Oct. 2 / 2017

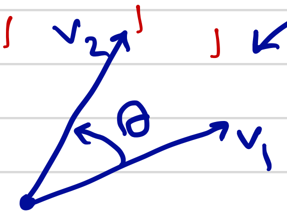
Quiz is from 7.4 arc length & 7.6 applications in physics

Last time: Dot product

$v_1 = (x_1, y_1, z_1) \qquad v_2 = (x_2, y_2, z_2)$

$$v_1 \cdot v_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

Fact / theorem: $$v_1 \cdot v_2 = |v_1|\,|v_2| \cos(\theta)$$



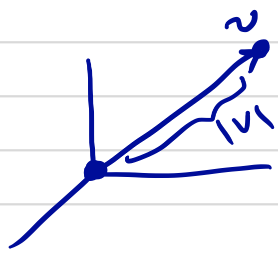
(It doesn't matter angle from v_1 to v_2 or v_2 to v_1)

$\cos(\theta) = \cos(-\theta)$

- Length & angle of vectors can be computed using coordinates.

Rem: $v = (x, y, z) \qquad |v| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{v \cdot v}$



$$|v|^2 = v \cdot v$$

Angle between v_1 & v_2 :

$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1|\,|v_2|} \rightsquigarrow \theta = \cos^{-1}\left(\frac{v_1 \cdot v_2}{|v_1|\,|v_2|}\right)$$

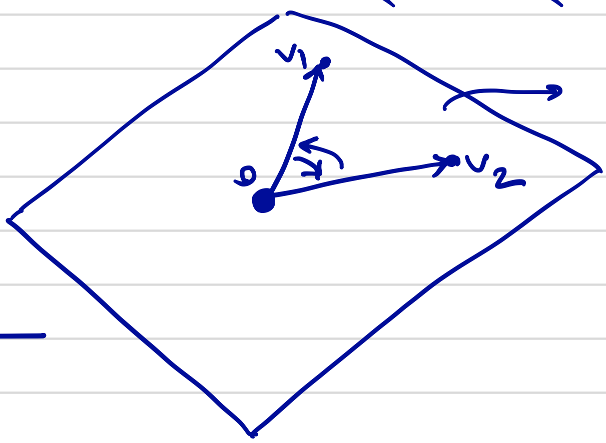
<u>Ex.</u> Angle between $v = (1,1,1)$ & x-axis ?
$(1,0,0)$

$$(1,1,1)\cdot(1,0,0) = 1$$

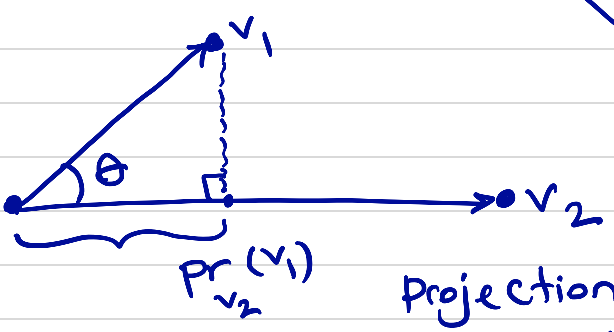
$$|(1,1,1)| = \sqrt{3} \qquad |(1,0,0)| = 1$$

$$\cos(\theta) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \rightsquigarrow \boxed{\theta = \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)}$$

$$0 \leq \theta \leq \pi$$



θ & $-\theta$ angles have same cos.



Projection of v_1 on v_2.
(shadow of v_1 on v_2)

length of proj. of v_1 on v_2 $= |v_1| \cos(\theta)$.

$v_1 \cdot v_2 = |v_2|$ times length of proj. of v_1 on v_2 .

Ex. What is the length of proj. of

$v_1=(1,1,1)$ on $v_2=(1,3,4)$?

Answer: $$v_1\cdot v_2 = 1+3+4 = 8$$

$$|v_2| = \sqrt{1+9+16} = \sqrt{26}$$

length of proj. of v_1 on v_2 $= \dfrac{8}{\sqrt{26}}$

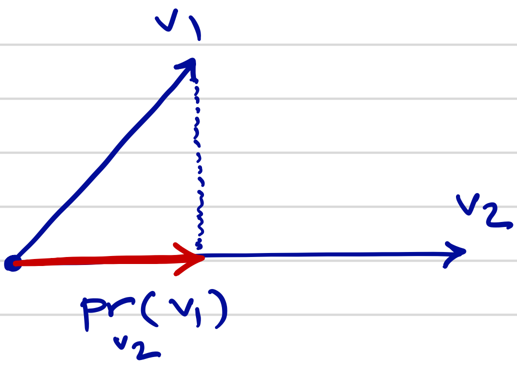
$$= \frac{|v_1|\,\cancel{|v_2|}\,\cos(\theta)}{\cancel{|v_2|}}$$

