Nov. 6 / 2017 Quiz tomorrow 8.3 - 8.4 (ratio test)

Ratio test $\sum a_n$ Conv. or div ?

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ abs. conv.
- ---------------- $L > 1$ --- $\sum a_n$ div.

Rem If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ the ratio test does not say anything.

Example: • $a_n = \frac{1}{n}$ $\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim \frac{n}{n+1} = 1$ RT claims nothing.

we know $\sum \frac{1}{n}$ div.

• $a_n = \frac{1}{n^2}$ $\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim\left(\frac{n}{n+1}\right)^2 = 1$ " " .

we know $\sum \frac{1}{n^2}$ conv.

Example $\sum_{n=0}^{\infty}\frac{1}{n!} = \overset{=1}{\frac{1}{0!}} + \frac{1}{1!} + \frac{1}{2!} + \cdots$

$a_n = \frac{1}{n!} \rightsquigarrow \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{\cancel{1\times 2\times\cdots\times n}}{\cancel{1\times 2\times\cdots\times n}\times(n+1)}$

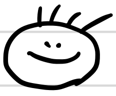
$= \frac{1}{n+1}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$ ☺

So $\sum_{n=0}^{\infty}\frac{1}{n!}$ conv. $\rightsquigarrow$ in fact later we see that sum $= e$.

Ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ Conv. or div. ?

Ratio test: $\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{(n+1)^3}{3^{n+1}}\right)}{\left(\frac{n^3}{3^n}\right)}\right| = \frac{(n+1)^3}{3n^3}$

$\lim = \frac{1}{3} < 1$ 

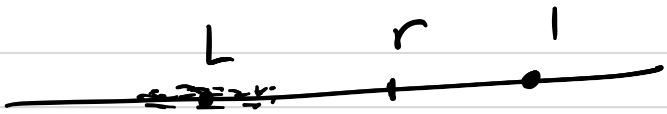
RT applies & series is abs. Conv. $\rightsquigarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ is conv.

Idea of proof of ratio test

Suppose $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$.

Then there is $L < r < 1$ such that for large enough $n \geq N$

$\displaystyle\frac{|a_{n+1}|}{|a_n|} < r < 1$



$n = N, N+1, N+2 \ldots$

$\displaystyle\frac{|a_{n+1}|}{|a_n|} < r \rightsquigarrow |a_{n+1}| < |a_n| r$

$|a_{N+1}| < |a_N| r$

$|a_{N+2}| < |a_{N+1}| r < |a_N| r^2$

$|a_{N+3}| < |a_{N+2}| r < |a_N| r^3$

$\vdots$

$|a_{N+1}| < |a_N| r$

$|a_{N+2}| < |a_N| r^2$

$|a_{N+3}| < |a_N| r^3$

$\longrightarrow \displaystyle\sum_{n=N}^{\infty} |a_n| < |a_N| \sum_{i=1}^{\infty} r^i$

geo. series conv. $r<1$

By Comparison we conclude $\sum |a_n|$ conv.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ Conv. or div. ?

RT

$$\frac{\overbrace{\frac{(n+1)^{(n+1)}}{(n+1)!}}^{a_{n+1}}}{\underbrace{\frac{n^n}{n!}}_{a_n}} = \frac{\overbrace{(n+1)^n \cdot \cancel{(n+1)}}^{(n+1)^{n+1}} \cdot n!}{n^n \quad \cancel{(n+1)!}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

RT applies

Series divergent!

(Recall: $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$ ☺)

The root test (also called Cauchy test)

- $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum a_n$ abs. Conv.
- ---------- > 1 ------ div.

(Rem: If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ test is inconclusive)

Ex. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{5n+1}\right)^n$ Conv. or div.

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+5}{5n+1}\right)^n} = \lim_{n\to\infty} \frac{2n+5}{5n+1} = \frac{2}{5} < 1$$ Conv. ☺

8.5 Power series ⇝ (infinite version of a polynomial)

or

polynomial with infinite number of terms

Polynomial: $1 + 3x + 4x^2$

$a_0 + a_1 x + a_2 x^2$

Power series: $1 + 2x + 3x^2 + 4x^3 + -----$

$a_0 + a_1 x + a_2 x^2 + -----$

$$\sum_{n=0}^{\infty} a_n x^n$$

Main question: Given coeff. a_n (i.e. given the power series) determine for which x, the power series is conv.

Sometimes we replace the variable x with $x - a$ ($\underline{a}$ some fixed number)

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

This is a power series at (or around) $\underline{\underline{a}}$.

Ex. $a_n = 1$

$$\sum_{n=0}^{\infty} x^n \rightsquigarrow \begin{cases} \text{Conv.} & 0 \leq |x| < 1 \quad (-1 < x < 1) \\ \text{div.} & 1 < |x| \end{cases}$$