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. Stewart Calc. online notes about diff. equ.

2nd order linear diff. equ.

$y(x)$ → variable

function we look for

2nd order

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = G(x)$$

Homogenous equation :

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \qquad (\star)$$

Easy fact

Suppose $y_1(x)$ & $y_2(x)$ are two solution of $(\star)$

Then any linear combination $c_1 y_1 + c_2 y_2$ (c_1, c_2 constants) is also a solution of $(\star)$.

Example $y'' + y = 0$ → homog. equ.

$$\left.\begin{aligned} y_1(x) &= \sin(x) \\ y_2(x) &= \cos(x) \end{aligned}\right\} \rightsquigarrow$$

$y(x) = c_1 \sin(x) + c_2 \cos(x)$ is also a solution.

$$y(x) = \sin(x)$$
$$y'(x) = \cos(x)$$
$$y''(x) = -\sin(x)$$

$$\left(y''(x) = -c_1 \sin(x) - c_2 \cos(x) \right).$$

Linearly independent solution :

$y_1(x)$ & $y_2(x)$ solutions of $(\star)$.

y_1 & y_2 are lin. ind. if they are <u>not</u> constant multiple of eachother. That is,

there is no c_1 const. $c_1 y_1(x) = y_2(x)$ on

- - - - - - - - - - - - c_2 - - - - - $c_2 y_2(x) = y_1(x)$.

<u>Example</u> $y'' + y = 0$

$y_1 = \sin(x)$ & $y_2 = \cos(x)$ are lin. ind.

But $\sin(x)$ & $5\sin(x)$ are lin. dependent.

(Harder) Fact : If $y_1(x)$ & $y_2(x)$ are two <u>linear independent</u> <u>Sol.</u> of :

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \qquad (\star)$$

& moreover $P(x)$ is never <u>0</u>, then :

<u>every</u> solution of $(\star)$ is of the form

$c_1 y_1 + c_2 y_2$ for const. c_1, c_2.

- Basically, if you know two lin. ind. sol. then you know all solutions.

Now we talk Case of "Constant Coeff."

2nd order lin. diff. equ : homog.

homog. equ.

$$\boxed{a y'' + b y' + c y = 0}$$

a, b, c Const.

<u>Rem</u> These Const. Coeff. 2nd. ord. lin. diff. equ. have many features similar to 2 lin. equ. 2 unknowns in linear alg.

Today's goal : Find solutions of this equ.

We guess that a sol. is of the form $y(x) = e^{rx}$.
(r const.)

We want to find r.

$$y(x) = e^{rx} \qquad y'(x) = r e^{rx} \qquad y''(x) = r^2 e^{rx}$$

$$a r^2 e^{rx} + b r e^{rx} + c e^{rx} = 0$$

characteristic equ.

$$(ar^2 + br + c)\, e^{rx} = 0 \iff \boxed{ar^2 + br + c = 0}$$

Need to find roots of $ar^2 + br + c = 0$

Quad. formula 

Ex. $y'' + y' - 6y = 0$.

Look for r $y = e^{rx}$ is a sol.

$\boxed{r^2 + r - 6 = 0}$ char. equ.

$r_1 = \dfrac{-1 + \sqrt{1+24}}{2} = 2$ $\qquad r_2 = \dfrac{-1 - \sqrt{1+24}}{2} = -3$

Conclusion: $c_1 e^{2x} + c_2 e^{-3x}$

(Note e^{2x} & e^{-3x} are lin. ind.)

$ay'' + by' + cy = 0$ Our equation

$ar^2 + br + c = 0$ char. equ.

① Char. equ. has two real roots r_1 & r_2 (as in the above example)
(i.e. $b^2 - 4ac > 0$)

General sol. : $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ basic sol.

② Char. equ. has only one real root r.
(i.e. $b^2 - 4ac = 0$)

General sol. : $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ basic sol.

Ex. $4y'' + 12y' + 9y = 0$.

$4r^2 + 12r + 9 = 0 \rightsquigarrow (2r+3)^2 = 0$

$r = -\frac{3}{2}$

$$\frac{-12 \pm \sqrt{\cancel{12^2 - 4.4.9}}^{\,0}}{8} = \frac{-12}{8} = \frac{-3}{2}.$$

There is only one sol. $(b^2 - 4ac = 0)$.

Basic sol. : $y_1(x) = e^{-\frac{3}{2}x}$ $\quad y_2(x) = x e^{-\frac{3}{2}x}$

(Please as a boring exercise plug-in y_1 & y_2 in $4y'' + 12y' + 9y$ & see you get 0.)

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3 Char. equ. has no real roots.
$(b^2 - 4ac < 0)$.

In this case suppose the roots of char. equ. are :

$r_1 = \alpha + i\beta$
$r_2 = \alpha - i\beta$

$i = \sqrt{-1}$
α, β real number

$$\underbrace{\frac{-b}{2a}}_{\alpha} \pm \underbrace{\frac{\sqrt{b^2 - 4ac}}{2a}}_{i\beta}$$

$$y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right).$$