Rem v_1 and v_2 are orthogonal (i.e. angle $= \pi/2$)

$$\Longleftrightarrow \quad v_1\cdot v_2 = 0$$

(v_1 & v_2 non-zero vectors)

(because $\cos(\theta)=0$ if $\theta = \pi/2$ or $-\pi/2$).



we want coor. of the projected vector.

$$pr_{v_2}(v_1) = \frac{v_1\cdot v_2}{|v_2|}\;\frac{v_2}{|v_2|}$$

length of proj.

unit vector

$$pr_{v_2}(v_1) = \frac{v_1\cdot v_2}{v_2\cdot v_2}\; v_2$$

scalar

vector

<u>Ex.</u> $v_1 = (1,1,1)$ $\quad v_2 = (1, 2, -1)$

$$pr_{v_2}(v_1) = \frac{v_1 \cdot v_2}{v_2 \cdot v_2}\, v_2 \qquad \begin{array}{l} v_1\cdot v_2 = 2 \\ v_2 \cdot v_2 = 6 \end{array}$$

$$= \frac{1}{3}(1, 2, -1) = \left(\frac{1}{3}, \frac{2}{3}, \frac{-1}{3}\right).$$

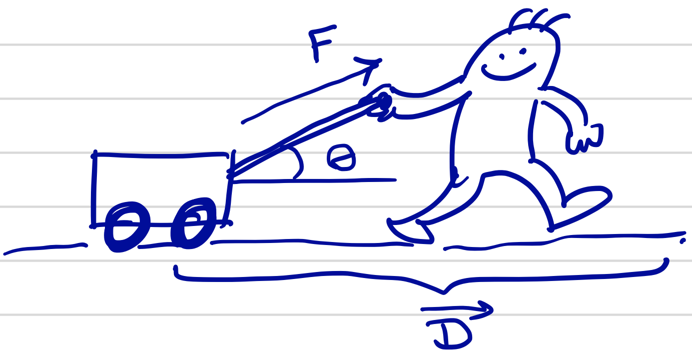
I can do geometry!

Work (in physics)

$$\underbrace{W}_{\text{scalar}} = \underset{\vec{F}}{\text{Force vector}} \underbrace{\,\bullet\,}_{\text{dot product}} \text{displacement vector } \vec{D}$$

$$W = |\vec{F}|\,|\vec{D}| \cos(\theta)$$

$$= |\vec{D}| \text{ times } \underbrace{\left(\text{Component of } \vec{F} \text{ along } \vec{D}\right)}_{\text{in other words proj. of } \vec{F} \text{ on } \vec{D}}$$



$$W = \vec{F} \cdot \vec{D} = |\vec{F}|\,|\vec{D}| \cos(\theta).$$

10.4

Cross product

v_1, v_2 3D vectors $\rightsquigarrow$ $v_1 \times v_2$ 3D vector

($v_1 \cdot v_2$ was a scalar)

$v_1 \cdot v_2$

→ Alg. $v_1 \cdot v_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$

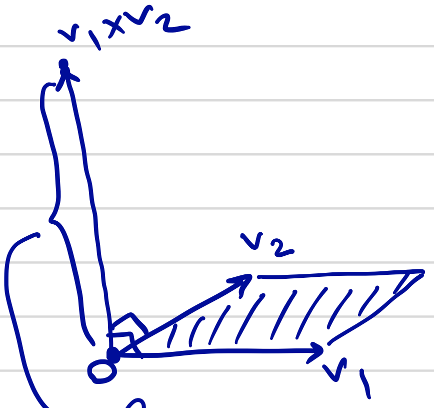
→ Geo. $v_1 \cdot v_2 = |v_1||v_2| \cos(\theta)$

$v_1 \times v_2$

→ Alg. $\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = (y_1 z_2 - z_1 y_2)\vec{i} - (x_1 z_2 - z_1 x_2)\vec{j} + (x_1 y_2 - y_1 x_2)\vec{k}$

→ Geo. $v_1 \times v_2$ is a vec. orthogonal to v_1 & v_2 & $|v_1 \times v_2| = |v_1||v_2||\sin(\theta)|$

($|v_1 \times v_2|$ = area of parallelogram formed by v_1, v_2)



length of $|v_1 \times v_2|$ = shaded area

Righthand rule (to decide direction of $v_1 \times v_2$)