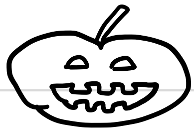
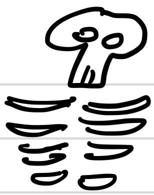
Oct. 27 /2017

Seq. and Series

$a_1, a_2, a_3, \dots$ a sequence

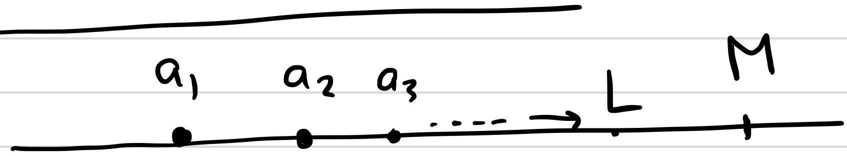
- If there exist M all a_n's $\leq M$
  then $\{a_n\}$ is <u>bounded from above</u> (by M).
- " " " m $m \leq$ all a_n's
  " " " <u>bounded from below</u> (by m)
- Bounded above & below $\Rightarrow$ <u>bounded</u>.

<u>Theorem</u> (Montonic Seq. theorem)

Let $\{a_n\}$ be a seq.

Suppose $\{a_n\}$ is <u>bounded</u> and <u>monotone</u>

then $\lim_{n\to\infty} a_n$ exist (or $\{a_n\}$ is Convergent).



a_n increasing

Intuitively it is clear a_n should converge to some $L \leq M$.

Sec. 8.2

Series

$a_1, a_2, a_3, \ldots$ Seq. $\rightsquigarrow$ Series

. $a_1 + a_2 + a_3 + \cdots$

. $\sum_{n=1}^{\infty} a_n$

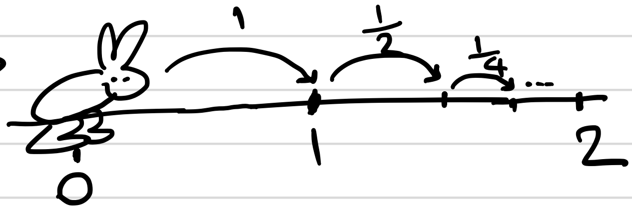
By $\sum_{n=1}^{\infty} a_n$ we mean the limit of the seq.

$a_1, \; a_1+a_2, \; a_1+a_2+a_3, \; a_1+a_2+a_3+a_4, \; \ldots$

Partial sums of the series.

Xenon $\rightsquigarrow$

Greek philosopher



Question $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = ?$

$\sum_{n=0}^{\infty} \frac{1}{2^n} = ?$ $\rightsquigarrow$ example "geometric" series

General alg. identity:

$$(1-a)\left(1 + a + a^2 + \cdots + a^m\right) = 1 - a^{m+1}$$

Proof

$$1 + \cancel{a} + \cancel{a^2} + \cdots + a^m - \cancel{a} - \cancel{a^2} - \cdots - a^{m+1} = 1 - a^{m+1}$$ ☺

Rem Very useful for all financial computations about compounding interest.

$$\boxed{1 + a + \cdots + a^m = \frac{1 - a^{m+1}}{1 - a}}$$

$$\sum_{n=0}^{\infty} a^n = \lim_{m \to \infty} \frac{1 - a^{m+1}}{1 - a}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \lim_{m \to \infty} \frac{1 - \left(\frac{1}{2}\right)^{m+1}}{1 - \frac{1}{2}} = \frac{1 - 0}{1/2} \qquad a = \frac{1}{2}$$

$$= 2$$ ☺

<u>Fact / thm</u>

$$\sum_{n=0}^{\infty} a^n \longrightarrow \begin{cases} \text{Converges } \boxed{\lim = \frac{1}{1-a}} & \text{if } 0 \leq |a| < 1 \\ \text{diverges} & \text{if } 1 \leq |a| \end{cases}$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad (\text{if } 0 \leq |a| < 1).$$

Little more general case:

c & a fixed numbers.

$$c + ca + ca^2 + \cdots + ca^m = \frac{c(1 - a^{m+1})}{1 - a}$$

$$\boxed{\sum_{n=0}^{\infty} ca^n = \frac{c}{1-a}} \rightsquigarrow 0 \leq |a| < 1$$

Ex. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots$

$0 < \frac{2}{3} < 1$

$$= \underbrace{\left(\frac{2}{3}\right)}_{\frac{2}{3}} \cdot \underbrace{\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots\right)}_{\frac{1}{1-\frac{2}{3}}} = \left(\frac{2}{3}\right)\cdot 3 = 2.$$

Ex. $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \cdots$

$$\frac{3}{2} = \frac{1}{1-\frac{1}{3}} = 1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\left(\frac{1}{3}\right)^3 + \cdots\right)$$

$$\frac{3}{2} - \left(1 - \frac{1}{3} - \frac{1}{9}\right) = \sum_{n=3}\left(\frac{1}{3}\right)^n$$

$$\sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^3 \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{27}\cdot\frac{1}{1-\frac{1}{3}}$$

$$= \frac{1}{27}\cdot\frac{3}{2} = \frac{1}{18}$$

Ex. (not a geo. series)

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots = ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$$

Trick : $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ ⤳ Similar to partial fraction

(Sometimes $\frac{1}{\text{quad. expression}} = \frac{1}{\text{linear}} \pm \frac{1}{\text{linear}}$.)

$$\to \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots$$

$$= 1$$

$$\sum_{n=0}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n} - \sum_{n=0}^{\infty} \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad = 1$$

Next time

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Ex. $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$ divergent $p=1$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ Convergent $\left(= \frac{\pi^2}{6}\right)$!!!

goes back to Leonard Euler