Nov. 3 / 2017 ☺

Ex (from quiz)

$$a_n = \tan\left(2n \cdot \pi \cdot \frac{1}{1+8n}\right).$$

$$\lim_{n\to\infty} a_n = ?$$

Note:

$a_1, a_2, a_3, \ldots \longrightarrow ?$ is very different from

$a_1 + a_2 + a_3 + \cdots \longrightarrow ?$

Look at $f(x) = \tan\left(\frac{2x \cdot \pi}{1+8x}\right)$.

$$\lim_{x\to\infty} \tan\left(\frac{2\pi x}{1+8x}\right) = \tan\left(\frac{\pi}{4}\right) = 1.$$

(Continuity of tan at $\frac{\pi}{4}$).

$$\lim_{x\to\infty} \frac{2\pi x}{1+8x} = \frac{2\pi}{8} = \frac{\pi}{4}$$

Back to series convergence tests.

- Alternating series.

$b_1, b_2, b_3, \ldots \qquad b_n \geq 0$

$b_1 - b_2 + b_3 - b_4 + b_5 \cdots \longrightarrow ?$

Ex. $b_n = \frac{1}{n}$ (alt. harmonic series)

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ Conv.

$1 + \frac{1}{2} + \frac{1}{3} + \cdots$ div. (harmonic series)

Notation

$$b_1 - b_2 + b_3 - b_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

$(-1)^n$ $-1, +1, -1, \cdots$

$(-1)^{n+1} = (-1)^{n-1}$ $+1, -1, +1, -1, \cdots$

Alt. Series test $\rightsquigarrow$ Leibintz

$b_1, b_2, b_3, \cdots$

1. $b_n > 0$ for all n.
2. $b_1 \geq b_2 \geq b_3 \cdots$ decreasing
3. $\lim_{n\to\infty} b_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$

is Convergent

Ex. $b_n = \frac{1}{n}$

① $1, \frac{1}{2}, \frac{1}{3}, \cdots > 0$

② decreasing

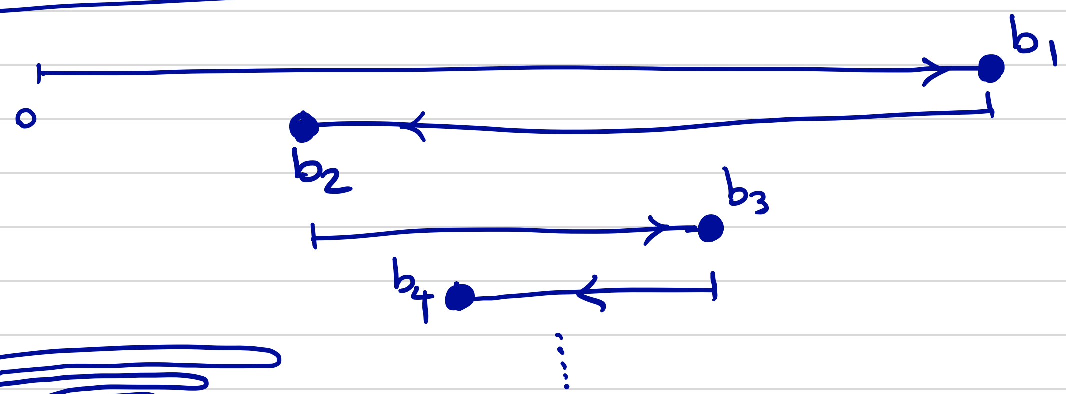
③ $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges (by alt. series test).

(In fact, we will see later using Taylor series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2)$$).

Justification of alt. series test



b_1

$b_1 - b_2$

$b_1 - b_2 + b_3$

$b_1 - b_2 + b_3 - b_4$

$\vdots$

limit of alt. series!

Ex. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{2n}{1+n}$ $\rightsquigarrow$ $b_n = \frac{-2n}{1+n}$

$$\lim_{n\to\infty} \frac{-2n}{1+n} = -2 \neq 0$$

So alt. series test does not apply.

Ex. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

1. $\dfrac{n^2}{n^3+1} > 0$
2. $\displaystyle\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$
3. dec. ?

$$f(x) = \frac{x^2}{x^3+1} \rightsquigarrow f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2}$$

$$= \frac{x(2-x^3)}{(x^3+1)^2} \rightsquigarrow x > 2 \text{ then } f'(x) < 0.$$

(It is OK if decreasing starting from $n=3, \dots$)

Absolute Convergence (of a series)

$a_1, a_2, \dots$ $\quad$ a_n positive or negative.

Def. $\sum_n a_n$ is "absolutely convergent"

if $\sum_n |a_n|$ is convergent.

Ex. $\sum_n (-1)^{n+1} \frac{1}{n}$ is convergent

but <u>not</u> absolutely convergent.

($\sum_n \frac{1}{n}$ is div.).

. $\sum_n a_n$ is "Conditionally Conv." if it is conv. but not abs. Conv.

Theorem If $\sum_n |a_n|$ is Conv. then $\sum_n a_n$ is Conv.

Or if abs. Conv. $\Longrightarrow$ conv. (in usual sense).

Next time: Ratio test.

Theorem (Ratio test):

$\sum_n a_n$

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ then $\sum_n a_n$ abs. Conv.
