Oct. 13 / 2017   Friday 13th

Cross product ⤳ Sir William Hamilton
19th century

- Hamiltonian mechanics
- Quaternions

. multi. of real numbers ⤳ 1-dim vectors

. " " Complex " ⤳ 2-dim vec.

? " " 3-dim vec. ⤳ Cross product

(no division ☹ for this product)

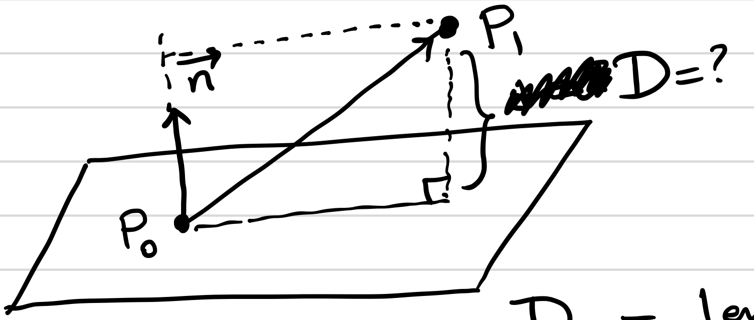
" " 4-dim vec. ⤳ Hamilton found Quaternions

Continue with equ. of lines & planes Sec. 10.5:

Problem   Plane $ax + by + cz + d = 0$

$P_1 = (x_1, y_1, z_1)$ point in 3D

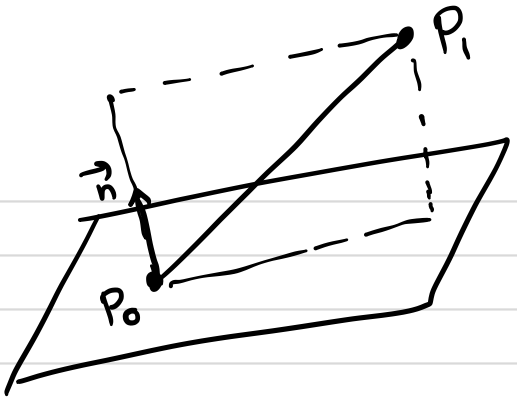
Find distance of this point P_1 to the plane.



P_0 some point on the plane

$\vec{n} = (a, b, c)$

D = length of projection of $\overrightarrow{P_0P_1}$ onto the normal vec. $\vec{n}$.



$$\vec{P_0P_1} \cdot \vec{n} = \underbrace{|\vec{n}|}_{\sqrt{a^2+b^2+c^2}} \; \underbrace{|\text{Proj}_{\vec{n}} \vec{P_0P_1}|}_{d \text{ we want}}$$

$$\vec{P_0P_1} = (x_1 - x_0,\ y_1 - y_0,\ z_1 - z_0)$$

$P_0 = (x_0, y_0, z_0)$ on the plane

$P_1 = (x_1, y_1, z_1)$

$$D = \frac{\vec{P_0P_1} \cdot \vec{n}}{|\vec{n}|} = \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2+b^2+c^2}}$$

$$D = \frac{ax_1 + by_1 + cz_1 - \overbrace{(ax_0 + by_0 + cz_0)}^{-d}}{\sqrt{a^2+b^2+c^2}}$$

$$\boxed{D = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}}$$

Ex. $P_1 = (1,1,1)$ $\quad 2x + 3y + z - 1 = 0$

$D =$ distance from P_1 to the plane

$$= \frac{2 \cdot 1 + 3 \cdot 1 + 1 \cdot 1 - 1}{\sqrt{4+9+1}} = \frac{5}{\sqrt{14}}$$

Parametric Curves Sec. 9.1

Parametric curve in 2D

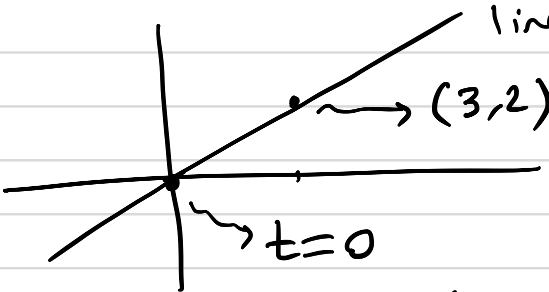
$(x(t), y(t))$ t parameter varying in a range $a \le t \le b$

Ex. $\begin{cases} x = 2t \\ y = 3t \end{cases}$ $\frac{x}{2} = t$ $-\infty < t < \infty$ any number

$(2t, 3t)$

You can solve y in terms of x (get rid of t)

$y = 3t = 3\left(\frac{x}{2}\right) = \frac{3}{2}x$.



line $y = \frac{3}{2}x$

$(3,2)$ $t=1$

$t=0$

Note: When we draw a parametric curve we do not see the parameter t.

Why the letter t? ⇝ because of t.

