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Sequences & series

(infinite)
Series $\quad a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

- Power series (with coeff. $a_0, a_1, a_2, \ldots$)
(a function in variable x)

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Famous example : $\quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$2!$: $2 \leftarrow 2!$, $\quad 6 \leftarrow 3!$

Recall $n! = 1 \times 2 \times \cdots \times n$ $\quad$ (Convention $0! = 1$)

Theorem For every number x, this power series Converges and :

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

(Take derivative of $1 + x + \frac{x^2}{2!} + \cdots \rightsquigarrow$

$$0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots = 1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots$$

$\left(\frac{\cancel{n}}{1 \times 2 \cdots \times \cancel{n}} = \frac{1}{(n-1)!} \right)$ $\quad$ agrees with $(e^x)' = e^x$.)

Goal (goes back to Newton & Leibnitz) :
Express different functions (such as $\sin(x)$, $\cos(x)$, $\ln(x)$, $\ldots$) as power series.

Approximately Compute number e :

$$e = e^1 = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

magic!

. There are power series for $\sin(x)$, $\cos(x)$ ...

. Taylor series / polynomial

$\underline{\underline{f}}$ any (good) function

$$f(x) \longleftrightarrow \underbrace{\underset{a_0}{f(0)} + \underset{a_1}{f'(0)}\, x + \underset{a_2}{\frac{f''(0)}{2!}}\, x^2 + \cdots}_{\text{Taylor series of } f \text{ at } \underline{0}}$$

<u>Back to sequences</u>

<u>Rem</u> f some function e.g. $f(x) = \dfrac{x}{x+1}$

& if $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} a_n = L$

where $a_n = f(n)$.

$$\lim_{x\to\infty} \frac{x}{x+1} = 1 \qquad \left(\text{because } \frac{x+1-1}{x+1} = 1 - \underset{\to 0}{\cancel{\frac{1}{x+1}}}\right)$$

or

$$\text{L'Hospital } \frac{x'}{(x+1)'} = \frac{1}{1}$$

Then limit of the sequence $\frac{n}{n+1}$ is also 1.

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \longrightarrow 1.$

$f(1) \quad f(2) \quad f(3)$

So sometimes we can use methods from Calc. 1 to find limit of sequences. For example L'Hospital.

<u>Ex.</u> Find $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

$f(x) = \frac{\ln x}{x} \longrightarrow \lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to\infty} x = \infty$

L'Hospital $\frac{\ln'(x)}{x'} = \frac{\frac{1}{x}}{1}$ $\quad \lim_{x\to\infty} \frac{1}{x} = 0$

<u>Ex.</u> $\lim_{n\to\infty} \frac{(-1)^n}{n} = ?$ Here n should be <u>integer</u>.

(Can not use calc. 1) methods

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = 0.$$

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

$\longrightarrow 0.$

(Useful)

- General fact / theorem :

. $a_1, a_2, a_3, \dots \to L$ $\qquad \lim_{n\to\infty} a_n = L$

. f continuous function

Then: $f(a_1), f(a_2), f(a_3), \dots \longrightarrow f(L)$

Ex. $b_n = \sin\left(\frac{\ln(n)}{n}\right)$

$\lim_{n\to\infty} b_n = ?$

Let $a_n = \frac{\ln(n)}{n}$ we know $\lim_{n\to\infty} a_n = 0$

Then $\lim_{n\to\infty} \sin(a_n) = \sin(0) = 0$

Some useful limits

$\lim_{n\to\infty} \left(\frac{1}{n}\right)^p = 0 \qquad p>0$

$\lim_{n\to\infty} \left(\frac{1}{n}\right)^p = +\infty \qquad p<0$

Fix r

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & r>1 \\ \text{no limit} & r<-1 \\ 1 & r=1 \\ 0 & \boxed{-1<r<1} \to |r|<1 \end{cases}$$

Some simple definitions

- If $a_1 \leq a_2 \leq a_3 \leq \cdots$ increasing seq.
- If $a_1 \geq a_2 \geq a_3 \geq \cdots$ decreasing seq.
- If either dec. or inc. then monotone seq.