiz tomorrow from 8.1 & 8.2 (Seq. & series)

General fact / theorem

Suppose $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Ex.

$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \cdots \longrightarrow \frac{1}{1-a}$ Convergent

$|a|<1 \quad -1<a<1$

$\lim_{n\to\infty} a^n = 0$

Conclusion (Convergence test): If $\lim_{n\to\infty} a_n$ does not exist or $\neq 0$

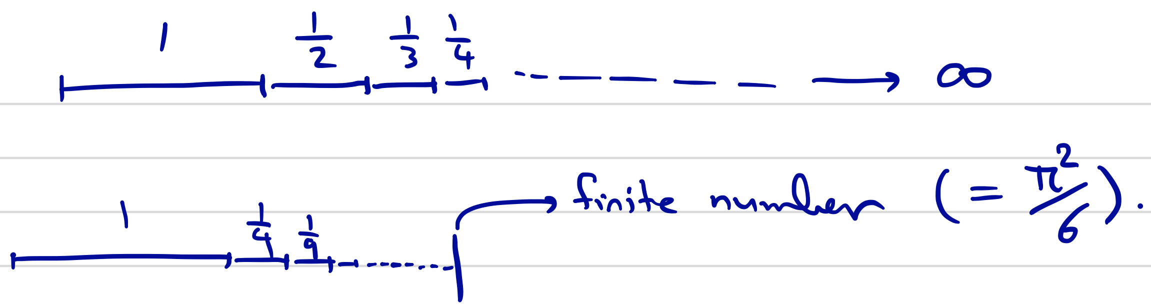
then $\sum_{n=1}^{\infty} a_n$ is divergent.

But if $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ may or may not be convergent.

$\sum_{n=1}^{\infty} a_n$ Convergent depends on how fast $a_n \xrightarrow[n\to\infty]{} 0$.

Famous example:

- $a_n = \frac{1}{n}$ $\quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots \longrightarrow \infty$ diverges
- $a_n = \frac{1}{n^2}$ $\quad 1 + \frac{1}{4} + \frac{1}{9} + \cdots$ Converges



- Operations on sums & series:

- $$\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n \pm b_n)$$

$$\left(\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) \neq \sum_{n=1}^{\infty} a_n b_n \right)$$

- $$c\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} c a_n .$$

- Changing index variable n:

$$\sum_{n=1}^{\infty} a_n = \sum_{m=0}^{\infty} a_{m+1} = \sum_{n=0}^{\infty} a_{n+1}$$

<u>Ex.</u>

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1}$$

$$= 1 + \sum_{n=2}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1} = 1 + \cancel{\sum_{n=1}^{N-1} \frac{1}{n+1}} - \cancel{\sum_{n=1}^{N-1} \frac{1}{n+1}} - \frac{1}{N+1}$$

$$= \boxed{1 - \frac{1}{N+1}}$$

Some examples

- Sequences.

Ex. Is the seq. $\dfrac{3^n}{n!}$ convergent ?

$$\frac{3\times 3\times 3\times \cdots \times 3}{1\times 2\times 3\times \cdots \times n} = \frac{\overbrace{3\times3\times3}^{3}}{1\times 2}\times \frac{\overbrace{3\times 3\times 3\times \cdots \times 3}^{n-3}}{3\times 4\times 5\times \cdots \times n-1}\times \frac{1}{n}$$

$$= \frac{27}{2}\cdot\left(\frac{3}{3}\right)\cdot\left(\frac{3}{4}\right)\cdot\left(\frac{3}{5}\right)\cdots\left(\frac{3}{n-1}\right)\cdot\frac{1}{n} \leq \frac{27}{2}\cdot\frac{1}{n}$$

$$0 < \frac{3^n}{n!} \leq \frac{27}{2}\cdot\frac{1}{n}$$ But $\lim_{n\to\infty} \frac{27}{2}\cdot\frac{1}{n} = 0$

Squeeze thm. implies that $\lim_{n\to\infty} \frac{3^n}{n!} = 0.$

Ex. $a_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$

Is $\{a_n\}$ convergent?

$$-1 \leq \sin(2n) \leq 1$$

$$\frac{-1}{1+\sqrt{n}} \leq a_n \leq \frac{1}{1+\sqrt{n}}$$

$$0$$

By squeeze thm.

$\lim_{n\to\infty} a_n = 0$

Ex. $\quad a_n = \dfrac{n^2}{e^n}$

$$\lim_{n\to\infty} \frac{n^2}{e^n} = 0 \qquad \text{by L'Hospital.}$$

$$\downarrow$$

$$\frac{2n}{e^n}$$

$$\downarrow$$

$$\lim_{n\to\infty} \frac{2}{e^n} = 0$$

<u>Ex.</u> (telescopic series $\longrightarrow$ lots of cancelations)

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} \qquad\qquad \frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$\sum_{n=2}^{M} \frac{1}{n-1} - \frac{1}{n+1} = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{M-1} - \frac{1}{M+1}\right).$$

$$= \left(1 + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\cdots} + \cancel{\frac{1}{M-1}}\right) - \left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\cdots} + \cancel{\frac{1}{M-1}} + \frac{1}{M} + \frac{1}{M+1}\right)$$

$$= 1 + \frac{1}{2} - \cancel{\frac{1}{M}} - \cancel{\frac{1}{M+1}} \to 0$$

$$M \to 0$$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = 1 + \frac{1}{2} = \frac{3}{2}$$