Note: There are many ways of parametrizing the same curve.

Ex. $(3t, 2t)$ $\boxed{(6t, 4t)}$

parametrize line $y = \frac{3}{2}x$

Ex. Graph of any function $y = f(x)$
can be though of a parametric curve.

$(x, f(x)) \rightsquigarrow$ point on the graph of f

$x = t$
$y = f(t)$

Ex. $y = x^2 \rightsquigarrow (t, t^2)$

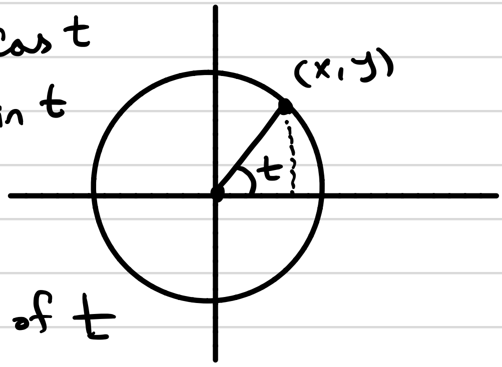
Ex. (Circle) $x^2 + y^2 = r^2$

$r > 0$ radius
Center $= (0,0)$

$$\boxed{\begin{aligned} x &= r\cos t \\ y &= r\sin t \end{aligned}}$$

$\frac{x}{r} = \cos t$

$\frac{y}{r} = \sin t$



$t =$ angle $\rightsquigarrow$ Sometimes θ instead of t

$0 \leq t < 2\pi$

Note: parametric representation of a curve
we may traverse the curve a bunch of times
or the curve can intersect itself ...

Variations of para. of circle:

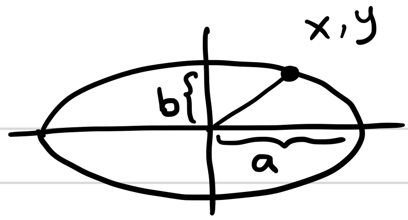
① $x = r\cos(-\theta)$ $\quad 0 \leq \theta < 2\pi$
$y = r\sin(-\theta)$

$\rightarrow$ $x = r\cos(\theta)$
$y = -r\sin(\theta)$

② $x = r\sin(\theta)$
$y = r\cos(\theta)$

③ $x = r\cos(2\theta)$
$y = r\sin(2\theta)$
$0 \leq \theta < \pi$

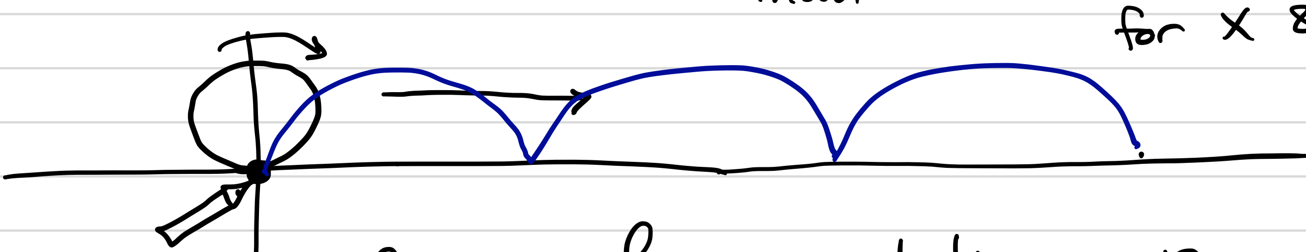
Ex. (Ellipse)



$$\left.\begin{aligned} x &= a\cos(\theta)\\ y &= b\sin(\theta)\end{aligned}\right\} \longleftrightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Ex. (cycloid) $\rightsquigarrow$ easier to write parametrically than to write equ. for x & y.



θ = angle of rotation $\quad$ r = radius (fixed)

$$\left.\begin{aligned} x &= r(\theta - \sin\theta)\\ y &= r(1-\cos\theta)\end{aligned}\right)$$ next time

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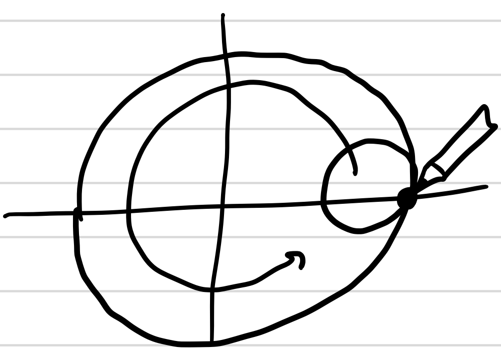
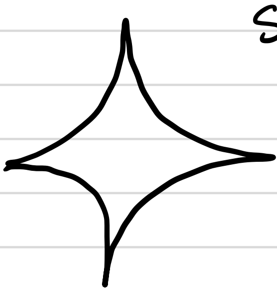


$\rightsquigarrow$



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