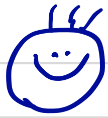
Nov. 17 / 2017



Quick review of Taylor series / poly.

f $\quad x = a$ (given to us)

- Supposedly we ~~know~~ Compute $f(a)$ & its derivative $f'(a)$, $f''(a)$, ...

- Compute or approx $f(x)$ for all x or at least x near a.

Answer: Best deg $\underline{\underline{n}}$ polynomial approx. to $f(x)$ around $\underline{\underline{x=a}}$ is :

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$$\cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

equals to $f(x)$ (in many situations).

Ex. $f(x) = e^x$ $a = 0$.

$$f(x) = T(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$ for all x.

Ex. $f(x) = e^{(x^2)}$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

Error of Taylor approx.

for some $a < z < x$

Theorem error

$$f(x) - T_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

This helps to estimate the error.

Sec. 8.8 $f(x) = \sin(x)$ at $a = 0$.

$\sin' = \cos$
$\sin'' = -\sin$
$\sin''' = -\cos$
$\sin'''' = \sin$
$\sin^{(5)} = \cos$
$\sin^{(6)} = -\sin$

Ex. deg 1 Taylor poly.

$$T_1(x) = \sin(0) + \sin'(0)x$$

$$= 0 + 1x = x$$

First deg. approx. $\sin(x) \approx x$

deg 2 $T_2(x) = \sin(0) + \sin'(0)x + \sin''(0)x^2$

$\sin''(0) = 0$

$= x$

:

deg 5 $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

$$T(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$\sin(x) = T(x)$ for all x.

<u>Question</u> For <u>what values</u> of x, (How close to 0)

$$\left|\sin(x) - T_5(x)\right| \le \frac{1}{10} \; ?$$

$$\left| \frac{\sin^{(6)}(z)}{6!} x^6 \right| = \frac{|-\sin(z)|}{6!} |x|^6 \le \boxed{\frac{|x|^6}{6!} \le \frac{1}{10}}$$

For some $0 \le z \le x$

Solve to find how small $|x|$ should be:

$$|x|^6 \le \frac{6!}{10} \Rightarrow |x| \le \sqrt[6]{\frac{6!}{10}}.$$

<u>Ex.</u> $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3} x^{\left(\frac{1}{3}-1\right)} = \frac{x^{-\frac{2}{3}}}{3} = \frac{1}{3\sqrt[3]{x^2}}$$

Taylor series/poly at $a=0$.

$f'(0)$ undefined 

So <u>no</u> Taylor series/poly at $a=0$.

Pick another <u>a</u>.

$a = 8.$ $\qquad f(8) = 2.$

$f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$ $\qquad f'(8) = \dfrac{1}{12}.$

$f''(x) = -\dfrac{2}{9} x^{-5/3}.$ $\qquad f''(8) = -\dfrac{1}{144}.$

$$T_2(x) = 2 + \frac{1}{12}(x-8) + \frac{-1}{144}\frac{(x-8)^2}{2}.$$

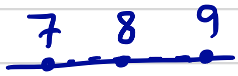
<u>Question</u> Estimate the error $\left|\sqrt[3]{x} - T_2(x)\right|$

if $7 \leq x \leq 9$ $\rightsquigarrow 0 \leq |x-8| \leq 1$

Answer: We need $f'''(x) = \dfrac{10}{27} x^{-8/3}.$

$$\left|\sqrt[3]{x} - T_2(x)\right| = \left|\frac{10}{27} z^{-8/3}\right| \frac{|x-8|^3}{3!}.$$

For some $\left.\begin{array}{c} 8 \leq z \leq x \\ \text{or} \\ x \leq z \leq 8 \end{array}\right\} \longrightarrow 7 \leq z \leq 9$



$|z|^{-8/3} \leq 7^{-8/3}$ $\qquad |x-8| \leq 1$

$$\left|\sqrt[3]{x} - T_2(x)\right| \leq \boxed{\frac{10}{27} \cdot 7^{-8/3} \cdot \frac{1}{3!}}$$

A list of useful Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \rightsquigarrow \text{Taylor series at } a=0.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

For people familiar with complex number:

$$\boxed{e^{ix} = \cos(x) + i\sin(x).}$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$(1+x)^2 = 1 + 2x + x^2$$

$$(1+x)^m = \underbrace{\sum_{n=0}^{m} \binom{m}{n} x^n}$$ $m = 1, 2, 3 \ldots$

$$\binom{m}{n} = m \text{ choose } n = \frac{m!}{(n-m)!\,n!}.$$ (From finite math)

$(1+x)^m$ has Taylor series of the form for when $\underline{\underline{m}}$ is not necessarily integer.
